

# A Note on Stability (Hyers-Ulam) of Functional Equations in 2-Banach spaces

## Abstract

This study presented the modifications to the results reported in the paper of S.C. Chung W. Park (6). The modified theorems strengthen the theoretical framework of functional equations in 2-Banach spaces and ensure consistency in their applications to stability theory

*Keywords:* Stability (Hyers-Ulam), 2-Banach space, Functional equation (Additive)

2010 Mathematics Subject Classification: Primary 46K05; Secondary 46H05.

## 1 Introduction

In this note, we would like to bring out some modifications in the Theorems in the paper of S. Chung and W. Park (6).

S. Gähler (2) has given the notion of linear 2-normed spaces in the 1960s.

**Definition 1.1.** Let  $Z$  be a linear space over  $\mathbb{R}$  with  $\dim Z > 1$  and let  $\|\cdot, \cdot\| : Z \times Z \rightarrow \mathbb{R}$  be a function satisfying the following axioms:

1.  $\|x_1, y_1\| = 0$  if and only if  $x_1$  and  $y_1$  are linearly dependent,
2.  $\|x_1, y_1\| = \|y_1, x_1\|$ ,
3.  $\|kx_2, y_2\| = |k|\|x_2, y_2\|$ ,
4.  $\|x_1, y_1 + z_1\| \leq \|x_1, y_1\| + \|x_1, z_1\|$

for each  $x_1, y_1, z_1, x_2, y_2 \in Z$  and  $k \in \mathbb{R}$ . Then the function  $\|\cdot, \cdot\|$  is called a 2-norm on  $Z$  and  $(Z, \|\cdot, \cdot\|)$  is called a 2-normed space.

From this we can say that if  $\|\cdot, \cdot\|$  is a 2-norm on  $X$  then  $\|\cdot, \cdot\| : X \times X \rightarrow \mathbb{R}$  is a function having some properties. In order to find  $\|x_1, y_1\|$ , it is necessary to take  $x_1, y_1 \in Z$ . Throughout the paper (6) authors have taken  $Z$ , a normed linear space and  $Y$ , a 2-Banach space.

## 2 Main Results

In paper (6), Theorem (2.1) is as under.

Let  $\eta_1, \theta_1 \in [0, \infty)$ ,  $p_1, q_1 \in (0, 1)$  and let  $g : Z \rightarrow Y$  be a mapping satisfying

$$\|g(x_5 + y_5) - g(x_5) - g(y_5), z_5\| \leq \eta_1 \|x_5\|^{p_1} + \theta_1 \|y_5\|^{q_1} \quad (2.1)$$

for all  $y_5, x_5 \in Z$  and all  $z_5 \in Y$ . Then there is a unique additive mapping  $A_1 : X \rightarrow Y$  such that

$$\|g(x_6) - A_1(x_6), y_6\| \leq \frac{\eta_1 \|x_6\|^{p_1}}{2 - 2^{p_1}} + \frac{\theta_1 \|x_6\|^{q_1}}{2 - 2^{q_1}}$$

for each  $x_6 \in Z$  and all  $y_6 \in Y$ .

*Remark 2.1.* No such function  $g : Z \rightarrow Y$  satisfying (2.1) exists. If such function  $g : Z \rightarrow Y$  exists, then for each  $n \in \mathbb{N}$ ,  $z_2 \in Y \Rightarrow nz_2 \in Y$ . By (2.1) we have

$$\|g(x_0 + y_0) - g(x_0) - g(y_0), nz_2\| \leq \eta_1 \|x_0\|^{p_1} + \theta_1 \|y_0\|^{q_1}$$

each  $y_0, x_0 \in Z$  and each  $z_2 \in Y$ . Therefore

$$n \leq \frac{\eta_1 \|x_0\|^{p_1} + \theta_1 \|y_0\|^{q_1}}{\|g(x_0 + y_0) - g(x_0) - g(y_0), z_2\|}$$

for each  $n \in \mathbb{N}$ , where  $\|g(x_0 + y_0) - g(x_0) - g(y_0), z_2\| \neq 0$ . So  $\mathbb{N}$  would be bounded, which is absurd. Therefore, no such function exists.

We modified Theorem 2.1. of (6) as follow.

Suppose  $\eta_2, \theta_2 \in [0, \infty)$ ,  $r_2 \in (0, \infty)$ ,  $p_2, q_2 \in (0, 1)$  and  $(Z, \|\cdot\|)$  is norm linear space,  $(Z, \|\cdot, \cdot\|)$  is a 2-Banach space and  $h : (Z, \|\cdot, \cdot\|) \rightarrow (Z, \|\cdot, \cdot\|)$  is a function satisfying

$$\|h(x_b + y_b) - h(x_b) - h(y_b), z_b\| \leq (\eta_2 \|x_b\|^{p_2} + \theta_2 \|y_b\|^{q_2}) \|z_b\|^{r_2}$$

for all  $x_b, y_b \in Z$ . Then find unique additive function  $A_2 : Z \rightarrow Z$  such that

$$\|h(x_a) - A_2(x_a), y_a\| \leq \left( \frac{\eta_2 \|x_a\|^{p_2}}{2 - 2^{p_2}} + \frac{\theta_2 \|x_a\|^{q_2}}{2 - 2^{q_2}} \right) \|y_a\|^{r_2}$$

for every  $x_a, y_a \in Z$ .

