Structural Characterization of T-Normal Graphs

Abstract

Graph-theoretic properties pertaining to topological separation are important in order to understand how graph elements are structurally related. In this paper, we introduce the concept of T-normal graphs and investigate some of their basic properties. Specifically, we derive conditions under which the complete graph K_n is T-normal, and consider how the T-normal aspect behaves under the union and Cartesian product of two graphs. Finally, we provide a characterization of T-normal graphs using complements of graphs, providing a more in-depth understanding of their structural and topological behavior.

Keywords: T-regular graph, Complete Graph, Union, Cartesian product, Complement of a graph, Clique.

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1. Introduction

All graphs are finite and simple in this paper. Let the set of vertices of G be denoted by $V(G)$, the set of edges of G denoted by $E(G)$, the maximum degree of G denoted by $\Delta(G)$, and the minimum degree by $\delta(G)$.

Given a vertex v in G, the *degree*[1] of v in G, denoted by $d_G(v)$,is defined as the number of edges incident with v. A *pendant vertex* [2] is a vertex of degree one in G . A vertex of degree 0 is *isolated*[3] An *empty graph* [4] is a graph with no edges. A simple graph is *complete* [5] if every pair of distinct vertices of G are adjacent in G. A complete graph on n vertices is denoted by K_n. Given two graphs, G and H, we say H is an *induced subgraph*[6] of G if $V(H) \subseteq V(G)$, such that two vertices of H are adjacent if and only if they are adjacent in G. In this case if $V(H) = S$, we write $H = G[S]$ or $H = \langle S \rangle$. An *edge cover*[7] of G shall mean a set F of edges of G such that each vertex of G is contained in at least one edge $e \in F$. The *union* [8] of two graphs G_1 and G_2, which is represented as $G_1 \cup G_2$ is the graph that has edge set $E(G_1) \cup E(G_2)$ and vertex set $V(G_1) \cup V(G_2)$. If v is a vertex of a graph G, then $G - v$ is the graph obtained from G by deleting the vertex v. In general, if S is a set of vertices in G, then $G - S$ is the graph derived from G by deleting all the vertices in S, and all the edges incident with any one of the vertices in S[9].

Given any two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, their *Cartesian product*[10] $G \Box H$ is the graph with vertex set $V(G) \times V(H)$ where the vertex (u_1, v_1) is adjacent to the vertex (u_2, v_2) whenever $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$. The An *independent edge set*[11] (stable set) of a graph G is a subset of the edges of G such that no two edges in the subset share a vertex of G. A property P is said to be a *hereditary property*[12] of a graph G, if a graph G has the property P, then every subgraph of G also has the property P.

Lemma 1.1. Let Q be a clique in $G_1 \Box G_2$ then, either the first coordinates of all vertices of Q are same and the subgraph of G induced by the second coordinates of all vertices of Q is a clique in G_2, or the second coordinates of all vertices of Q are same and the the subgraph of G induced by the first coordinates of all vertices of Q is a clique in G_1.

The remainder of this manuscript is ordered as follows. Section 2 presents the concept of T-normal graphs, provides examples, remarks, and results regarding their characterization using graph complements, as well as the normality of the union of two graphs. Section 3 provides conditions into which the Cartesian product of two graphs is T-normal. We conclude with final thoughts and some possibilities for future work.

2. T-normal Graphs

This section defines T-normal graphs and provides some examples. We also give remarks and some basic results on the characterization of T-normality using graph complements and the T-normality of the union of two graphs.

Definition 2.1. A graph G is said to be T-normal if for any two disjoint cliques Q_1 and Q_2 in G there exists covers $\mathcal{D}_1$ and $\mathcal{D}_2$ of Q_1 and Q_2 respectively in G such that $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) = \emptyset$.

Remark 2.1. Let Q_1 and Q_2 be two disjoint cliques in G_1 and G_2 respectively with $|V(Q_1)| = m$ and $|V(Q_2)| = n$. If both m and n are even then both Q_1 and Q_2 can be covered by its own edges. It is a trivial case. Hence it is enough to consider the cases in which one of m and n are odd; and in which both m and n are odd.

Example 2.1. Path on three vertices, P_3 is not T-normal.

Consider the path $G = P_3$ with vertices $v_1 - v_2 - v_3$. Consider disjoint cliques $Q_1 = \{v_1\}$ and $Q_2 = \{v_3\}$. Any edge cover of Q_1 must contain v_1v_2, so $V(\mathcal{D}_1) \supseteq \{v_1, v_2\}$. Any edge cover of Q_2 must contain v_2v_3, so $V(\mathcal{D}_2) \supseteq \{v_2, v_3\}$. Thus $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) \supseteq \{v_2\} \neq \varnothing$, and no disjoint endpoint sets are possible. Therefore P_3 is not T-normal.

