Hyperbolic Extensions of Generalized Pandita Numbers

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Abstract. This paper introduces the framework of generalized hyperbolic Pandita numbers constructed over the bidimensional Clifford algebra of hyperbolic numbers, contributing a novel class of structured sequences to the expanding domain of number theory. Anchored in the principles of hyperbolic systems, these constructs pave the way for exploring algebraic symmetries and recursive behaviors beyond classical formulations.

Special attention is devoted to notable cases, including the hyperbolic Pandita and hyperbolic Pandita-Lucas numbers, whose properties are meticulously examined. To deepen understanding and facilitate computation, we derive explicit closed-form representations using Binet-type formulas, construct generating functions through formal power series, and establish summation expressions with broad applicability. Additionally, matrix-based representations are developed to offer an algebraic lens through which structural dynamics can be modeled and analyzed.

These formulations not only enrich the theoretical foundations of discrete mathematics and symbolic computation but also highlight promising applications in engineering disciplines—particularly in the modeling of iterative systems, signal transformations, the analysis of complex networks, and cryptographic systems.

The insights presented herein lay the groundwork for future exploration into hybrid sequence systems and their role in interdisciplinary problem solving.

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1. Introduction

The hypercomplex numbers systems, [27], are extensions of real numbers. Some commutative examples of hypercomplex number systems are complex numbers,

$$\mathbb{C}=\{z=a+ib:a,b\in\mathbb{R},i^2=-1\},$$

hyperbolic (double, split-complex) numbers, [24],

$$\mathbb{H}=\{h=a+jb:a,b\in\mathbb{R},j^2=1,j\neq\pm1\},$$

and dual numbers, [43],

$$\mathbb{D}=\{d=a+\varepsilon b:a,b\in\mathbb{R},\varepsilon^2=0,\varepsilon\neq0\},$$

Some non-commutative examples of hypercomplex number systems are quaternions, [78],

$$\mathbb{H}_{\mathbb{Q}}=\{q=a_0+ia_1+ja_2+ka_3:a_0,a_1,a_2,a_3\in\mathbb{R},i^2=j^2=k^2=ijk=-1\},$$

octonions [28] and sedenions [46]. The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are real algebras obtained from the real numbers $\mathbb{R}$ by a doubling procedure called the Cayley-Dickson Process. This doubling process can be extended beyond the sedenions to form what are known as the 2^n-ions (see for example [15], [39], [22]).

Quaternions were invented by Irish mathematician W. R. Hamilton (1805-1865) [78] as an extension to the complex numbers. Hyperbolic numbers with complex coefficients are introduced by J. Cockle in 1848, [29]. H. H. Cheng and S. Thompson [25] introduced dual numbers with complex coefficients and called complex dual numbers. Akar, Yüce and Şahin [40] introduced dual hyperbolic numbers.

A dual hyperbolic number is a hyper-complex number and is defined by

$$q=(a_0+ja_1)+\varepsilon(a_2+ja_3)=a_0+ja_1+\varepsilon a_2+\varepsilon ja_3,$$

where a_0,a_1,a_2 and a_3 are real numbers.

The set of all dual hyperbolic numbers are denoted by

$$\mathbb{H}_{\mathbb{D}}=\{a_0+ja_1+\varepsilon a_2+\varepsilon ja_3:a_0,a_1,a_2,a_3\in\mathbb{R},\ j^2=1,j\neq\pm1,\varepsilon^2=0,\varepsilon\neq0\}.$$

The base elements $\{1,j,\varepsilon,\varepsilon j\}$ of dual hyperbolic numbers satisfy the following properties (commutative multiplications):

$$\begin{aligned}1.\varepsilon&=\varepsilon,1.j=j,\ \varepsilon^2=\varepsilon.\varepsilon=(j\varepsilon)^2=0,\ j^2=j.j=1,\\ \varepsilon.j&=j.\varepsilon,\ \varepsilon.(\varepsilon j)=(\varepsilon j).\varepsilon=0,\ j(\varepsilon j)=(\varepsilon j)j=\varepsilon,\end{aligned}$$

where ε denotes the pure dual unit ($\varepsilon^2=0,\varepsilon\neq0$), j denotes the hyperbolic unit ($j^2=1$), and εj denotes the dual hyperbolic unit ($(j\varepsilon)^2=0$).

