A Note on Hyperbolic Generalized Pierre Numbers

Abstract. In this study, we present a new family of number sequences called the generalized hyperbolic Pierre numbers, defined within the bidimensional Clifford algebra of hyperbolic numbers. This algebraic framework enables the extension of classical sequence theory into the hypercomplex domain, offering both structural and analytical enrichment. As notable special cases, we examine the hyperbolic Pierre numbers and hyperbolic Pierre Lucas numbers, analyzing their algebraic properties, characteristic behaviors, and mutual relationships in detail.

We derive closed-form expressions through Binet-type formulas, construct generating functions that reflect the recursive nature of the sequences, and establish summation identities that reveal deeper arithmetic patterns. Furthermore, we develop matrix representations for each sequence, providing a compact and elegant algebraic tool for modeling and manipulating their evolution.

This research contributes to the broader theory of hypercomplex number sequences and proposes a novel approach to generalizing classical sequences within Clifford algebraic systems. The results have potential applications not only in pure mathematics but also in fields such as cryptography, numerical analysis, modeling of symmetric structures, solving differential equations, and algebraic representation of physical systems. In this context, the study lays a solid foundation for advanced investigations into the reinterpretation of number sequences in hyperbolic spaces.

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1. Introduction

Hypercomplex number systems, as introduced in [60], constitute algebraic extensions of the real numbers. Among the commutative examples are:

- **Complex numbers**

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\},$$

- **Hyperbolic numbers** (also known as double or split-complex numbers)[73]

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\},$$

- **Dual numbers** [56]

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\},$$

In contrast, non-commutative hypercomplex systems include:

- **Quaternions** [57]

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\},$$

- **Octonions** [49]
- **Sedenions** [75].

The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions), and $\mathbb{S}$ (sedenions) are real algebras constructed from the real numbers $\mathbb{R}$ via a recursive doubling procedure known as the Cayley–Dickson process. This process can be extended beyond the sedenions to generate higher-dimensional algebras referred to as 2^n-ions (see, for instance, [50], [58], [67]).

Quaternions were introduced by the Irish mathematician W. R. Hamilton (1805–1865) [57] as a four-dimensional extension of complex numbers. Hyperbolic numbers with complex coefficients were first proposed by J. Cockle in 1848 [53]. Later, H. H. Cheng and S. Thompson[52] extended the concept of dual numbers to the complex domain, referring to them as complex dual numbers. More recently, Akar, Yüce, and Şahin introduced the notion of [47] dual hyperbolic numbers, further enriching the landscape of hypercomplex number systems.

A **dual hyperbolic** number is a type of hypercomplex number defined as

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon j a_3,$$

where a_0, a_1, a_2 and a_3 are real numbers.

The set of all dual hyperbolic numbers is denoted by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon j a_3 : a_0, a_1, a_2, a_3 \in \mathbb{R},\ j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The basis elements$\{1, j, \varepsilon, \varepsilon j\}$ satisfy the following commutative multiplication rules:

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j,\ \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0,\ j^2 = j.j = 1, \\ \varepsilon.j &= j.\varepsilon,\ \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0,\ j(\varepsilon j) = (\varepsilon j)j = \varepsilon, \end{aligned}$$

Here, ε denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j is the hyperbolic unit ($j^2 = 1$), and εj is the dual hyperbolic unit

(with $(j\varepsilon)^2 = 0$).

The product of two dual hyperbolic numbers $q = a_0 + ja_1 + \varepsilon a_2 + j\varepsilon a_3 p = b_0 + jb_1 + \varepsilon b_2 + j\varepsilon b_3$ is given by

$$qp = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0) + \varepsilon(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + j\varepsilon(a_0b_3 + a_1b_2 + a_2b_1 + b_0a_3).$$

Addition of dual hyperbolic numbers is defined componentwise.

The dual hyperbolic numbers form a commutative ring, a real vector space and an algebra. But $\mathbb{H}_{\mathbb{D}}$ is not field because every dual hyperbolic numbers doesn't have an inverse. For more information on the dual hyperbolic numbers, see [47].

Here we use the set of hyperbolic numbers. The set of hyperbolic numbers $\mathbb{H}$ can be described as

$$\mathbb{H} = \{z = x + hy \mid h \notin \mathbb{R},\ h^2 = 1, x, y \in \mathbb{R}\}.$$

Here,h is the hyperbolic unit satisfying $h^2 = 1$, and x, y are real components.

The hyperbolic ring $\mathbb{H}$ constitutes a bidimensional Clifford algebra; for further details, see [63]. In the mathematical literature, hyperbolic numbers have appeared under various names, including Lorentz numbers, double numbers, duplex numbers, split-complex numbers, and perplex numbers. These numbers are particularly useful in modeling distances within the Lorentz space-time plane (see Sobczyk [73]). For additional insights into the structure and applications of

hyperbolic numbers,refer to [59,62,68,74] .

Addition, substraction and multiplication of any two hyperbolic numbers z_1 and z_2 are defined by

$$\begin{aligned} z_1 \pm z_2 &= (x_1 + hy_1) \pm (x_2 + hy_2) = (x_1 \pm x_2) + h(y_1 \pm y_2), \\ z_1 \times z_2 &= (x_1 + hy_1) \times (x_2 + hy_2) = x_1x_2 + y_1y_2 + h(x_1y_2 + y_1x_2) \end{aligned}$$

and the division of two hyperbolic numbers are given by

$$\frac{z_1}{z_2} = \frac{x_1 + hy_1}{x_2 + hy_2} = \frac{(x_1 + hy_1)(x_2 - hy_2)}{(x_2 + hy_2)(x_2 - hy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 - y_2^2} + h\frac{x_1y_2 + y_1x_2}{x_2^2 - y_2^2}.$$

It is easy to see that this algebra of hyperbolic numbers is commutative and contains zero divisors. The hyperbolic conjugation of $z = x + hy$ is defined by

$$\overline{z} = z^{\dagger} = x - hy.$$

Note that $\overline{\overline{z}} = z$. Note also that for any hyperbolic numbers z_1, z_2, z we have

$$\begin{aligned} \overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2}, \\ \overline{z_1 \times z_2} &= \overline{z_1} \times \overline{z_2}, \\ \|z\|^2 &= z \times \overline{z} = x^2 - y^2. \end{aligned}$$

Let us now revisit the definition of generalized Pierre numbers.

A generalized Pierre sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations

$$W_n = 2W_{n-1} - W_{n-4}, \tag{1.1}$$

with the initial values W_0, W_1, W_2, W_3 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-3)} - W_{-(n-4)},$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

The initial values of the generalized Pierre numbers for both positive and negative subscripts are presented in Table 1.

Table 1. A few generalized Pierre numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$2W_2 - W_3$
2	W_2	$2W_1 - W_2$
3	W_3	$2W_0 - W_1$
4	$2W_3 - W_0$	$4W_2 - W_0 - 2W_3$
5	$4W_3 - W_1 - 2W_0$	$4W_1 - 4W_2 + W_3$
6	$8W_3 - 2W_1 - W_2 - 4W_0$	$4W_0 - 4W_1 + W_2$
7	$15W_3 - 4W_1 - 2W_2 - 8W_0$	$W_1 - 4W_0 + 8W_2 - 4W_3$
8	$28W_3 - 8W_1 - 4W_2 - 15W_0$	$W_0 + 8W_1 - 12W_2 + 4W_3$
9	$52W_3 - 15W_1 - 8W_2 - 28W_0$	$8W_0 - 12W_1 + 6W_2 - W_3$
10	$96W_3 - 28W_1 - 15W_2 - 52W_0$	$6W_1 - 12W_0 + 15W_2 - 8W_3$
11	$177W_3 - 52W_1 - 28W_2 - 96W_0$	$6W_0 + 15W_1 - 32W_2 + 12W_3$
12	$326W_3 - 96W_1 - 52W_2 - 177W_0$	$15W_0 - 32W_1 + 24W_2 - 6W_3$
13	$600W_3 - 177W_1 - 96W_2 - 326W_0$	$24W_1 - 32W_0 + 24W_2 - 15W_3$

If we set $W_0 = 0, W_1 = 1, W_2 = 2, W_3 = 4$ then $\{W_n\}$ is the well-known Pierre sequence and if we set $W_0 = 4, W_1 = 2, W_2 = 4, W_3 = 8$ then $\{W_n\}$ is the well-known Pierre -Lucas sequence. In other words, Pierre sequence $\{P_n\}_{n\geq 0}$ and Pierre -Lucas sequence $\{C_n\}_{n\geq 0}$ are defined by the second-order recurrence relations

$$P_n = 2P_{n-1} - P_{n-4}, \qquad P_0 = 0, P_1 = 1, P_2 = 2,\ P_3 = 4, \quad n \geq 4, \tag{1.2}$$

and

$$C_n = 2C_{n-1} - C_{n-4}, \qquad C_0 = 4, C_1 = 2, C_2 = 4, C_3 = 8, \quad n \geq 4. \tag{1.3}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = 2P_{-(n-3)} - P_{-(n-4)},$$

and

$$C_{-n} = 2C_{-(n-3)} - C_{-(n-4)},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.2) and (1.3) hold for all integer n.

