

ON STABILITY OF GENERALIZED QUADRATIC FUNCTIONAL EQUATION IN 2-BANACH SPACES

ABSTRACT. Stability of functional equations was initiated by Ulam [17]. The stability problem for functional equations have been extensively investigated by a number of great mathematicians [5, 11, 16]. During the last five decades, lots of research articles and research monographs have been published on various generalizations and applications of the Hyers-Ulam stability for several functional equations. In this research article, we find the general solution of functional equation

$$h(ba + c) + h(a - bc) = (b^2 + 1)h(a) + (b^2 + 1)h(c)$$

$a, c \in X$, $b \in \mathbb{N}$ and we prove the generalized Hyers-Ulam Stability of above functional equation in 2-Banach spaces.

1. INTRODUCTION

Various types of functional equations for functions from normed space to Banach space have been extensively studied by lots of mathematician. (see [5, 11, 16]). Hyers-Ulam stability of a functional equation for a function from a normed space to a Banach space has been studied by Hyers [8]. Stability of a functional equation for a function from a normed space to a 2-Banach space have been studied by B.M. Patel and A.B. Patel in [2], [3], [4]. Hyers-Ulam stability of the functional equation

$$(1.1) \quad g(a + b) + g(a - b) = 2g(a) + 2g(b).$$

has been proved by Skof [15]. He shown that for a function $f : X \rightarrow Y$, a function between norm linear space X to Banach space Y satisfying

$$\|g(a + b) + g(a - b) - 2g(a) + 2g(b)\| \leq \alpha$$

for $a, b \in X$ and $\alpha > 0$, there exists a unique quadratic function $Q_1 : X \rightarrow Y$ such that

$$\|g(x) - Q_1(x)\| < \frac{\alpha}{2}.$$

Definition 1.1. Suppose X and Y are real vector spaces. A function $g : X \rightarrow Y$ is said to be quadratic function if it satisfies (1.1) Every solution of function g is said to be quadratic functional equation.

Consider the functional equation

$$(1.2) \quad h(bx + y) + h(x - by) = (b^2 + 1)h(x) + (b^2 + 1)h(y)$$

for all $x, y \in X$, $b \in \mathbb{N}$.

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Our motive is to study Hyers-Ulam stability of the functional equation (1.2).

Definition 1.2. Let L be a linear space over \mathbb{R} and $\dim L > 1$. Suppose $\|\cdot, \cdot\| : L \times L \rightarrow \mathbb{R}$ is a function satisfying the following properties:

- (1) $\|a, b\| = 0$ if and only if a and b are linearly dependent,
- (2) $\|a, b\| = \|a, b\|$,
- (3) $\|xa, b\| = |x|\|a, b\|$,
- (4) $\|a, b + c\| \leq \|a, b\| + \|a, c\|$

$a, b, c \in L$ and $x \in \mathbb{R}$. Then the function $\|\cdot, \cdot\|$ is called a 2-norm on L and $(L, \|\cdot, \cdot\|)$ is called a 2-normed space.

We introduce some basic property of 2-normed spaces. Let $(X, \|\cdot, \cdot\|)$ be a 2-normed space, $a \in X$ and $\|a, b\| = 0$ for each $b \in X$. Suppose $a \neq 0$, as $\dim X > 1$, choose $b \in X$ such that $\{a, b\}$ is linearly independent so we get $\|a, b\| \neq 0$, which is a contradiction. Hence, we have the following lemma.

Lemma 1.3. Let $(X, \|\cdot, \cdot\|)$ be a 2-normed space. If $a \in X$ and $\|a, b\| = 0$ for $\forall b \in X$, then $a = 0$.

Remark 1.4. Let $(X, \|\cdot, \cdot\|)$ be a 2-normed space. Note that the conditions (2) and (4) imply that

$$\|a + b, c\| \leq \|a, c\| + \|b, c\|$$

$a, b, c \in X$. Putting $d = a + b$, we have $\|d, c\| \leq \|a, c\| + \|d - a, c\|$ for $\forall a, b, c \in X$. So $\|d, c\| - \|a, c\| \leq \|d - a, c\|$ for $\forall a, b, c \in X$. Replacing d by a in the above inequality to get $\|a, c\| - \|d, c\| \leq \|a - d, c\|$ for $\forall a, b, c \in X$. Thus, we have

$$(1.3) \quad \left| \|a, c\| - \|b, c\| \right| \leq \|a - b, c\|$$

for each $a, b, c \in X$. Hence the function $a \rightarrow \|a, b\|$ is continuous from X into \mathbb{R} for $b \in X$.