The product of two dual hyperbolic numbers $q=a_0+ja_1+\varepsilon a_2+j\varepsilon a_3$ and $p=b_0+jb_1+\varepsilon b_2+j\varepsilon b_3$ is

$$qp=a_0b_0+a_1b_1+j(a_0b_1+a_1b_0)+\varepsilon(a_0b_2+a_2b_0+a_1b_3+a_3b_1)+j\varepsilon(a_0b_3+a_1b_2+a_2b_1+b_0a_3)$$

and addition of dual hyperbolic numbers is defined as componentwise.

The dual hyperbolic numbers form a commutative ring, real vector space and an algebra. But $\mathbb{H}_{\mathbb{D}}$ is not field because every dual hyperbolic numbers doesn't have an inverse. For more information on the dual hyperbolic numbers, see [40].

Here we use the set of hyperbolic numbers. The set of hyperbolic numbers $\mathbb{H}$ can be described as

$$\mathbb{H} = \{z = x + hy \mid h \notin \mathbb{R},\ h^2 = 1, x, y \in \mathbb{R}\}.$$

The hyperbolic ring $\mathbb{H}$ is a bidimensional Clifford algebra, see [31] for details. Hyperbolic numbers has been called in the mathematical literature with different names: Lorentz numbers, double numbers, duplex numbers, split complex numbers and perplex numbers. Hyperbolic numbers are useful for measuring distances in the Lorentz space-time plane (see Sobczyk [24]). For more information on hyperbolic numbers, see also [26,30,41,45].

Addition, substraction and multiplication of any two hyperbolic numbers z_1 and z_2 are defined by

$$\begin{aligned} z_1 \pm z_2 &= (x_1 + hy_1) \pm (x_2 + hy_2) = (x_1 \pm x_2) + h\,(y_1 \pm y_2)\,, \\ z_1 \times z_2 &= (x_1 + hy_1) \times (x_2 + hy_2) = x_1x_2 + y_1y_2 + h\,(x_1y_2 + y_1x_2)\,. \end{aligned}$$

and the division of two hyperbolic numbers are given by

$$\frac{z_1}{z_2} = \frac{x_1 + hy_1}{x_2 + hy_2} = \frac{(x_1 + hy_1)(x_2 - hy_2)}{(x_2 + hy_2)(x_2 - hy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 - y_2^2} + h\frac{x_1y_2 + y_1x_2}{x_2^2 - y_2^2}.$$

It is easy to see that this algebra of hyperbolic numbers is commutative and contains zero divisors. The hyperbolic conjugation of $z = x + hy$ is defined by

$$\overline{z} = z^{\dagger} = x - hy.$$

Note that $\overline{\overline{z}} = z$. Note also that for any hyperbolic numbers z_1, z_2, z we have

$$\begin{aligned} \overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2}, \\ \overline{z_1 \times z_2} &= \overline{z_1} \times \overline{z_2}, \\ \|z\|^2 &= z \times \overline{z} = x^2 - y^2. \end{aligned}$$

Now let us recall the definition of generalized Pandita numbers.

A generalized Pandita sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations

$$W_n = 2W_{n-1} - W_{n-2} + W_{n-3} - W_{n-4} \tag{1.1}$$

with the initial values W_0, W_1, W_2, W_3 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-1)} - W_{-(n-2)} + W_{-(n-3)} - W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

Table 1 presents the initial set of generalized Pandita numbers with both positive and negative subscripts.