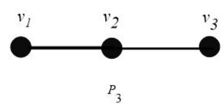

Figure 1: Path P_3

Proposition 2.1. The Cycle on 5 vertices, C_5 is T-normal

Proof. Considerthe cycle $G = C_5$ with vertices v_0, v_1, v_2, v_3, v_4. As C_5 has no triangles, all cliques are either singletons or edges.

Case 1. Two singletons.

For $Q_1 = \{v_0\}$ and $Q_2 = \{v_2\}$, choose $\mathcal{D}_1 = \{v_0v_4\}$ and $\mathcal{D}_2 = \{v_2v_3\}$. Then $V(\mathcal{D}_1) = \{v_0, v_4\}$ and $V(\mathcal{D}_2) = \{v_2, v_3\}$, which are disjoint.

Case 2. Singleton and edge

Let $Q_1 = \{v_0\}$ and $Q_2 = \{v_2, v_3\}$. Take $\mathcal{D}_1 = \{v_0v_4\}$ and $\mathcal{D}_2 = \{v_2v_3\}$; the endpoint sets are again disjoint.

Case 3. Two disjoint edges.

For $Q_1 = \{v_0, v_1\}$ and $Q_2 = \{v_2, v_3\}$, choose $\mathcal{D}_1 = \{v_0v_1\}$ and $\mathcal{D}_2 = \{v_2v_3\}$. Here $V(\mathcal{D}_1) = \{v_0, v_1\}$ and $V(\mathcal{D}_2) = \{v_2, v_3\}$, which are disjoint.

Thus every pair of disjoint cliques in C_5 admits covers with disjoint vertex sets, and hence C_5 is T-normal. □

Proposition 2.2. Then complete graph on n vertices, K_n is T-normal if and only if $n = 1$.

Proof. If $n = 1$, there are no two cliques in K_1 that are distinct and nonempty. Thus, the condition holds vacuously.

Now suppose $n \geq 2$, fix $x \in V(K_n)$ and consider disjoint cliques $Q_1 = \{x\}$ and $Q_2 = V(K_n) \setminus \{x\}$. Any cover $\mathcal{D}_2$ of Q_2 must contain all vertices of Q_2 as end vertices, so $V(\mathcal{D}_2) = Q_2$. Any cover $\mathcal{D}_1$ of Q_1 uses an edge xy with $y \in Q_2$, hence $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) \supseteq \{y\} \neq \emptyset$. Therefore, K_n is not T-normal. □

Theorem 2.1. Give graph G is T-normal if and only if its complement $\overline{G}$ satisfies the following property: Given any two disjoint stable sets $I_1, I_2 \subseteq V(\overline{G})$, there exist stable-set covers $\mathcal{E}_1, \mathcal{E}_2$ of I_1 and I_2 in $\overline{G}$ such that

$$V(\mathcal{E}_1) \cap V(\mathcal{E}_2) = \emptyset$$

Here, an *stable-set cover*[13] of I_i means a collection of stable sets in $\overline{G}$ whose union contains I_i.

Proof. Necessary Part. Suppose that G is a T -normal graph. Then, for any two disjoint cliques Q_1 and Q_2 in G there exists covers $\mathcal{D}_1$ and $\mathcal{D}_2$ of Q_1 and Q_2 respectively in G such that $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) = \emptyset$. Let I_1, I_2 be two disjoint stable sets in $\overline{G}$. Then I_1 and I_2 are disjoint cliques in G. As G is T-normal, there exist clique covers $\mathcal{C}_1$ and $\mathcal{C}_2$ of I_1 and I_2 in G, respectively, satisfying $V(\mathcal{C}_1) \cap V(\mathcal{C}_2) = \emptyset$. Each clique in the cover $\mathcal{C}_i$ becomes a stable set in $\overline{G}$, so $\mathcal{E}_i = \{\overline{C} : C \in \mathcal{C}_i\}$ forms an stable-set cover of I_i in $\overline{G}$, preserving vertex disjointness. Hence, $\overline{G}$ holds the required property.