We can list some important properties of generalized Pierre numbers that are needed.

- Binet formula of generalized Pierre sequence can be calculated using its characteristic equation which is given as

$$z^4 - 2z^3 + 1 = (z^3 - z^2 - z - 1)(z - 1) = 0.$$

The roots of characteristic equation are

$$\begin{aligned}
\alpha &= \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3}, \\
\beta &= \frac{1 + \omega\sqrt[3]{19 + 3\sqrt{33}} + \omega^2\sqrt[3]{19 - 3\sqrt{33}}}{3}, \\
\gamma &= \frac{1 + \omega^2\sqrt[3]{19 + 3\sqrt{33}} + \omega\sqrt[3]{19 - 3\sqrt{33}}}{3}, \\
\delta &= 1,
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Using these roots and the recurrence relation, Binet formula can be given as

$$\begin{aligned}
W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \quad (1.4) \\
&= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-1)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-1)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-1)} + \frac{p_4}{(1-\alpha)(1-\beta)(1-\gamma)} \\
&= A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4
\end{aligned}$$

where p_1, p_2, p_3 and p_4 are given below

$$\begin{aligned} p_1 &= W_3 - (\beta+\gamma+\delta)W_2 + (\beta\gamma+\beta\delta+\gamma\delta)W_1 - \beta\gamma\delta W_0, \quad (1.5) \\ p_2 &= W_3 - (\alpha+\gamma+\delta)W_2 + (\alpha\gamma+\alpha\delta+\gamma\delta)W_1 - \alpha\gamma\delta W_0, \\ p_3 &= W_3 - (\alpha+\beta+\delta)W_2 + (\alpha\beta+\alpha\delta+\beta\delta)W_1 - \alpha\beta\delta W_0, \\ p_4 &= W_3 - W_2 - W_1 - W_0 \end{aligned}$$

and

$$\begin{aligned} A_1 &= \frac{W_3 - (\beta+\gamma+\delta)W_2 + (\beta\gamma+\beta\delta+\gamma\delta)W_1 - \beta\gamma\delta W_0}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}, \quad (1.6) \\ A_2 &= \frac{W_3 - (\alpha+\gamma+\delta)W_2 + (\alpha\gamma+\alpha\delta+\gamma\delta)W_1 - \alpha\gamma\delta W_0}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\ A_3 &= \frac{W_3 - (\alpha+\beta+\delta)W_2 + (\alpha\beta+\alpha\delta+\beta\delta)W_1 - \alpha\beta\delta W_0}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\ A_4 &= \frac{W_3 - W_2 - W_1 - W_0}{-2}. \end{aligned}$$

Binet formula of Pierre and Pierre Lucas sequences are

$$P_n = \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} - \frac{1}{2},$$

and

$$C_n = \alpha^n + \beta^n + \gamma^n + 1,$$

respectively.

The generating function for generalized Pierre numbers is

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 2W_0)x + (W_2 - 2W_1)x^2 + (W_3 - 2W_2)x^3}{1 - 2x + x^4}. \quad (1.7)$$

In the following section, we introduce the dual hyperbolic generalized Pierre numbers and investigate several of their fundamental properties.

Next, we give the exponential generating function of $\sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence W_n.

Lemma 1. *[69, Lemma 1.4]. Suppose that $f_{GW_n}(x) = \sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is the exponential generating function of the generalized Pierre sequence $\{W_n\}$. Then*

$$\begin{aligned} \sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \frac{(\alpha W_3 - \alpha(2-\alpha)W_2 + (-\alpha^2+\alpha+1)W_1 - W_0)}{2\alpha^2 + 2\alpha - 2} e^{\alpha x} \\ &+ \frac{(\beta W_3 - \beta(2-\beta)W_2 + (-\beta^2+\beta+1)W_1 - W_0)}{2\beta^2 + 2\beta - 2} e^{\beta x} \\ &+ \frac{(\gamma W_3 - \gamma(2-\gamma)W_2 + (-\gamma^2+\gamma+1)W_1 - W_0)}{2\gamma^2 + 2\gamma - 2} e^{\gamma x} + (\frac{W_3 - W_2 + W_1 - W_0}{-2}) e^x. \end{aligned}$$

The previous Lemma gives the following results as particular examples.

Corollary 2. *Exponential generating function of Pierre and Pierre-Lucas numbers are*

a): $\sum\limits_{n=0}^{\infty} P_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} (\frac{(\alpha^2+\alpha+1)\alpha^n}{2(\alpha^2+\alpha-1)} + \frac{(\beta^2+\beta+1)\beta^n}{2(\beta^2+\beta-1)} + \frac{(\gamma^2+\gamma+1)\gamma^n}{2(\gamma^2+\gamma-1)} - \frac{1}{2})\frac{x^n}{n!}$
$= \frac{(\alpha^2+\alpha+1)}{2(\alpha^2+\alpha-1)}e^{\alpha x} + \frac{(\beta^2+\beta+1)}{2(\beta^2+\beta-1)}e^{\beta x} + \frac{(\gamma^2+\gamma+1)\gamma^n}{2(\gamma^2+\gamma-1)}e^{\gamma x} - \frac{1}{2}e^x.$

b): $\sum\limits_{n=0}^{\infty} C_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} (\alpha^n+\beta^n+\gamma^n+1)\frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^{\gamma x} + e^x.$

Next, we provide an overview of selected publications in the literature that pertain to hyperbolic numbers.

- Cockle [53] presented the hyperbolic numbers with complex coefficients.
- Akar at al [47] introduced the dual hyperbolic numbers.
- Aydın [48] ntroduced the concept of hyperbolic Fibonacci numbers, defined by the following expression:

$$\widetilde{F}_n = F_n + hF_{n+1},$$

where Fibonacci numbers are given by $F_{n+2} = F_{n+1} + F_n$, with the initial conditation $F_0 = 0$, $F_1 = 1$.

- Soykan and Taşdemir [71] studied hyperbolic generalized Jacobsthal numbers given by

$$\widetilde{V}_n = V_n + hV_{n+1}$$

where generalized Jacobsthal numbers are $V_{n+2} = V_{n+1} + 2V_n$ with the initial conditation $V_0 = a$, $V_1 = b$.

- Dikmen and Altınsoy, [51] introduced On Third Order Hyperbolic Jacobsthal Numbers are

$$\begin{aligned}\widehat{J}_n^{(3)} &= J_n^{(3)} + hJ_{n+1}^{(3)},\\ \widehat{j}_n^{(3)} &= j_n^{(3)} + hj_{n+1}^{(3)}\end{aligned}$$

where Jacobsthal numbers, respectively, given by $J_n^{(3)} = J_{n-1}^{(3)} + J_{n-2}^{(3)} + 2J_{n-3}^{(3)}$, $J_0^{(3)} = 0$, $J_1^{(3)} = 1$, $J_2^{(3)} = 1$, $j_n^{(3)} = j_{n-1}^{(3)} + j_{n-2}^{(3)} + 2j_{n-3}^{(3)}$, $j_0^{(3)} = 2$, $j_1^{(3)} = 1$, $j_2^{(3)} = 5$.

- Yılmaz and Soykan , [70] studied hyperbolic generalized Guglielmo numbers given by

$$HW_n = Wn + jW_{n+1}$$

where generalized Guglielmo numbers are $W_n = 3W_{n-1} - 3W_{n-2} + W_{n-3}$ with the initial conditation $W_0, W_1, W_2 \quad (n \geq 2)$.

- Ayrılma and Soykan , [55] introduced On Hyperbolic Edouard Numbers are

$$HE_n = 7HE_{n-1} - 7HE_{n-2} + HE_{n-3}$$

where generalized Edouard numbers are $E_n = 7E_{n-1} - 7E_{n-2} + E_{n-3}$ with the initial conditation $E_0 = 0, E_1 = 1, E_2 = 7$

Following this, we provide details on dual hyperbolic sequences as they are presented in literature.