Table 1. Initial Values of Generalized Pandita Numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$W_0 - W_1 + 2W_2 - W_3$
2	W_2	$W_1 + W_2 - W_3$
3	W_3	$W_0 + W_1 - W_2$
4	$W_1 - W_0 - W_2 + 2W_3$	$2W_0 - 2W_1 + 2W_2 - W_3$
5	$W_1 - 2W_0 - W_2 + 3W_3$	$3W_2 - 2W_3$
6	$W_1 - 3W_0 - 2W_2 + 5W_3$	$3W_1 - 2W_2$
7	$2W_1 - 5W_0 - 4W_2 + 8W_3$	$3W_0 - 2W_1$
8	$3W_1 - 8W_0 - 6W_2 + 12W_3$	$W_0 - 3W_1 + 6W_2 - 3W_3$
9	$4W_1 - 12W_0 - 9W_2 + 18W_3$	$5W_1 - 2W_0 - W_2 - W_3$
10	$6W_1 - 18W_0 - 14W_2 + 27W_3$	$3W_0 + W_1 - 5W_2 + 2W_3$
11	$9W_1 - 27W_0 - 21W_2 + 40W_3$	$4W_0 - 8W_1 + 8W_2 - 3W_3$
12	$13W_1 - 40W_0 - 31W_2 + 59W_3$	$4W_1 - 4W_0 + 5W_2 - 4W_3$
13	$19W_1 - 59W_0 - 46W_2 + 87W_3$	$9W_1 - 12W_2 + 4W_3$

If we set $W_0 = 0, W_1 = 1, W_2 = 2, W_3 = 3$ then $\{W_n\}$ is the well-known Pandita sequence and if we set $W_0 = 4, W_1 = 2, W_2 = 2, W_3 = 5$ then $\{W_n\}$ is the well-known Pandita-Lucas sequence. In other words, Pandita sequence $\{P_n\}_{n\geq 0}$ and Pandita-Lucas sequence $\{S_n\}_{n\geq 0}$ are defined by the second-order recurrence relations

$$P_n = 2P_{n-1} - P_{n-2} + P_{n-3} - P_{n-4}, \qquad P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 3, \qquad n \geq 4, \tag{1.2}$$

and

$$S_n = 2S_{n-1} - S_{n-2} + S_{n-3} - S_{n-4}, \qquad S_0 = 4, S_1 = 2, S_2 = 2, S_3 = 5, \qquad n \geq 4 \tag{1.3}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{S_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = P_{-(n-1)} - P_{-(n-2)} + 2P_{-(n-3)} - P_{-(n-4)}$$

and

$$S_{-n} = S_{-(n-1)} - S_{-(n-2)} + 2S_{-(n-3)} - S_{-(n-4)},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.2) and (1.3) hold for all integer n.

Several key properties of the generalized Pandita numbers, essential for further analysis, are listed below.

- Binet formula of generalized Pandita sequence can be calculated using its characteristic equation which is given as

$$x^4 - 2x^3 + x^2 - x + 1 = (x^3 - x^2 - 1)(x - 1) = 0$$

The roots of characteristic equation are

$$\begin{aligned}
\alpha &= \frac{1}{3}+\left(\frac{29}{54}+\sqrt{\frac{31}{108}}\right)^{1/3}+\left(\frac{29}{54}-\sqrt{\frac{31}{108}}\right)^{1/3}, \\
\beta &= \frac{1}{3}+\omega\left(\frac{29}{54}+\sqrt{\frac{31}{108}}\right)^{1/3}+\omega^2\left(\frac{29}{54}-\sqrt{\frac{31}{108}}\right)^{1/3}, \\
\gamma &= \frac{1}{3}+\omega^2\left(\frac{29}{54}+\sqrt{\frac{31}{108}}\right)^{1/3}+\omega\left(\frac{29}{54}-\sqrt{\frac{31}{108}}\right)^{1/3}, \\
\delta &= 1,
\end{aligned}$$

where

$$\omega = \frac{-1+i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Using these roots and the recurrence relation, Binet formula can be given as

$$\begin{aligned}
W_n &= \frac{z_1\alpha^n}{3\alpha-2}+\frac{z_2\beta^n}{3\beta-2}+\frac{z_3\gamma^n}{3\gamma-2}+z_4 \\
&= A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4,
\end{aligned}$$

where z_1, z_2 and z_3 are given below

$$\begin{aligned}
z_1 &= (\alpha W_3-\alpha(2-\alpha)W_2+(-\alpha^2+\alpha+1)W_1-W_0), \\
z_2 &= (\beta W_3-\beta(2-\beta)W_2+(-\beta^2+\beta+1)W_1-W_0), \\
z_3 &= (\gamma W_3-\gamma(2-\gamma)W_2+(-\gamma^2+\gamma+1)W_1-W_0), \\
z_4 &= -W_3+W_2+W_0.
\end{aligned}$$

and

$$\begin{aligned}
A_1 &= \frac{z_1}{3\alpha-2}, \\
A_2 &= \frac{z_2}{3\beta-2}, \\
A_3 &= \frac{z_3}{3\gamma-2}, \\
A_4 &= z_4.
\end{aligned} \tag{1.4}$$

Binet formula of Pandita and Pandita-Lucas sequences are

$$P_n = \frac{\alpha^{n+3}}{3\alpha-2}+\frac{\beta^{n+3}}{3\beta-2}+\frac{\gamma^{n+3}}{3\gamma-2}-1,$$

and

$$S_n = \alpha^n+\beta^n+\gamma^n+1,$$

respectively.