The notion of 2-Banach space was given by S. Gähler and A. White in 1960s.

Definition 1.5. A sequence $\{c_n\}$ in a 2-normed space X is called a 2-Cauchy sequence if

$$\lim_{m, n \rightarrow \infty} \|c_n - c_m, a\| = 0$$

$\forall a \in X$.

Definition 1.6. A sequence $\{c_n\}$ in a 2-normed space X is called a 2-convergent sequence if $\exists c \in X$ such that

$$\lim_{n \rightarrow \infty} \|c_n - c, a\| = 0$$

for each $a \in X$. If $\{c_n\}$ converges to c , we write $\lim_{n \rightarrow \infty} c_n = c$.

Definition 1.7. If every 2-Cauchy sequence in 2-normed space X is 2-convergent in X , then $(X, \|\cdot, \cdot\|)$ is called a 2-Banach space.

Following lemma tells us that $\|\cdot, \cdot\|$ is continuous in every component.

Lemma 1.8. [2] For a convergent sequence $\{c_n\}$ in a 2-normed space X ,

$$\lim_{n \rightarrow \infty} \|c_n, a\| = \left\| \lim_{n \rightarrow \infty} c_n, a \right\|$$

for every $a \in X$.

2. Solution of Equation (1.2)(GENERAL)

Theorem 2.1. *Let X and Y be linear spaces over \mathbb{R} . A mapping $h : X \longrightarrow Y$ satisfies equation (1.1) if and only if $h : X \longrightarrow Y$ satisfies equation (1.2).*

Proof. First assume that $f : X \longrightarrow Y$ satisfies equation (1.2), that is

$$(2.1) \quad h(ba + c) + h(a - bc) = [h(a) + h(c)][(b^2 + 1)]$$

Take $b = 1$ in (2.1), we get

$$h(a + c) + h(a - c) = 2h(a) + 2h(c)$$

Therefore h satisfies equation (1.1).

Conversely assume that h satisfies (1.1). Therefore $h(a+c)+h(a-c) = 2f(a)+2f(c)$.

We show that h satisfies (1.2). We shall prove the result by principle of mathematical induction on b .

Let $b = 1$ then (1.2) reduces to (1.1).

Let $b = 2$ then

$$\begin{aligned} h(2a + c) + h(a - 2c) &= h(a + c + a) + h(a - c - c) \\ &= 2h(a + c) + 2h(a) - h(c) + 2h(a - c) + 2h(c) - h(a) \\ &= 2[h(a + c) + h(a - c)] + h(a) + h(c) \\ &= 4h(a) + 4h(c) + h(a) + h(c) \\ &= 5h(a) + 5h(c) \end{aligned}$$

for $a, c \in X$. Therefore h satisfies (1.2), for $b = 2$.

Next assume that h satisfies equation (1.2), for all $b < m$. For $m \in \mathbb{N}$

$$\begin{aligned} h(ma + c) + h(a - mc) &= h[(m - 1)a + c + a] + h[a - (m - 1)c - c] \\ &= 2h((m - 1)a + c) + 2h(a) - h((m - 2)a + c) \\ &\quad + 2h(a - (m - 1)c) + 2h(c) - h(a - (m - 2)c) \\ &= 2[h((m - 1)a + c) + h(a - (m - 1)c)] \\ &\quad + 2h(a) + 2h(c) - [h((m - 2)a + c) + h(a - (m - 2)c)] \\ &= 2[((m - 1)^2 + 1)h(a) + ((m - 1)^2 + 1)h(c)] + 2h(a) + 2h(c) \\ &\quad - [((m - 2)^2 + 1)h(a) + ((m - 2)^2 + 1)h(c)] \\ &= [2(m - 1)^2 + 2 + 2 - (m - 2)^2 - 1]h(a) \\ &\quad + [2(m - 1)^2 + 2 + 2 - (m - 2)^2 - 1]h(c) \\ &= (m^2 + 1)h(a) + (m^2 + 1)h(c) \end{aligned}$$

for each $a, c \in X$. Hence by the principle of mathematical induction, h satisfies (1.2), for each $b \in \mathbb{N}$, for $a, c \in X$. \square

3. Main Results

In this section consider, X as 2-Banach space. For function $h : X \longrightarrow X$, define $B_h : X \times X \longrightarrow X$ by

$$B_h(x, y) = h(ax + y) + h(x - ay) - (a^2 + 1)h(x) - (a^2 + 1)h(y)$$

for each $x, y \in X, a \in \mathbb{N}$.