- Demirci and Soykan, [54] studied dual hyperbolic generalized Adrien numbers given by

$$\hat{A}_n = 3\hat{A}_{n-1} - \hat{A}_{n-2} - \hat{A}_{n-4}$$

where generalized Adrien numbers are $A_n = 3A_{n-1} - A_{n-2} + A_{n-4}$ with the initial conditation $A_0 = 0, A_1 = 1, A_2 = 3, A_3 = 8,\ \ n \geq 4$.

- Kalca and Soykan , [61] studied dual hyperbolic generalized Pandita numbers given by

$$\hat{P}_n = 2\hat{P}_{n-1} - \hat{P}_{n-2} + \hat{P}_{n-3} - \hat{P}_{n-4}$$

where generalized Pandita numbers are $P_n = 2P_{n-1} - P_{n-2} + P_{n-3} - P_{n-4}$ with the initial conditation $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 3,\ \ n \geq 4$.

- [64]The Narayana hybrid sequence NH_n and Narayana–Lucas hybrid sequenceUH_n are defined as follows:

$NH_n = N_n + \imath N_{n+1} + \epsilon N_{n+2} + hN_{n+3}$, $UH_n = U_n + \imath U_{n+1} + \epsilon Un_{+2} + hU_{n+3}$,where $\imath, \epsilon, h$ are operators such that $\imath^2 = -1$,

$\epsilon^2 = 0, h^2 = 1, \imath h = -h_1 = \epsilon + 1$.

- In [72], the outhers introduce the dual generalized Fibonacci matrices.

In this paper, we define the hyperbolic generalized Pierre numbers in the next section and give some properties of them.

Next, the hyperbolic Fibonacci sequence will be introduced, followed by an explanation of its relationship with Pierre numbers. Subsequently, the practical applications and significance of Pierre numbers in daily life will be discussed.

2. Hyperbolic Fibonacci Numbers

Hyperbolic extensions of classical recursive sequences provide deeper algebraic and geometric interpretations, particularly within the context of hypercomplex systems, combinatorics, and theoretical physics. Among these, the hyperbolic formulation of the Fibonacci sequence has attracted attention due to its analytical richness and structural elegance.

Hyperbolic Fibonacci numbers generalize the classical Fibonacci sequence using hyperbolic functions. One such formulation involves the hyperbolic sine function defined as:

$$\sinh_F(x) = \frac{\phi^x - \psi^x}{\phi - \psi}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$ are the golden ratio and its conjugate, respectively. This expression yields values closely related to the classical Fibonacci numbers for integer inputs:

$$\sinh_F(n) \approx F_n$$

Further generalizations include hyperbolic cosine and tangent functions that encode Fibonacci-related ratios. These constructions are useful in analytic number theory and combinatorial identities[66]

3. Applications and Relevance of Pierre Numbers

Pierre numbers, introduced as a generalization of classical recursive sequences such as Narayana's Cows, exhibit rich algebraic structures that render them suitable for both theoretical investigation and practical modeling. Although primarily studied within pure mathematics, their inherent properties—such as fourth-order recurrence relations, matrix formulations, and closed-form expressions—enable interdisciplinary applications across various scientific fields.

3.1. Potential Applications in Applied Contexts. While Pierre numbers have yet to achieve widespread recognition in engineering or consumer technologies, their structural parallels with well-established sequences such as Fibonacci, Narayana, and Lucas numbers suggest promising avenues for future utilization:

(1) **Digital Signal Processing:** Recursive sequences like Pierre numbers can be employed in waveform modeling, filter design, and compression algorithms, particularly in systems characterized by layered or fourth-order recurrence behavior.

(2) **Cryptography and Coding Theory:** The algebraic and modular properties of Pierre numbers may contribute to the development of secure key generation protocols and robust error-correcting codes.

(3) **Pattern Recognition and Image Processing:** Matrix representations and transformation schemes derived from Pierre sequences can be adapted for feature extraction in visual data, especially in contexts involving periodic or recursive structures.

(4) **Biological and Fluid Modeling:** Recent studies indicate that generalized number systems—including Pierre-Narayana variants—can effectively model cilia-driven flow, microorganism propulsion, and mucus transport in low Reynolds number environments.

3.2. Contributions to Mathematical Research. Pierre numbers also support advanced modeling frameworks in several mathematical disciplines:

(1) **Hypercomplex Systems:** Their extension into Gaussian, hyperbolic, and Clifford algebras facilitates simulations in non-Euclidean geometries and relativistic contexts.

(2) **Special Functions and Combinatorics:** Pierre sequences yield novel identities, generating functions, and summation formulas that enrich the landscape of analytic number theory.

(3) **Numerical Methods:** The recurrence structure of Pierre numbers aligns with finite-difference and iterative schemes commonly employed in computational fluid dynamics and bioengineering simulations.

In the following section, we introduce the hyperbolic generalized Pierre numbers and establish several of their fundamental properties.

4. Hyperbolic Generalized Pierre Numbers and their Generating Functions and Binet's Formulas

In this section, we introduce the hyperbolic generalized Pierre numbers and derive their corresponding generating functions and Binet formulas. We now define the hyperbolic generalized Pierre numbers over the algebra $\mathbb{H}_{\mathbb{D}}$of dual hyperbolic numbers.The nth hyperbolic generalized Pierre number is

$$HW_n = W_n + jW_{n+1}. \tag{4.1}$$

The sequence $\{HW_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$HW_{-n} = W_{-n} + jW_{-n+1},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (4.2) holds for all integer n.

Note that

$$\begin{aligned} HW_0 &= W_0 + jW_1 = W_0 + jW_1, \\ HW_1 &= W_1 + jW_2 = W_1 + jHW_2, \\ HW_2 &= W_2 + jW_3 = W_2 + jHW_3, \\ HW_3 &= W_3 + jW_4 = W_3 + j(2HW_3 - HW_0). \end{aligned}$$

It can be easily shown that

$$HW_n = 2HW_{n-1} - HW_{n-4} \tag{4.2}$$

and

$$HW_{-n} = 2HW_{-(n-3)} - HW_{-(n-4)}.$$

The initial values of the hyperbolic generalized Pierre numbers for both positive and negative subscripts are listed in Table 2.

A few hyperbolic generalized Pierre numbers

n	HW_n	HW_{-n}
0	HW_0	HW_0
1	HW_1	$2HW_2 - HW_3$
2	HW_2	$2HW_1 - HW_3$
3	HW_3	$2W_0 - W_1$
4	$2HW_3 - HW_0$	$4HW_2 - HW_0 - 2HW_3$
5	$4HW_3 - HW_1 - 2HW_0$	$4HW_1 - 4HW_2 + HW_3$
6	$8HW_3 - HW_2 - 2HW_1 - 4HW_0$	$4HW_0 - 4HW_1 + HW_2$
7	$15HW_3 - 2HW_2 - 4HW_1 - 8HW_0$	$HW_1 - 4HW_0 + 8HW_2 - 4HW_3$
8	$28HW_3 - 4HW_2 - 8HW_1 - 15HW_0$	$HW_0 + 8HW_1 - 12HW_2 + 4HW_3$
9	$52HW_3 - 8HW_2 - 15HW_1 - 28HW_0$	$8HW_0 - 12HW_1 + 6HW_2 - HW_3$
10	$96HW_3 - 15HW_2 - 28HW_1 - 52HW_0$	$6HW_1 - 12HW_0 + 15HW_2 - 8HW_3$
11	$177HW_3 - 15HW_2 - 28HW_1 - 96HW_0$	$6HW_0 + 15HW_1 - 32HW_2 + 12HW_3$
12	$326HW_3 - 52HW_2 - 96HW_1 - 177HW_0$	$15HW_0 - 32HW_1 + 24HW_2 - 6HW_3$
13	$600HW_3 - 96HW_2 - 177HW_1 - 326HW_0$	$24HW_1 - 32HW_0 + 24HW_2 - 15HW_3$

As special cases, the nth dual hyperbolic Pierre numbers and the nth dual hyperbolic Pierre Lucas numbers are given as

$$HP_n = P_n + jP_{n+1} \tag{4.3}$$

and

$$HC_n = C_n + jC_{n+1} \tag{4.4}$$

respectively. The sequences $\{HP_n\}_{n\geq 0}$ and $\{HC_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$HP_{-n} = 2P_{-(n-3)} - P_{-(n-4)},$$

and

$$HC_{-n} = 2C_{-(n-3)} - C_{-(n-4)},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (4.3) and (4.4) holds for all integer n.