Sufficent Part. Suppose $\overline{G}$ satisfies the stated property. Now we have to show that $G$is T-normal. Let Q_1 and Q_2 be two disjoint cliques in G. Then Q_1 and Q_2 are disjoint stable sets in $\overline{G}$. By assumption there exist stable-set covers $\mathcal{E}_1, \mathcal{E}_2$ of Q_1, Q_2 in $\overline{G}$ with $V(\mathcal{E}_1) \cap V(\mathcal{E}_2) = \emptyset$. Each stable set in $\overline{G}$ corresponds to a clique in G, so $\mathcal{C}_i = \{\overline{D} : D \in \mathcal{E}_i\}$ are clique covers of Q_i in G, which are vertex disjoint. Therefore, G is a T -normal graph. □

Proposition 2.3. If G_1 and G_2 are two disjoint T-normal graphs then $G = G_1 \cup G_2$ is T-normal.

Proof. Given that $G_1$and G_2 are disjoint T- normal graphs. Therefore,$V(G_1) \cap V(G)_2 = \emptyset$..Let $G = G_1 \cup G_2$.Then G is the disjoint union of two components G_1 and G_2.Let Q_1 and Q_2 be two cliquesin G.

Case 4. The cliques Q_1 and Q_2 lie in different components.

Suppose that $Q_1 \subseteq G_1$ and $Q_2 \subseteq G_2$. Since G_1 and G_2 are T-normal, there exist clique covers $\mathcal{D}_1$ of Q_1 in G_1 and $\mathcal{D}_2$ of Q_2 in G_2 with $V(\mathcal{D}_1)$ is contained in $V(G_1)$ and $V(\mathcal{D}_2)$ is contained in $V(G_2)$, respectively. Hence $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) = \emptyset$.

Case 5. Both cliques Q_1 and Q_2 lie in the same component of G, say G_1.

As $G_1$is T-normal, there exist disjoint clique covers for Q_1 and Q_2 within G_1 itself. Hence disjointness condition holds in G.

Therefore,in all cases given any two disjoint cliques Q_1, Q_2 in G, there exist disjoint clique covers $\mathcal{D}_1$ and $\mathcal{D}_2$ in G. Therefore G is T-normal. □

Remark 2.2. T normality in graphs is not a hereditary property.That is if G is T-normal then every induced subraph of G need not be normal.

3. *T*-normal in Cartesian Product of Graphs

In this section, we explore the conditions that allow for the T-normal behaviour of a Cartesian of two graphs. Theorems and proofs are presented below that characterize these conditions.

Theorem 3.1. Let G_1 and G_2 be two graphs with $\delta(G_1)$ and $\delta(G_2) \geq 2$, then $G_1 \square G_2$ is T-normal.

Proof. Let Q_1 and Q_2 be two disjoint cliques in G_1 and G_2 respectively. Let $|V(Q_1)| = m$ and $|V(Q_2)| = n$. By Lemma 1.1 either the first coordinate of vertices of Q_1 are same and the span of second coordinate of vertices of Q_1 forms a clique in G_2, or the second coordinate of vertices of Q_1 are same and the span of second coordinate of vertices of Q_1 forms a clique in G_1. Similarly, the case of Q_2 also. First of all suppose that second coordinate of each vertex of Q_1 is x and that of Q_2 is y. Assume that $n \geq m$. Let $V(Q_1) = \{(u_1, x), (u_2, x), (u_3, x), \ldots, (u_m, x)\}$ and $V(Q_2) = \{(v_1, y), (v_2, y), (v_3, y), \ldots, (v_n, y)\}$. As $\delta(G_1) \geq 2$, $m \geq 2$ and $n \geq 2$. Therefore, we can suppose $v_1 \neq u_m$. To get covers $\mathcal{D}_1$ and $\mathcal{D}_2$ of Q_1 and Q_2 respectively in G such that $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) = \emptyset$, consider the following two cases.

Case 1. m is even and n is odd

As m is even Q_1 can be covered by its own edges. As $\delta(G_2) \geq 2$, there exists a vertex w different from x such that y is adjacent to w. Then $\{(v_1, y)(v_2, y), (v_3, y)(v_4, y), \ldots, (v_{n-2}, y)(v_{n-1}, y),$ $(v_n, y)(v_n, w)\}$ is a cover of Q_2 which is not incident with any edges of Q_1.

Case 2. Both m and n are odd

As $\delta(G_2) \geq 2$, there exists a vertex w of G_2 different from y such that w is adjacent to x. Similarly, there exists a vertex z of G_2 different from x such that z is adjacent to y. Then $\mathcal{D}_1 = \{(u_1, x)(u_2, x), (u_3, x)(u_4, x), \ldots, (u_m, x)(u_m, w)\}$ and $\mathcal{D}_2 = \{(v_1, z)(v_1, y), (v_2, y)(v_3, y), (v_4, y)(v_5, y), \ldots, (v_{n-3}, y)(v_{n-2}, y), (v_{n-1}, y)(v_n, y)\}$ are covers of Q_1 and Q_2 respectively and $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) = \emptyset$.