Theorem 3.1. *Let $G : X \times X \times X \rightarrow [0, \infty)$ be a function such that*

$$(3.1) \quad H(x, y, z) := \sum_{n=0}^{\infty} \frac{1}{a^{2n}} G(a^n x, a^n y, z) < \infty \text{ and } \lim_{n \rightarrow \infty} \frac{1}{a^{2n}} H(a^n x, 0, z) = 0$$

for each $x, y, z \in X, a \in \mathbb{N}$. Let $h : X \rightarrow X$ be a function satisfying

$$(3.2) \quad \|B_h(x, y), z\| \leq G(x, y, z)$$

for each $x, y \in X, z \in X$. Then there exists a unique quadratic function $Q : X \rightarrow X$ such that

$$(3.3) \quad \|h(x) - Q(x), z\| \leq \frac{1}{a^2} H(x, 0, z)$$

for each $x \in X, z \in X$.

Proof. Letting $y = 0$ in (3.2), we get

$$(3.4) \quad \|h(ax) - a^2 h(x), z\| \leq G(x, 0, z)$$

for each $x \in X, z \in X$. Therefore

$$(3.5) \quad \left\| \frac{h(ax)}{a^2} - h(x), z \right\| \leq \frac{1}{a^2} G(x, 0, z)$$

for each $x \in X, z \in X$. Replacing x by ax in (3.5), we get

$$(3.6) \quad \left\| \frac{h(a^2 x)}{a^2} - h(ax), z \right\| \leq \frac{1}{a^2} G(ax, 0, z)$$

for each $x \in X, z \in X$. By (3.5) and (3.6), we get

$$\begin{aligned} \left\| \frac{h(a^2 x)}{a^4} - h(x), z \right\| &= \left\| \frac{h(a^2 x)}{a^4} - \frac{h(ax)}{a^2} + \frac{h(ax)}{a^2} - h(x), z \right\| \\ &\leq \frac{1}{a^2} \left\| \frac{h(a^2 x)}{a^2} - h(ax), z \right\| + \left\| \frac{h(ax)}{a^2} - h(x), z \right\| \\ &\leq \frac{1}{a^2} \left[\frac{1}{a^2} G(ax, 0, z) \right] + \frac{1}{a^2} G(x, 0, z) \\ &= \frac{1}{a^2} \left[\frac{1}{a^2} G(ax, 0, z) + G(x, 0, z) \right] \end{aligned}$$

for all $x \in X, z \in X$. Using induction on $n \in \mathbb{N}$, we get

$$(3.7) \quad \left\| \frac{h(a^n x)}{a^{2n}} - h(x), z \right\| \leq \frac{1}{a^2} \sum_{j=0}^{n-1} \frac{1}{a^{2j}} G(a^j x, 0, z)$$

for each $x \in X, z \in X$. For $m, n \in \mathbb{N}$

$$\begin{aligned}
\left\| \frac{h(a^{m+n}x)}{a^{2(m+n)}} - \frac{h(a^m x)}{a^{2m}}, z \right\| &= \frac{1}{a^{2m}} \left\| \frac{h(a^{m+n}x)}{a^{2n}} - h(a^m x), z \right\| \\
&\leq \frac{1}{a^{2m}} \left[\frac{1}{a^2} \sum_{j=0}^{n-1} \frac{1}{a^{2j}} G(a^{m+j}x, 0, z) \right] \\
&= \frac{1}{a^2} \sum_{j=0}^{n-1} \frac{1}{a^{2(m+j)}} G(a^{m+j}x, 0, z) \\
&\leq \frac{1}{a^2} \sum_{j=0}^{\infty} \frac{1}{a^{2(m+j)}} G(a^{m+j}x, 0, z) \\
&= \frac{1}{a^2} \sum_{j=m}^{\infty} \frac{1}{a^{2j}} G(a^j x, 0, z) \\
&\rightarrow 0 \text{ as } m \rightarrow \infty
\end{aligned}$$

for all $x \in X, z \in X$. From this $\left\{ \frac{h(a^m x)}{a^{2m}} \right\}$ is a 2-Cauchy sequence in X , for $x \in X$.