For hyperbolic Pierre numbers (taking $W_n = P_n,\ \ P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 4,$) we get

$$\begin{aligned} HP_0 &= j, \\ HP_1 &= 2j + 1, \\ HP_2 &= 4j + 2, \end{aligned}$$

and for hyperbolic Pierre Lucas numbers (taking $W_n = C_n,\ \ C_0 = 4, C_1 = 2, C_2 = 4, C_3 = 8,$) we get

$$\begin{aligned} HC_0 &= 2j+4, \\ HC_1 &= 4j2, \\ HC_2 &= 8j+4. \end{aligned}$$

Selected values of the hyperbolic Pierre numbers and hyperbolic Pierre Lucas numbers for both positive and negative subscripts are presented in Table 3 and Table 4, respectively.

Table 3.Hyperbolic Pierre numbers

n	HP_n	HP_{-n}
0	j	j
1	$2j+1$	0
2	$4j+2$	0
3	$8j+4$	-1
4	$15j+8$	$-j$
5	$28j+15$	0

Table 4. Hyperbolic Pierre- Lucas numbers

n	HC_n	HC_{-n}
0	$2j+4$	$2j+4$
1	$4j+2$	$4j$
2	$8j+4$	0
3	$12j+8$	6
4	$22j+12$	$-4+6j$
5	$40j+22$	$-4j$

We now present the Binet formula for the hyperbolic generalized Pierre numbers, and for the remainder of the paper, we adopt the following notational conventions.

$$\widehat{\alpha} = 1 + j\alpha, \tag{4.5}$$

$$\widehat{\beta} = 1 + j\beta, \tag{4.6}$$

$$\widehat{\gamma} = 1 + j\gamma \tag{4.7}$$

$$\widehat{\delta} = \widehat{1} = 1 + j, \tag{4.8}$$

Note that we have the following identities:

$$\begin{aligned}
\widehat{\alpha}^2 &= 1+\alpha^2+2\alpha j, \\
\widehat{\beta}^2 &= 1+\beta^2+2j\beta, \\
\widehat{\alpha}\widehat{\beta} &= 1+\alpha\beta+(\alpha+\beta)j, \\
\widehat{\gamma}^2 &= 1+\gamma^2+2j\gamma, \\
\widehat{\delta}^2 &= \widehat{1}^2 = 2+2j, \\
\widehat{\gamma}\widehat{\delta} &= 1+\gamma+j+j\gamma.
\end{aligned}$$

Theorem 3. *(Binet's Formula) For any integer n, the nth hyperbolic generalized Pierre number is*

$$HW_n = \widehat{\alpha}A_1\alpha^n + \widehat{\beta}A_2\beta^n + \widehat{\gamma}A_3\gamma^n + \widehat{\delta}A_4 \tag{4.9}$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}, \widehat{\delta}$ are given as (4.5)-(4.8)

Proof. Using Binet's formula of the generalized Pierre numbers given below

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4$$

where A_1, A_2, A_3, A_4 are given (1.6) we get

$$\begin{aligned}
HW_n &= W_n + jW_{n+1}, \\
&= A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4 + (A_1\alpha^{n+1} + A_2\beta^{n+1} + A_3\gamma^{n+1} + A_4)j \\
&= \alpha A_1\alpha^n + \widehat{\beta}A_2\beta^n + \widehat{\gamma}A_3\gamma^n + \widehat{\delta}A_4.
\end{aligned}$$

This proves (4.9).

As special cases, for any integer n, the Binet's Formula of nth hyperbolic Pierre number is

$$HP_n = \frac{(\alpha^2+\alpha+1)\alpha^n\widehat{\alpha}}{2(\alpha^2+\alpha-1)} + \frac{(\beta^2+\beta+1)\beta^n\widehat{\beta}}{2(\beta^2+\beta-1)} + \frac{(\gamma^2+\gamma+1)\gamma^n\widehat{\gamma}}{2(\gamma^2+\gamma-1)} - \frac{\widehat{1}}{2} \tag{4.10}$$

and the Binet's Formula of nth hyperbolic Pierre Lucas number is

$$HC_n = \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n + \widehat{\gamma}\gamma^n + \widehat{1}. \tag{4.11}$$

Next, we present generating function.

Theorem 4. *The generating function for the hyperbolic generalized Pierre numbers is*

$$f_{HW_n}(x) = \sum_{n=0}^{\infty} HW_n x^n = \frac{HW_0 + (HW_1 - 2HW_0)x + (HW_2 - 2HW_1)x^2 + (HW_3 - 2HW_2)x^3}{1-2x+x^4}.$$

Proof. We assume that $f_{HW_n}(x)$ is the generating function of the hyperbolic generalized Pierre numbers and then we can write

$$f_{HW_n}(x) = \sum_{n=0}^{\infty} HW_n x^n.$$

Then, using the definition of the hyperbolic generalized Pierre numbers, and substracting $xf(x)$ and $x^2f(x)$ from $f(x)$, we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
(1-2x+x^4)f_{HW_n}(x) &= \sum_{n=0}^{\infty} HW_n x^n - 2x\sum_{n=0}^{\infty} HW_n x^n + x^4\sum_{n=0}^{\infty} HW_n x^n \\
&= \sum_{n=0}^{\infty} HW_n x^n - 2\sum_{n=0}^{\infty} HW_n x^{n+1} + \sum_{n=0}^{\infty} HW_n x^{n+4} \\
&= \sum_{n=0}^{\infty} HW_n x^n - 2\sum_{n=1}^{\infty} HW_{(n-1)} x^n + \sum_{n=4}^{\infty} HW_{(n-4)} x^n \\
&= (HW_0 + HW_1 x + HW_2 x^2 + HW_3 x^3) - 2(HW_0 x + HW_1 x^2 + HW_2 x^3) \\
&\quad + \sum_{n=4}^{\infty}(HW_n - 2HW_{n-1} + HW_{n-4})x^n \\
&= HW_0 + (HW_1 - 2HW_0)x + (HW_2 - 2HW_1)x^2 + (HW_3 - 2HW_2)x^3.
\end{aligned}$$

As special cases, the generating functions for the hyperbolic Pierre and hyperbolic Pierre Lucas numbers are

$$\sum_{n=0}^{\infty} HP_n x^n = \frac{j+x}{1-2x+x^4}$$

and

$$\sum_{n=0}^{\infty} HC_n x^n = \frac{2j+4-6x-4jx^3}{1-2x+x^4}$$

respectively.

Next, we give the exponential generating function of $\sum_{n=0}^{\infty} HW_n \frac{x^n}{n!}$ of the sequence HW_n.

Lemma 5. *Suppose that $f_{HW_n}(x) = \sum_{n=0}^{\infty} HW_n \frac{x^n}{n!}$ is the exponential hyperbolic generating function of the generalized Pierre sequence $\{ HW_n\}$.*

Then $\sum_{n=0}^{\infty} HW_n \frac{x^n}{n!}$ is given by

$$\sum_{n=0}^{\infty} HW_n \frac{x^n}{n!} == A_1 e^{\alpha x}\widehat{\alpha} + A_2 e^{\beta x}\widehat{\beta} + A_3 e^{\gamma x}\widehat{\gamma} + A_4 e^{x}\widehat{1}.$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}, \widehat{\delta}$ are given as (4.5)-(4.8)

Proof. Using Binet's formula

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4.$$

where A_1, A_2, A_3, A_4 are given as in (1.6) we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\widehat{W}_n\frac{x^n}{n!} &= \sum_{n=0}^{\infty}W_n\frac{x^n}{n!}+j\sum_{n=0}^{\infty}W_{n+1}\frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty}(A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4)\frac{x^n}{n!}+j\sum_{n=0}^{\infty}(A_1\alpha^{n+1}+A_2\beta^{n+1}+A_3\gamma^{n+1}+A_4)\frac{x^n}{n!} \\
&= (A_1e^{\alpha x}+A_2e^{\beta x}+A_3e^{\gamma x}+A_4e^{x})+j(A_1\alpha e^{\alpha x}+A_2\beta e^{\beta x}+A_3\gamma e^{\gamma x}+A_4e^{x}) \\
&= A_1e^{\alpha x}(1+j\alpha)+A_2e^{\beta x}(1+j\beta)+A_3e^{\gamma x}(1+j\gamma)+A_4e^{x}(1+j) \\
&= A_1e^{\alpha x}\widehat{\alpha}+A_2e^{\beta x}\widehat{\beta}+A_3e^{\gamma x}\widehat{\gamma}+A_4e^{x}\widehat{1}
\end{aligned}$$

This proves (5). □

The previous Lemma gives the following results as particular examples.