Modification in Theorem (2.2) of (6) is as follow.

Let  $\eta_3, \theta_3 \in [0, \infty)$ ,  $r_3 \in (0, \infty)$ ,  $1 < p_3, q_3 < \infty$  and  $(Z, \|\cdot, \cdot\|)$  be a normed space and  $(Z, \|\cdot, \cdot\|)$  be a 2-Banach space and  $h_1 : (Z, \|\cdot, \cdot\|) \rightarrow (Z, \|\cdot, \cdot\|)$  be a mapping satisfying

$$\|h_1(x_s + y_s) - h_1(x_s) - h_1(y_s), z_s\| \leq (\eta_3 \|x_s\|^{p_3} + \theta_3 \|y_s\|^{q_3}) \|z_s\|^{r_3} \quad (2.2)$$

for all  $x_s, y_s, z_s \in Z$ . Then  $\exists$  unique additive mapping  $A_3 : Z \rightarrow Z$  such that

$$\|h_1(x_r) - A_3(x_r), y_r\| \leq \left( \frac{\eta_3 \|x_r\|^{p_3}}{2^{p_3} - 2} + \frac{\theta_3 \|x_r\|^{q_3}}{2^{q_3} - 2} \right) \|y_r\|^{r_3}$$

for all  $x_r, y_r, z_r \in Z$ .

Revision in Theorem (3.1) of (6) is as follow.

Suppose  $\theta_4, \eta_4 \in [0, \infty), r_4 \in (0, \infty), 0 < p_4, q_4 < 1, (Z, \|\cdot\|)$  is norm linear space and  $(Z, \|\cdot, \cdot\|)$  2-Banach space,  $h_2 : (Z, \|\cdot\|) \rightarrow (Z, \|\cdot, \cdot\|)$  be a mapping satisfying  $h_2(0) = 0$  and

$$\left\| 2h_2\left(\frac{x_d + y_d}{2}\right) - h_2(x_d) - h_2(y_d), z_d \right\| \leq (\eta_4 \|x_d\|^{p_4} + \theta_4 \|y_d\|^{q_4}) \|z_d\|^{r_4}$$

for all  $x_d, y_d, z_d \in Z$ . Then there exists unique Jensen function  $J_4 : Z \rightarrow Z$  such that

$$\|h_2(x) - J_4(x), y\| \leq \left( \frac{2\eta_4}{3 - 3^{p_4}} \|x\|^{p_4} + \theta_4 \frac{1 + 3^{q_4}}{3 - 3^{q_4}} \|x\|^{q_4} \right) \|y\|^{r_4}$$

for all  $x, y, z \in Z$ .

New version of Theorem (3.2) of (6) is as follow.

Let  $\theta_5, \eta_5 \in [0, \infty), r_5 \in (0, \infty), p_5, q_5 \in (1, \infty)$  and  $(Z, \|\cdot\|)$  be a normed space and  $(Z, \|\cdot, \cdot\|)$  be a 2-Banach space and  $g_3 : (Z, \|\cdot\|) \rightarrow (Z, \|\cdot, \cdot\|)$  be a mapping satisfying  $g_3(0) = 0$  and

$$\left\| 2g_3\left(\frac{x_h + y_h}{2}\right) - g_3(x_h) - g_3(y_h), z_h \right\| \leq (\eta_5 \|x_h\|^{p_5} + \theta_5 \|y_h\|^{q_5}) \|z_h\|^{r_5}$$

for all  $x_h, y_h, z_h \in Z$ . Then  $\exists$  unique Jensen mapping  $J' : Z \rightarrow Z$  such that

$$\|g_3(x_j) - J'(x_j), y_j\| \leq \left( \frac{2\eta_5}{3^{p_5} - 3} \|x_j\|^{p_5} + \theta_5 \frac{1 + 3^{q_5}}{3^{q_5} - 3} \|x_j\|^{q_5} \right) \|y_j\|^{r_5}$$

for all  $x_j, y_j \in Z$ .

In paper (6), Theorem (4.1) is as follow.