Now, assume the second coordinate of each vertex of Q_1 is x and the first coordinate of each vertex of Q_2 is u. Also assume $n \geq m$. Let $V(Q_1) = \{(u_1, x), (u_2, x), (u_3, x), \ldots, (u_m, x)\}$ and $V(Q_2) = \{(u, v_1), (u, v_2), (u, v_3), \ldots, (u, v_n)\}$. The vertex u may or may not belongs to $\{u_1, u_2, \ldots, u_m\}$. If $u \in \{u_1, u_2, \ldots, u_m\}$, let $u = u_1$. Assume $y \notin \{v_1, v_2, \ldots, v_n\}$. To get covers $\mathcal{D}_1$ and $\mathcal{D}_2$ of Q_1 and Q_2 respectively in G such that $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) = \emptyset$, consider the following two cases.

Case 1. m is even and n is odd

Since m is even, Q_1 can be covered by its own edges. As $\delta(G_1) \geq 2$, there exists a vertex w different from u_1 such that u is adjacent to w. Then, $\{(w, v_1)(u, v_1), (u, v_2)(u, v_3), (u, v_4)(u, v_5), \ldots, (u, v_{n-1})(u, v_n)\}$ is a cover of Q_2 which is not incident with any edges of Q_1.

Case 2. Both m and n are odd

As $\delta(G_1) \geq 2$, there exists a vertex w of G_1 different from u_1 such that w is adjacent to u. As $\delta(G_2) \geq 2$, there exists a vertex z of G_2 different from v_1 such that z is adjacent to y. Then $\mathcal{D}_1 = \{(u_1, z)(u_1, y), (u_2, y), (u_3, y), (u_4, y)(u_5, y), \ldots, (u_{m-1}, y)(u_m, y)\}$ and $\mathcal{D}_2 = \{(w, v_1)(u, v_1), (u, v_2)(u, v_3), (u, v_4)(u, v_5), \ldots, (u, v_{n-1})(u, v_n)\}$ are covers of Q_1 and Q_2 respectively and $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) = \emptyset$.

Now, assume $y \in \{v_1, v_2, \ldots, v_n\}$. Let $y = v_1$. To get covers $\mathcal{D}_1$ and $\mathcal{D}_2$ of Q_1 and Q_2 respectively in G such that $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) = \emptyset$, consider the following two cases.

Case 1. m is even and n is odd

Since m is even, Q_1 can be covered by its own edges. As $\delta(G_1) \geq 2$, there exists a vertex w different from u_1 such that u is adjacent to w. Then $\{(u, v_1)(u, v_2), (u, v_3)(u, v_4), (u, v_5)(u, v_6), \ldots, (u, v_{n-2})(u, v_{n-1}), (u, v_n)(w, v_n)\}$ is a cover of Q_2 which is not incident with any edges of Q_1.

Case 2. Both m and n are odd

As $\delta(G_1) \geq 2$, there exists a vertex w of G_1 different from u_1 such that w is adjacent to u. As $\delta(G_2) \geq 2$, there exists a vertex z of G_2 different from v_1 such that z is adjacent to y. Then $\mathcal{D}_1 = \{(u_1, z)(u_1, y), (u_2, y), (u_3, y), (u_4, y)(u_5, y), \ldots, (u_{m-1}, y)(u_m, y)\}$ and $\mathcal{D}_2 = \{(u, v_1)(u, v_2), (u, v_3)(u, v_4),$

$(u, v_5)(u, v_6), \ldots, (u, v_{n-2})(u, v_{n-1}), (u, v_n)(w, v_n)\}$ are covers of Q_1 and Q_2 respectively and $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) = \emptyset$.

Hence the theorem. □

4. Conclusion

In this paper, we proposed the concept of T -normal graphs and clarified the notion by using some examples and a few counterexamples. Specifically, we proved that cycles C_5 is T -normal while P_3, any complete graph K_n where $n \geq 2$ is not T -normal because there is forced overlap in any pair of edge covers. We provided commentary and results relating T -normality to properties of complements of graphs and also analyzed when the union of two graph produces a T -normal. Finally, we explored the conditions under which the Cartesian product is T -normal. Future work may focus on obtaining a complete characterization of T-normal graphs and identifying minimal forbidden subgraphs for this class. It would also be interesting to investigate the behavior of T-normality under other graph operations such as the lexicographic and strong products, and to examine possible algorithmic approaches for recognizing T-normal graphs efficiently.

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