We have give that X is 2-Banach space, $\left\{ \frac{h(a^m x)}{a^{2m}} \right\}$ 2-converges, for $x \in X$. Define $Q : X \rightarrow X$ as

$$Q(x) := \lim_{m \rightarrow \infty} \frac{f(a^m x)}{a^{2m}}$$

for each $x \in X$. From (3.7),

$$\lim_{m \rightarrow \infty} \left\| \frac{h(a^m x)}{a^{2m}} - h(x), z \right\| \leq \frac{1}{a^2} \sum_{j=0}^{\infty} \frac{1}{a^{2j}} G(a^j x, 0, z)$$

$\forall x \in X, z \in X$. Therefore

$$(3.8) \quad \|f(x) - Q(x), z\| \leq \frac{1}{a^2} H(x, 0, z)$$

$x \in X, z \in X$. Next we shall prove that Q satisfies (1.2), for $x, y \in X$.

$$\begin{aligned}
\|B_Q(x, y), z\| &= \lim_{n \rightarrow \infty} \frac{1}{a^{2n}} \|B_h(a^n x, a^n y), z\| \\
&\leq \lim_{n \rightarrow \infty} \frac{1}{a^{2n}} G(a^n x, a^n y, z) \\
&= 0
\end{aligned}$$

for each $z \in X$. Therefore $B_Q(x, y) = 0, x, y \in X$. Now we prove the uniqueness of Q . Suppose $Q_1 : X \rightarrow X$ is another quadratic function satisfying (3.3). Now Q and Q_1 are quadratic mappings, $Q(a^n x) = a^{2n} Q(x), Q_1(a^n x) = a^{2n} Q_1(x)$, for all $x \in X$.

$$\begin{aligned}
\|Q(x) - Q_1(x), z\| &= \frac{1}{a^{2n}} \|Q(a^n x) - Q_1(a^n x), z\| \\
&\leq \frac{1}{a^{2n}} [\|Q(a^n x) - h(a^n x), z\| + \|h(a^n x) - Q_1(a^n x), z\|] \\
&\leq \frac{2}{a^{2n}} \frac{1}{a^2} H(a^n x, 0, z) \\
&\rightarrow 0 \text{ as } n \rightarrow \infty
\end{aligned}$$

for all $z \in X$. Hence $Q(x) = Q_1(x)$, $x \in X$. \square

Corollary 3.2. *Let $\|\cdot, \cdot\|$ be a 2-Banach space, $0 \leq u, v < 2$. . Let h be a function from $(X, \|\cdot, \cdot\|)$ to $(X, \|\cdot, \cdot\|)$ satisfying*

$$(3.9) \quad \|B_h(x, y), z\| \leq \|x, z\|^u + \|y, z\|^v$$

for every $x, y \in X, z \in Y$. Then \exists a unique quadratic function $Q_2 : X \rightarrow X$ such that

$$(3.10) \quad \|h(x) - Q_2(x), z\| \leq \frac{\|x, z\|^u}{a^2 - a^u}$$

for all $x, z \in X$.

Theorem 3.3. *Let $G : X \times X \times X \rightarrow [0, \infty)$ be a function such that*

$$(3.11) \quad H(x, y, z) := \sum_{n=0}^{\infty} a^{2n} G\left(\frac{x}{a^n}, \frac{y}{a^n}, z\right) < \infty \text{ and } \lim_{n \rightarrow \infty} a^{2n} H\left(\frac{x}{a^n}, 0, z\right) = 0$$

for each $x, y \in X, z \in X, a \in \mathbb{N}$. Let $h : X \rightarrow X$ be a mapping satisfying

$$(3.12) \quad \|B_h(x, y), z\| \leq G(x, y, z)$$

for each $x, y \in X, z \in X$. Then there exists a unique quadratic function $Q : X \rightarrow X$ such that

$$(3.13) \quad \|h(x) - Q(x), z\| \leq H\left(\frac{x}{a}, 0, z\right)$$

for every $x \in X, z \in X$.