Suppose  $\eta_6, \theta_6 \in [0, \infty)$  and  $p_6, q_6 \in (0, 2)$  and let  $f_q : X \rightarrow Y$  is a function satisfying

$$\|f_q(x_k + y_k) + f_q(x_k - y_k) - 2f_q(x_k) - 2f_q(y_k), z_k\| \leq \eta_6 \|x_k\|^{p_6} + \theta_6 \|y_k\|^{q_6} \quad (2.3)$$

for all  $x_k, y_k, z_k \in X$ . Then there is a unique quadratic mapping  $Q_9 : X \rightarrow Y$  such that

$$\|f_q(x_k) - Q_9(x_k), y_k\| \leq \frac{\eta_6 \|x_k\|^{p_6}}{4 - 2^{p_6}} + \frac{\theta_6 \|x_k\|^{q_6}}{4 - 2^{q_6}} + \frac{1}{3} \|f_q(0), y_k\|$$

for all  $x_k \in X$  and all  $y_k \in Y$ .

Modification in Theorem (4.1) of paper (6) is as under.

Let  $\eta_8, \theta_8 \in [0, \infty), r_8 \in (0, \infty), 0 < p_8, q_8 < 2$  and  $(Z, \|\cdot\|)$  be a normed space and  $(Z, \|\cdot, \cdot\|)$  be a 2-Banach space and  $h_6 : (Z, \|\cdot\|) \rightarrow (Z, \|\cdot, \cdot\|)$  be a mapping satisfying

$$\|h_6(x_q + y_q) + h_6(x_q - y_q) - 2h_6(x_q) - 2h_6(y_q), z_q\| \leq (\eta_8 \|x_q\|^{p_8} + \theta_8 \|y_q\|^{q_8}) \|z_q\|^{r_8} \quad (2.4)$$

for all  $x_q, y_q, z_q \in Z$ . Then there exists unique quadratic function  $Q_9 : Z \rightarrow Z$  such that

$$\|h_6(x_p) - Q_9(x_p), y_p\| \leq \left( \frac{\eta_8 \|x_p\|^{p_8}}{4 - 2^{p_8}} + \frac{\theta_8 \|x_p\|^{q_8}}{4 - 2^{q_8}} \right) \|y_p\|^{r_8}$$

for all  $x_p, y_p \in Z$ .

**Remark 2.2.** By taking  $x_p = y_p = 0$  in (2.4), we get  $\|2h_6(0), z_p\| = 0$ , for each  $z_p \in X$ . Therefore  $h_6(0) = 0$ .

Revised version of Theorem (4.2) of paper (6) is as under.

Let  $\eta_a, \theta_a \in [0, \infty), r_a \in (0, \infty), p_a, q_a \in (2, \infty)$  and  $(Z, \|\cdot\|)$  be a normed space and  $(Z, \|\cdot, \cdot\|)$  be a 2-Banach space and  $f_b : (Z, \|\cdot\|) \rightarrow (Z, \|\cdot, \cdot\|)$  be a mapping satisfying

$$\|f_b(x + y) + f_b(x - y) - 2f_b(x) - 2f_b(y), z\| \leq (\eta_a \|x\|^{p_a} + \theta_a \|y\|^{q_a}) \|z\|^{r_a}$$

for all  $x, y, z \in Z$ . Then there  $\exists$  unique quadratic mapping  $Q_a : Z \rightarrow Z$  such that

$$\|f_b(x) - Q_a(x), z\| \leq \left( \frac{\eta_a \|x\|^{p_a}}{2^{p_a} - 4} + \frac{\theta_a \|x\|^{q_a}}{2^{q_a} - 4} \right) \|z\|^{r_a}$$

for all  $x, z \in Z$ .

---

### 3 CONCLUSIONS

This study conclude that some modification in theorems of S. Chung and W. Park to strengthen the theoretical framework of functional equations in 2-Banach spaces and ensure consistency in their applications to stability theory. It will be useful in the theory of functional equations..

### References

- [1] Arumugam Ponmana Selvan, Abbas Najati, *Hyers–Ulam stability and hyperstability of a Jensen-type functional equation on 2-Banach spaces*, Journal of Inequalities and Applications Artical No 32(2022),1-11.
- [2] S. Gähler, *Lineare 2-normierte Räume*, Math. Nachr., 26(1963) 115-148.
- [3] Nazek Alessa and K. Tamilvanan, Orthogonal Stability and Nonstability of a Generalized Quartic Functional Equation in Quasi- $\beta$ -Normed Spaces, *Journal of Function Spaces*, Feb 2021, No. 2021(3).
- [4] B.M. Patel and A.B. Patel, Stability of Quartic Functional equations in 2-Banach space, *Int. Journal of Math. Analysis*, 7(2013), No. 23, 1097-1107.
- [5] W. Park, *Approximate Additive Mappings in 2-Banach spaces and Related Topics*, J. Math. Anal. Appl. 376(2011)193-202.
- [6] S. Chung, W. Park, *Hyers-Ulam Stability of Functional Equations in 2-Banach spaces*, Int. Jour. of Math. Anal. 6(2012)951-961.
- [7] K. Tamilvanan, G. Balasubramanian, Nazek Alessa, K. Loganathan, Hyers–Ulam Stability of Additive Functional Equation Using Direct and Fixed-Point Methods, *Journal of Mathematics*, Dec 2020, No. 2020(1).