COROLLARY 6. *Exponential generating function of hiperbolic Pierre and hiperbolic Pierre-Lucas numbers are*

a):

$$\begin{aligned}
\sum_{n=0}^{\infty}HP_n\frac{x^n}{n!} &= \Big(\frac{(\alpha^2+\alpha+1)}{2(\alpha^2+\alpha-1)}e^{\alpha x}+\frac{(\beta^2+\beta+1)}{2(\beta^2+\beta-1)}e^{\beta x}+\frac{(\gamma^2+\gamma+1)}{2(\gamma^2+\gamma-1)}e^{\gamma x}-\frac{1}{2}e^{x}\Big) \\
&\quad +j\Big(\frac{\alpha(\alpha^2+\alpha+1)}{2(\alpha^2+\alpha-1)}e^{\alpha x}+\frac{\beta(\beta^2+\beta+1)}{2(\beta^2+\beta-1)}e^{\beta x}+\frac{\gamma(\gamma^2+\gamma+1)}{2(\gamma^2+\gamma-1)}e^{\gamma x}-\frac{1}{2}e^{x}\Big)
\end{aligned}$$

b):

$$\sum_{n=0}^{\infty}HC_n\frac{x^n}{n!}=e^{\alpha x}+e^{\beta x}+e^{\gamma x}+e^{x}+j(\alpha e^{\alpha x}+\beta e^{\beta x}+\gamma e^{\gamma x}+e^{x}).$$

5. Obtaining Binet Formula From Generating Function

Next, we derive the Binet formula for the generalized hyperbolic Pierre numbers $\{HW_n\}$ by utilizing their corresponding generating function.

THEOREM 7. *Binet's formula of generalized hyperbolic Pierre numbers:*

$$HW_n=\frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}, \tag{5.1}$$

where

$$\begin{aligned}
q_1 &= HW_0\alpha^3+(HW_1-2HW_0)\,\alpha^2+(HW_2-2HW_1)\,\alpha+HW_3-2HW_2, \\
q_2 &= HW_0\beta^3+(HW_1-2HW_0)\,\beta^2+(HW_2-2HW_1)\,\beta+HW_3-2HW_2, \\
q_3 &= HW_0\gamma^3+(HW_1-2HW_0)\,\gamma^2+(HW_2-2HW_1)\,\gamma+HW_3-2HW_2, \\
q_4 &= HW_0\delta^3+(HW_1-2HW_0)\,\delta^2+(HW_2-2HW_1)\,\delta+HW_3-2HW_2.
\end{aligned}$$

Proof. Let

$$h(x) = x^4 - 2x + 1.$$

Then for some α, β, γ and δ we write

$$h(x) = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x).$$

i.e.,

$$x^4 - 2x + 1 = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x). \tag{5.2}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ and $\frac{1}{\delta}$ are the roots of $h(x)$. This gives α, β, γ and δ as the roots of

$$h(\frac{1}{x}) = h(\frac{1}{x}) = 1 - \frac{2}{x} + \frac{1}{x^4} = 0.$$

This implies $x^4 - 2x + 1 = 0$. Now, by it follows that

$$\sum_{n=0}^{\infty} HW_n x^n = \frac{(HW_3 - 2HW_2)\, x^3 + (HW_2 - 2HW_1)\, x^2 + (HW_1 - 2HW_0)\, x + HW_0}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}.$$

Then we write

$$\frac{(HW_3 - 2HW_2)\, x^3 + (HW_2 - 2HW_1)\, x^2 + (HW_1 - 2HW_0)\, x + HW_0}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)} = \frac{B_1}{(1-\alpha x)} + \frac{B_2}{(1-\beta x)} + \frac{B_3}{(1-\gamma x)} + \frac{B_4}{(1-\delta x)}. \tag{5.3}$$

So

$$\begin{aligned} & (HW_3 - 2HW_2)\, x^3 + (HW_2 - 2HW_1)\, x^2 + (HW_1 - 2HW_0)\, x + HW_0 \\ = \; & B_1(1-\beta x)(1-\gamma x)(1-\delta x) + B_2(1-\alpha x)(1-\gamma x)(1-\delta x) \\ & + B_3(1-\alpha x)(1-\beta x)(1-\delta x) + B_3(1-\alpha x)(1-\beta x)(1-\gamma x). \end{aligned}$$

If we consider $x = \frac{1}{\alpha}$, we get $HW_0 + \frac{1}{\alpha}(HW_1 - 2HW_0) + \frac{1}{\alpha^2}(HW_2 - 2HW_1) + \frac{1}{\alpha^3}(HW_3 - 2HW_2)$
$= -B_1\left(\frac{1}{\alpha}\beta - 1\right)\left(\frac{1}{\alpha}\gamma - 1\right)\left(\frac{1}{\alpha}\delta - 1\right)$.

This gives

$$\begin{aligned} B_1 &= \alpha^3(HW_0 + \frac{1}{\alpha^2}(HW_2 - 2HW_1) + \frac{1}{\alpha^3}(HW_3 - 2HW_2) + \frac{1}{\alpha}(HW_1 - 2HW_0) \\ &= \frac{HW_0\alpha^3 + (HW_1 - 2HW_0)\, \alpha^2 + (HW_2 - 2HW_1)\, \alpha + HW_3 - 2HW_2}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} B_2 &= \frac{HW_0\beta^3 + (HW_1 - 2HW_0)\, \beta^2 + (HW_2 - 2HW_1)\, \beta + HW_3 - 2HW_2}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\ B_3 &= \frac{HW_0\gamma^3 + (HW_1 - 2HW_0)\, \gamma^2 + (HW_2 - 2HW_1)\, \gamma + HW_3 - 2HW_2}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\ B_4 &= \frac{HW_0\delta^3 + (HW_1 - 2HW_0)\, \delta^2 + (HW_2 - 2HW_1)\, \delta + HW_3 - 2HW_2}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}. \end{aligned}$$

Thus (5.3) can be written as

$$\sum_{n=0}^{\infty} HW_n x^n = B_1(1-\alpha x)^{-1} + B_2(1-\beta x)^{-1} + B_3(1-\gamma x)^{-1} + B_4(1-\delta x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} HW_n x^n = B_1 \sum_{n=0}^{\infty} \alpha^n x^n + B_2 \sum_{n=0}^{\infty} \beta^n x^n + B_3 \sum_{n=0}^{\infty} \gamma^n x^n + B_4 \sum_{n=0}^{\infty} \delta^n x^n = \sum_{n=0}^{\infty} (B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n) x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$HW_n = B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n.$$

THEOREM 8. *For all integers m, n the following identities holds:*

$$HW_{m+n} = P_{m-2}HW_{n+3} - P_{m-5}HW_{n+2} - P_{m-4}HW_{n+1} - P_{m-3}HW_n.$$

Proof. First we assume that $m, n \geq 0$ theorem 8 can be proved by mathematical induction on m. If $m = 0$ we get

$$HW_n = P_{-2}HW_{n+3} - P_{-5}HW_{n+2} - P_{-4}HW_{n+1} - P_{-3}HW_n.$$

which is true since $P_{-2} = 0, P_{-3} = -1, P_{-4} = 0, P_{-5} = 0$. Suppose that the equality holds for $m \leq k$. For $m = k+1$, we obtain

$$\begin{aligned} HW_{k+1+n} &= 2HW_{n+k} + -HW_{n+k-3}, \\ &= 2\left(P_{k-2}HW_{n+3} - P_{k-5}HW_{n+2} - P_{k-4}HW_{n+1} - P_{k-3}HW_n\right) \\ &\quad - \left(2P_{k-5}HW_{n+3} - P_{k-8}HW_{n+2} - P_{k-6}HW_{n+1} - P_{k-6}HW_n\right) \end{aligned}$$

by mathematical induction on m, this proves Theorem 8.