Proof. Letting $y = 0$ in (3.12), we get

$$(3.14) \quad \|h(ax) - a^2h(x), z\| \leq G(x, 0, z)$$

for each $x \in X, z \in X$. Replacing x by $\frac{x}{a}$ in (3.14), we get

$$(3.15) \quad \left\|h(x) - a^2h\left(\frac{x}{a}\right), z\right\| \leq G\left(\frac{x}{a}, 0, z\right)$$

for each $x \in X, z \in X$. Replacing x by $\frac{x}{a}$ in (3.15), we get

$$(3.16) \quad \left\|h\left(\frac{x}{a}\right) - a^2h\left(\frac{x}{a^2}\right), z\right\| \leq G\left(\frac{x}{a^2}, 0, z\right)$$

for each $x \in X, z \in X$. By (3.15) and (3.16), we get

$$\begin{aligned} \left\|h(x) - a^4h\left(\frac{x}{a^2}\right), z\right\| &\leq \left\|h(x) - a^2h\left(\frac{x}{a}\right), z\right\| + \left\|a^2h\left(\frac{x}{a}\right) - a^4h\left(\frac{x}{a^2}\right), z\right\| \\ &\leq G\left(\frac{x}{a}, 0, z\right) + a^2G\left(\frac{x}{a^2}, 0, z\right) \end{aligned}$$

for each $x \in X, z \in X$. Using principal of mathematical induction on $n \in \mathbb{N}$, we get

$$(3.17) \quad \left\|h(x) - a^{2n}h\left(\frac{x}{a^n}\right), z\right\| \leq \sum_{j=0}^{n-1} a^{2j}G\left(\frac{x}{a^{j+1}}, 0, z\right)$$

for each $x \in X, z \in X$. For $m, n \in \mathbb{N}$

$$\begin{aligned} \left\| a^{2(m+n)} h\left(\frac{x}{a^{m+n}}\right) - a^{2m} h\left(\frac{x}{a^m}\right), z \right\| &= a^{2m} \left\| a^{2n} h\left(\frac{x}{a^{m+n}}\right) - h\left(\frac{x}{a^m}\right), z \right\| \\ &\leq a^{2m} \sum_{j=0}^{n-1} a^{2j} G\left(\frac{x}{a^{m+j+1}}, 0, z\right) \\ &= \sum_{j=0}^{n-1} a^{2(m+j)} G\left(\frac{x}{a^{m+j+1}}, 0, z\right) \\ &\leq \sum_{j=0}^{\infty} a^{2(m+j)} G\left(\frac{x}{a^{m+j+1}}, 0, z\right) \\ &\rightarrow 0 \text{ as } m \rightarrow \infty \end{aligned}$$

for each $x \in X, z \in X$. Hence $\left\{ a^{2m} h\left(\frac{x}{a^m}\right) \right\}$ is a 2-Cauchy sequence in X , for all $x \in X$. As X is a 2-Banach space, $\left\{ a^{2m} h\left(\frac{x}{a^m}\right) \right\}$ 2-converges, for each $x \in X$. Let us define $Q : X \rightarrow X$ as

$$Q(x) := \lim_{n \rightarrow \infty} a^{2n} h\left(\frac{x}{a^n}\right)$$

for every $x \in X$. By (3.17), we get

$$\lim_{n \rightarrow \infty} \left\| h(x) - a^{2n} h\left(\frac{x}{a^n}\right), z \right\| \leq \sum_{j=0}^{\infty} a^{2j} G\left(\frac{x}{a^{j+1}}, 0, z\right)$$

for each $x \in X, z \in X$. Therefore

$$\|h(x) - Q(x), z\| \leq H\left(\frac{x}{a}, 0, z\right)$$

for each $x \in X, z \in X$. Remaining part of the proof is similar to the proof of Theorem 3.1. \square

Corollary 3.4. Let $(X, \|\cdot, \cdot\|)$ be a 2-Banach space, $s, t > 2$. $f : (X, \|\cdot, \cdot\|) \rightarrow (X, \|\cdot, \cdot\|)$ be a mapping satisfying

$$(3.18) \quad \|B_f(x, y), z\| \leq \|x, z\|^s + \|y, z\|^t$$

for each $x, y \in X, z \in X$. Then there exists a unique quadratic function $Q : X \rightarrow X$ such that

$$(3.19) \quad \|f(x) - Q(x), z\| \leq \frac{\|x, z\|^s}{a^s - a^2}$$

for each $x, z \in X$.

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