The other cases of m, n can be proved smilarly for all integers m, n. □

Taking $HW_n = HP_n$ or $HW_n = HC_n$ in above Theorem, respectively, we obtain:

COROLLARY 9.

$$\begin{aligned} HP_{m+n} &= P_{m-2}HP_{n+3} - P_{m-5}HP_{n+2} - P_{m-4}HP_{n+1} - P_{m-3}HP_n, \\ HC_{m+n} &= P_{m-2}HC_{n+3} - P_{m-5}HC_{n+2} - P_{m-4}HC_{n+1} - P_{m-3}HC_n. \end{aligned}$$

6. SIMSON'S FORMULA

In this section, we present Simpson's formula for the hyperbolic generalized Pierre numbers, which constitutes a special case of [78,Theorem 4.1].

THEOREM 10. *(Simpson's formula for hyperbolic generalized Pierre numbers) For all integers n we have,*

$$\begin{vmatrix} HW_{n+3} & HW_{n+2} & HW_{n+1} & HW_{n} \\ HW_{n+2} & HW_{n+1} & HW_{n} & HW_{n-1} \\ HW_{n+1} & HW_{n} & HW_{n-1} & HW_{n-2} \\ HW_{n} & HW_{n-1} & HW_{n-2} & HW_{n-3} \end{vmatrix} = \begin{vmatrix} HW_{3} & HW_{2} & HW_{1} & HW_{0} \\ HW_{2} & HW_{1} & HW_{0} & HW_{-1} \\ HW_{1} & HW_{0} & HW_{-1} & HW_{-2} \\ HW_{0} & HW_{-1} & HW_{-2} & HW_{-3} \end{vmatrix}$$

$= (HW_3 - HW_2 - HW_1 - HW_0)(HW_3^3 - HW_2^3 - HW_1^3 - HW_0^3 + (-5HW_2 + HW_1 + HW_0)HW_3^2 + (7HW_3 - 3HW_0 - HW_1)HW_2^2$
$+(3HW_3+HW_2-HW_0)HW_1^2+(HW_3+HW_2+HW_1)HW_0^2+4(-HW_2HW_3-HW_0HW_3+HW_0HW_2)HW_1)$.

Proof. Using Theorem 10 it can be proved by using induction or use [78,Theorem 4.1]

From the Theorem 10 we get the following Corollary.

Corollary 11. *For all integers n, the Simson's formulas of dual hyperbolic Pierre numbers and dual hyperbolic Pierre Lucas numbers are given as,*

a):

$$n = 0$$

$$\begin{vmatrix} HP_{n+3} & HP_{n+2} & HP_{n+1} & HP_{n} \\ HP_{n+2} & HP_{n+1} & HP_{n} & HP_{n-1} \\ HP_{n+1} & HP_{n} & HP_{n-1} & HP_{n-2} \\ HP_{n} & HP_{n-1} & HP_{n-2} & HP_{n-3} \end{vmatrix} = \begin{vmatrix} HP_{3} & HP_{2} & HP_{1} & HP_{0} \\ HP_{2} & HP_{1} & HP_{0} & HP_{-1} \\ HP_{1} & HP_{0} & HP_{-1} & HP_{-2} \\ HP_{0} & HP_{-1} & HP_{-2} & HP_{-3} \end{vmatrix}$$

$$= \begin{vmatrix} 8j+4 & 4j+2 & 2j+1 & j \\ 4j+2 & 2j+1 & j & 0 \\ 2j+1 & j & 0 & 0 \\ j & 0 & 0 & -1 \end{vmatrix} = j^4 + 2j + 1$$

$$= 2 + 2j$$

b):

$$\begin{vmatrix} HC_{n+3} & HC_{n+2} & HC_{n+1} & HC_{n} \\ HC_{n+2} & HC_{n+1} & HC_{n} & HC_{n-1} \\ HC_{n+1} & HC_{n} & HC_{n-1} & HC_{n-2} \\ HC_{n} & HC_{n-1} & HC_{n-2} & HC_{n-3} \end{vmatrix} = \begin{vmatrix} HC_{3} & HC_{2} & HC_{1} & HC_{0} \\ HC_{2} & HC_{1} & HC_{0} & HC_{-1} \\ HC_{1} & HC_{0} & HC_{-1} & HC_{-2} \\ HC_{0} & HC_{-1} & HC_{-2} & HC_{-3} \end{vmatrix}$$

$$= \begin{vmatrix} 12j+8 & 8j+4 & 4j+2 & 2j+4 \\ 8j+4 & 4j+2 & 2j+4 & 4j \\ 4j+2 & 2j+4 & 4j & 0 \\ 2j+4 & 4j & 0 & 6 \end{vmatrix} = -176j^4 - 352j - 176$$

$$= -352 - 352j$$

respectively.

7. Linear Sums

In this section, we present the summation formulas for the hyperbolic generalized Pierre numbers corresponding to both positive and negative subscripts.

We now present the summation formulas for the generalized Pierre numbers.

Theorem 12. *For the dual hyperbolic Pierre numbers, we have the following formulas:*

(a): $\sum\limits_{k=0}^{n} W_k = \frac{1}{2}(-(n+3))W_{n+3}+(n+4)W_{n+2}+(n+3)W_{n+1}+(n+4)W_n+3W_3-4W_2-3W_1-2W_0).$

(b): $\sum\limits_{k=0}^{n} W_{2k} = \frac{1}{2}(-(n+2)W_{2n+2}+(n+3)W_{2n+1}+(n+3)W_{2n}+(n+2)W_{2n-1}+2W_3-2W_2-3W_1-W_0).$

(c): $\sum\limits_{k=0}^{n} W_{2k+1} = \frac{1}{2}(-(n+1)W_{2n+2}+(n+3)W_{2n+1}+(n+2)W_{2n}+(n+2)W_{2n-1}+2W_3-3W_2-W_1-2W_0).$

Proof. For the proof, see Soykan [76, Theorem 3.10]. □

Theorem 13. *For the hyperbolic Pierre numbers, we have the following formulas:*

(a): $\sum\limits_{k=0}^{n} HW_k = \frac{1}{2}(-(n+3))HW_{n+3}+(n+4)HW_{n+2}+(n+3)HW_{n+1}+(n+4)HW_n+3HW_3-4HW_2-3HW_1-2HW_0).$

(b): $\sum\limits_{k=0}^{n} HW_{2k} = \frac{1}{2}(-(n+2)HW_{2n+2}+(n+3)HW_{2n+1}+(n+3)HW_{2n}+(n+2)HW_{2n-1}+2HW_3-2HW_2-3HW_1-HW_0).$

(c): $\sum\limits_{k=0}^{n} HW_{2k+1} = \frac{1}{2}(-(n+1)HW_{2n+2}+(n+3)HW_{2n+1}+(n+2)HW_{2n}+(n+2)HW_{2n-1}+2HW_3-3HW_2-HW_1-2HW_0).$

Proof. Use Theorem 12 and the definition of HW_n. □

As a special case of Theorem 13, we state the following Corollary.

Corollary 14. *For $n \geq 0$, dual hyperbolic Pierre numbers have the following properties:*

(a): $\sum\limits_{k=0}^{n} HP_k = \frac{1}{2}(-(n+3))HP_{n+3}+(n+4)HP_{n+2}+(n+3)HP_{n+1}+(n+4)HP_n+1).$

(b): $\sum\limits_{k=0}^{n} HP_{2k} = \frac{1}{2}(-(n+2))HP_{2n+2}+(n+3)HP_{2n+1}+(n+3)HP_{2n}+(n+2)HP_{2n-1}+j+1).$

(c): $\sum\limits_{k=0}^{n} HP_{2k+1} = \frac{1}{2}(-(n+1)HP_{2n+2}+(n+3)HP_{2n+1}+(n+2)HP_{2n}+(n+2)HP_{2n-1}+1).$

As a second special case of the above theorem, we obtain the following summation formulas for the hyperbolic Pierre Lucas numbers:

Corollary 15. *For $n \geq 0$, the hyperbolic Pierre Lucas numbers satisfy the following properties.*

(a): $\sum\limits_{k=0}^{n} HC_k = \frac{1}{2}(-(n+3)HC_{n+3}+(n+4)HC_{n+2}+(n+3)HC_{n+1}+(n+4)HC_n-12j-6).$

(b): $\sum\limits_{k=0}^{n} HC_{2k} = \frac{1}{2}(-(n+2)HC_{2n+2}+(n+3)HC_{2n+1}+(n+3)HC_{2n}+(n+2)HC_{2n-1}-6j-2).$

(c): $\sum\limits_{k=0}^{n} HC_{2k+1} = \frac{1}{2}(-(n+1)HC_{2n+2}+(n+3)HC_{2n+1}+(n+2)HC_{2n}+(n+2)HC_{2n-1}-8j-6).$

Next, we present the ordinary generating functions corresponding to selected special cases of the hyperbolic generalized Pierre numbers.

THEOREM 16. *The ordinary generating functions of the sequences HW_{2n}, HW_{2n+1} are given as follows:*

(a): $\sum_{n=0}^{\infty} HW_{2n}x^n = \frac{HW_3(2x^2)+HW_2\left(x^3-4x^2+x\right)-HW_1\left(2x^3\right)+HW_0\left(x^2-4x+1\right)}{x^4+2x^2-4x+1}$.
(b): $\sum_{n=0}^{\infty} HW_{2n+1}x^n = \frac{HW_3\left(x^3+x\right)-HW_2\left(2x^3\right)-HW_1\left(x^2-4x+1\right)-HW_0(2x^2)}{x^4+2x^2-4x+1}$.

Proof. Similarly,the proof can be constructed as in[4]

From the preceding theorem, we derive the following Corollary, which provides a summation formula for the hyperbolic Pierre numbers. (Take $HW_n = HP_n$ with $HP_0 = j, HP_1 = 2j+1, HP_2 = 4j+2, HP_3 = 8j+4$)

COROLLARY 17. *$n \geq 0$ the hyperbolic Pierre numbers exhibit the following properties.*

(a): $\sum_{n=0}^{\infty} HP_{2n}x^n = \frac{j+2x+jx^2}{x^4+2x^2-4x+1}$.
(b): $\sum_{n=0}^{\infty} HP_{2n+1}x^n = \frac{(-2j-1)+(8+16j)x+(-4j-1)x^2}{x^4+2x^2-4x+1}$.

8. Matrices related with Hyperbolic Generalized Pierre Numbers

We define the square matrix A of order 4 as

$$A = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $detA = 1$. Note that

$$A^n = \begin{pmatrix} P_{n+1} & -P_{n-2} & -P_{n-1} & -P_n \\ P_n & -P_{n-3} & -P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-4} & -P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-5} & -P_{n-4} & -P_{n-3} \end{pmatrix}$$

for the proof see [77].

Then we give the following lemma.

LEMMA 18. *For $n \geq 0$ the following identitiy is true:*

$$\begin{pmatrix} HW_{n+3} \\ HW_{n+2} \\ HW_{n+1} \\ HW_n \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}.$$

Proof. The identitiy(18) can be proved by mathematical induction on n. If $n=0$ we obtain

$$\begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}$$

which is true. Assuming that the given identity holds for $n=k$, the following identity is consequently valid.

$$\begin{pmatrix} HW_{k+3} \\ HW_{k+2} \\ HW_{k+1} \\ HW_k \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}.$$

For $n=k+1$, we get

$$\begin{aligned} \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix} &= \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} HW_{k+3} \\ HW_{k+2} \\ HW_{k+1} \\ HW_k \end{pmatrix} \\ &= \begin{pmatrix} HW_{k+4} \\ HW_{k+3} \\ HW_{k+2} \\ HW_{k+1} \end{pmatrix}. \end{aligned}$$

Thus,the proof completed .□

We define

$$N_{HW} = \begin{pmatrix} HW_3 & HW_2 & HW_1 & HW_0 \\ HW_2 & HW_1 & HW_0 & HW_{-1} \\ HW_1 & HW_0 & HW_{-1} & HW_{-2} \\ HW_0 & HW_{-1} & HW_{-2} & HW_{-3} \end{pmatrix}, \tag{8.1}$$

$$E_{HW} = \begin{pmatrix} HW_{n+3} & HW_{n+2} & HW_{n+1} & HW_n \\ HW_{n+2} & HW_{n+1} & HW_n & HW_{n-1} \\ HW_{n+1} & HW_n & HW_{n-1} & HW_{n-2} \\ HW_n & HW_{n-1-} & HW_{n-2} & HW_{n-3} \end{pmatrix}. \tag{8.2}$$

Now, we have the following theorem with N_{HW} and E_{HW},

THEOREM 19. *Using N_{HW} and E_{HW} , we get*

$$A^n N_{HW} = E_{HW}.$$

Proof. Note that we get

$$\begin{aligned} A^n N_{HW} &= \begin{pmatrix} P_{n+1} & -P_{n-2} & -P_{n-1} & -P_n \\ P_n & -P_{n-3} & -P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-4} & -P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-5} & -P_{n-4} & -P_{n-3} \end{pmatrix} \begin{pmatrix} HW_3 & HW_2 & HW_1 & HW_0 \\ HW_2 & HW_1 & HW_0 & HW_{-1} \\ HW_1 & HW_0 & HW_{-1} & HW_{-2} \\ HW_0 & HW_{-1} & HW_{-2} & HW_{-3} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned}
a_{11} &= P_{n+1}HW_3 - P_{n-2}HW_2 - P_{n-1}HW_1 - P_nHW_0 = HW_{n+3}, \\
a_{12} &= P_{n+1}HW_2 - P_{n-2}HW_1 - P_{n-1}HW_0 - P_nHW_{-1} = HW_{n+2}, \\
a_{13} &= P_{n+1}HW_1 - P_{n-2}HW_0 - P_{n-1}HW_{-1} - P_nHW_{-2} = HW_{n+1}, \\
a_{14} &= P_{n+1}HW_0 - P_{n-2}HW_{-1} - P_{n-1}HW_{-2} - P_nHW_{-3} = HW_n, \\
a_{21} &= P_nHW_3 - P_{n-3}HW_2 - P_{n-2}HW_1 - P_{n-1}HW_0 = HW_{n+2}, \\
a_{22} &= P_nHW_2 - P_{n-3}HW_1 - P_{n-2}HW_0 - P_{n-1}HW_{-1} = HW_{n+1}, \\
a_{23} &= P_nHW_1 - P_{n-3}HW_0 - P_{n-2}HW_{-1} - P_{n-1}HW_{-2} = HW_n, \\
a_{24} &= P_nHW_0 - P_{n-3}HW_{-1} - P_{n-2}HW_{-2} - P_{n-1}HW_{-3} = HW_{n-1}, \\
a_{31} &= P_{n-1}HW_3 - P_{n-4}HW_2 - P_{n-3}HW_1 - P_{n-2}HW_0 = HW_{n+1}, \\
a_{32} &= P_{n-1}HW_2 - P_{n-4}HW_1 - P_{n-3}HW_0 - P_{n-2}HW_{-1} = HW_n, \\
a_{33} &= P_{n-1}HW_1 - P_{n-4}HW_0 - P_{n-3}HW_{-1} - P_{n-2}HW_{-2} = HW_{n-1}, \\
a_{34} &= P_{n-1}HW_0 - P_{n-4}HW_{-1} - P_{n-3}HW_{-2} - P_{n-2}HW_{-3} = HW_{n-2}, \\
a_{41} &= P_{n-2}HW_3 - P_{n-5}HW_2 - P_{n-4}HW_1 - P_{n-3}HW_0 = HW_n, \\
a_{42} &= P_{n-2}HW_2 - P_{n-5}HW_1 - P_{n-4}HW_0 - P_{n-3}HW_{-1} = HW_{n-1}, \\
a_{43} &= P_{n-2}HW_1 - P_{n-5}HW_0 - P_{n-4}HW_{-1} - P_{n-3}HW_{-2} = HW_{n-2}, \\
a_{44} &= P_{n-2}HW_0 - P_{n-5}HW_{-1} - P_{n-4}HW_{-2} - P_{n-3}HW_{-3} = HW_{n-3}.
\end{aligned}$$

Using the theorem (8) the proof is done. □

By taking $HW_n = HP_n$ with HP_0, HP_1, HP_2, HP_3 in (8.1) and (8.2)

$HW_n = C_n$ with HC_0, HC_1, HC_2, HC_3 in (8.1) and (8.2)

respectively, we get:

$$N_{HP} = \begin{pmatrix} 8j+4 & 4j+2 & 2j+1 & j \\ 4j+2 & 2j+1 & j & 0 \\ 2j+1 & j & 0 & 0 \\ j & 0 & 0 & -1 \end{pmatrix},$$

$$E_{HP} = \begin{pmatrix} HP_{n+3} & HP_{n+2} & HP_{n+1} & HP_n \\ HP_{n+2} & HP_{n+1} & HP_n & HP_{n-1} \\ HP_{n+1} & HP_n & HP_{n-1} & HP_{n-2} \\ HP_n & HP_{n-1} & HP_{n-2} & HP_{n-3} \end{pmatrix},$$

$$N_{HC} = \begin{pmatrix} 12j+8 & 8j+4 & 4j+2 & 2j+4 \\ 8j+4 & 4j+2 & 2j+4 & 4j \\ 4j+2 & 2j+4 & 4j & 0 \\ 2j+4 & 4j & 0 & 6 \end{pmatrix},$$

$$E_{HC} = \begin{pmatrix} HC_{n+3} & HC_{n+2} & HC_{n+1} & HC_n \\ HC_{n+2} & HC_{n+1} & HC_n & HC_{n-1} \\ HC_{n+1} & C_n & HC_{n-1} & HC_{n-2} \\ HC_n & HC_{n-1} & HC_{n-2} & HC_{n-3} \end{pmatrix}.$$

From Theorem [19], we can write the following corollary.

COROLLARY 20. *The following identities are hold:*

a): $A^n N_{HP} = E_{HP}$.

b): $A^n N_{HC} = E_{HC}$.

9. Conclusions

Recurrence relations define sequences where each term depends on previous ones. These sequences such as Fibonacci, Pell, Jacobsthal, Tribonacci, Padovan, Narayana's Cows, Leonardo, Tetranacci, and Pentanacci arise across fields including engineering, biology, mathematics, and physics. Below, we present their definitions with initial conditions using A_n notation and outline their real-world relevance.

- **Fibonacci Sequence:**

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, \quad F_1 = 1$$

- **Pell Sequence:**

$$P_n = 2P_{n-1} + P_{n-2}, \quad P_0 = 0, \quad P_1 = 1$$

- **Jacobsthal Sequence:**

$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 0, \quad A_1 = 1$$

- **Tribonacci Sequence:**

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \quad T_0 = 0, \quad T_1 = 1, \quad T_2 = 1$$

- **Padovan Sequence:**

$$P_n = P_{n-2} + P_{n-3}, \quad P_0 = P_1 = P_2 = 1$$

- **Narayana's Cows Sequence:**

$$N_n = N_{n-1} + N_{n-3}, \quad N_0 = N_1 = N_2 = 1$$

- **Leonardo Sequence:**

$$L_n = L_{n-1} + L_{n-2} + 1, \quad L_0 = 1, \quad L_1 = 1$$

- **Tetranacci Sequence:**

$$M_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4}, \quad M_0 = M_1 = M_2 = 0, \quad M_3 = 1$$

- **Pentanacci Sequence:**

$$P_n = P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5}, \quad P_0 = P_1 = P_2 = P_3 = 0, \quad P_4 = 1$$

These sequences demonstrate how mathematical recursions extend into the fabric of our world whether designing structures, analyzing algorithms, modeling nature, or probing the quantum realm. Their recursive beauty continues to inspire both theoretical and practical exploration.

Next, we explore several real-world applications of recurrence relations across disciplines.

- Engineering
  - **Fibonacci:** Models recursive filters in control systems and signal processing.
  - **Padovan and Perrin:** Guide architectural proportions using the plastic number.
  - **Jacobsthal:** Applied in digital circuits for counting and encoding.
- Science
  - **Tribonacci and Tetranacci:** Simulate biological systems with delayed reproduction.
  - **Leonardo:** Reflect branching in plants and trees.
  - **Fibonacci and Narayana's Cows:** Describe phyllotaxis and seed arrangement in botany.
- Mathematics
  - **Recurrence Relations:** Analyze algorithms like mergesort and quicksort.
  - **Pell:** Solve Diophantine equations and approximate square roots with continued fractions.
  - **Jacobsthal and Padovan:** Used in tiling and combinatorics problems.
- Physics

  - **Fibonacci and Tribonacci:** Appear in wave interference and quantum systems.
  - **Pentanacci:** Used in recursive models of particle interactions and fractals.
  - **Padovan:** Linked to equilibrium modeling via the plastic constant.

In this study, we extend the classical framework to fourth-order recurrence systems by introducing the hyperbolic Pierre numbers, along with two distinguished subclasses. For these novel sequences, we derive Binet-type formulas, ordinary and exponential generating functions, and generalized Simson-type identities. Our analysis also encompasses closed-form summation formulas, algebraic properties, recurrence behaviors, and matrix-based representations.

Recognizing the theoretical depth and real-world utility of recurrence-based sequences, we first revisit the applications of second-order sequences to establish context. We then position our fourth-order generalizations as a natural progression within this broader mathematical landscape—offering new insights and powerful tools for modeling, analysis, and optimization in both pure and applied settings.

- For the applications of Gaussian Fibonacci and Gaussian Lucas numbers to Pauli Fibonacci and Pauli Lucas quaternions, see [46].
- For the application of Pell Numbers to the solutions of three-dimensional difference equation systems, see [45].
- For the application of Jacobsthal numbers to special matrices, see [44].
- For the application of generalized k-order Fibonacci numbers to hybrid quaternions, see [43].
- For the applications of Fibonacci and Lucas numbers to Split Complex Bi-Periodic numbers, see [42].
- For the applications of generalized bivariate Fibonacci and Lucas polynomials to matrix polynomials, see [41].
- For the applications of generalized Fibonacci numbers to binomial sums, see [40].
- For the application of generalized Jacobsthal numbers to hyperbolic numbers, see [39].
- For the application of generalized Fibonacci numbers to dual hyperbolic numbers, see [38].
- For the application of Laplace transform and various matrix operations to the characteristic polynomial of the Fibonacci numbers, see [37].
- For the application of Generalized Fibonacci Matrices to Cryptography, see [36].
- For the application of higher order Jacobsthal numbers to quaternions, see [35].
- For the application of Fibonacci and Lucas Identities to Toeplitz-Hessenberg matrices, see [34].
- For the applications of Fibonacci numbers to lacunary statistical convergence, see [33].
- For the applications of Fibonacci numbers to lacunary statistical convergence in intuitionistic fuzzy normed linear spaces, see [31].
- For the applications of Fibonacci numbers to ideal convergence on intuitionistic fuzzy normed linear spaces, see [32].

- For the applications of k-Fibonacci and $k-$Lucas numbers to spinors, see [30].
- For the application of dual-generalized complex Fibonacci and Lucas numbers to Quaternions, see [29].
- For the application of special cases of Horadam numbers to Neutrosophic analysis see [28].
- For the application of Hyperbolic Fibonacci numbers to Quaternions, see [8].
- For the application of Pell numbers to Gaussian Hyperbolic numbers, see [21].

In the following, we explore several applications of third-order recurrence sequences across various mathematical and applied contexts.

- For the applications of third order Jacobsthal numbers and Tribonacci numbers to quaternions, see [27] and [26], respectively.
- For the application of Tribonacci numbers to special matrices, see [25].
- For the applications of Padovan numbers and Tribonacci numbers to coding theory, see [24] and [23], respectively.
- For the application of Pell-Padovan numbers to groups, see [22].
- For the application of adjusted Jacobsthal-Padovan numbers to the exact solutions of some difference equations, see [20].
- For the application of Gaussian Tribonacci numbers to various graphs, see [19].
- For the application of third-order Jacobsthal numbers to hyperbolic numbers, see [18].For the application of Narayan numbers to finite groups see [17].
- For the application of generalized third-order Jacobsthal sequence to binomial transform, see [16].
- For the application of generalized Generalized Padovan numbers to Binomial Transform, see [15].
- For the application of generalized Tribonacci numbers to Gaussian numbers, see [14].
- For the application of generalized Tribonacci numbers to Sedenions, see [13].
- For the application of Tribonacci and Tribonacci-Lucas numbers to matrices, see [11].
- For the application of generalized Tribonacci numbers to circulant matrix, see [12].
- For the application of Tribonacci and Tribonacci-Lucas numbers to hybrinomials, see [10].
- For the application of hyperbolic Leonardo and hyperbolic Francois numbers to quaternions, see [9].

In the following lists, we outline several applications of fourth-order recurrence sequences across theoretical and applied domains.

- For the application of Tetranacci and Tetranacci-Lucas numbers to quaternions, see [4].
- For the application of generalized Tetranacci numbers to Gaussian numbers, see [5].
- For the application of Tetranacci and Tetranacci-Lucas numbers to matrices, see [6].
- For the application of generalized Tetranacci numbers to binomial transform, see [7].

We now explore several applications of fifth-order sequences.

- For the application of Pentanacci numbers to matrices, see [3].
- For the application of generalized Pentanacci numbers to quaternions, see [2].
- For the application of generalized Pentanacci numbers to binomial transform, see [1]

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