- The generating function for generalized Pandita numbers is

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 2W_0)x + (W_2 - 2W_1 + W_0)x^2 + (W_3 - 2W_2 + W_1 - W_0)x^3}{1 - 2x + x^2 - x^3 + x^4}.$$

For further details on the generalized Pandita numbers, refer to [48].

Next, we give the exponential generating function of $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence W_n.

LEMMA 1. *[33, Lemma 1.4]Suppose that* $f_{W_n}(x) = \sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ *is the exponential generating function of the generalized Pandita sequence* $\{W_n\}$.

Then $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \frac{(\alpha W_3 - \alpha(2-\alpha)W_2 + (-\alpha^2 + \alpha + 1)W_1 - W_0)}{3\alpha - 2} e^{\alpha x} \\ &\quad + \frac{(\beta W_3 - \beta(2-\beta)W_2 + (-\beta^2 + \beta + 1)W_1 - W_0)}{3\beta - 2} e^{\beta x} \\ &\quad + \frac{(\gamma W_3 - \gamma(2-\gamma)W_2 + (-\gamma^2 + \gamma + 1)W_1 - W_0)}{3\gamma - 2} e^{\gamma x} \\ &\quad + (-W_3 + W_2 + W_0)e^x.\end{aligned}$$

The previous Lemma 1 gives the following results as particular examples.

COROLLARY 2. *Exponential generating function of Pandita and Pandita-Lucas numbers*

a): $\sum\limits_{n=0}^{\infty} P_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} (\frac{\alpha^{n+3}}{3\alpha - 2} + \frac{\beta^{n+3}}{3\beta - 2} + \frac{\gamma^{n+3}}{3\gamma - 2} - 1)\frac{x^n}{n!} = \frac{\alpha^3 e^{\alpha x}}{3\alpha - 2} + \frac{\beta^3 e^{\beta x}}{3\beta - 2} + \frac{\gamma^3 e^{\gamma x}}{3\gamma - 2} - e^x.$

b): $\sum\limits_{n=0}^{\infty} S_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} (\alpha^n + \beta^n + \gamma^n + 1)\frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^{\gamma x} + e^x.$

Next, we give some information on published papers related to hyperbolic and Dual hyperbolic numbers in literature.

1. Cockle [29] presented the hyperbolic numbers with complex coefficients.
2. Akar at al [40] introduced the dual hyperbolic numbers.
3. Cheng and Thompson[25] studied dual numbers with complex coefficients.

Next, we give some information related to dual hyperbolic sequences presented in literature.

1. Soykan at al [65] introduced dual hyperbolic generalized Pell numbers given by

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}$$

   where generalized Pell numbers are given by $V_n = 2V_{n-1} + V_{n-2}$, $V_0 = a, V_1 = b$ $(n \geq 2)$ with the initial values V_0, V_1 not all being zero
2. Cihan at al [8] studied dual hyperbolic Fibonacci and Lucas numbers given by, respectively,

$$DHF_n = F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3},$$

$$DHL_n = L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3},$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1, L_n = L_{n-1} + L_{n-2}, L_0 = 2, L_1 = 1.$

(3) Soykan at al [69] introduced dual hyperbolic generalized Jacopsthal numbers given by

$$\widehat{J}_n = J_n + jJ_{n+1} + \varepsilon J_{n+2} + j\varepsilon J_{n+3}$$

where $J_n = J_{n-1} + 2J_{n-2}, J_0 = a, J_1 = b$

4. Bród at al [3] studied dual hyperbolic generalized Balancing numbers are

$$DHB_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}$$

where $B_n = 6B_{n-1} - B_{n-2}, B_0 = 0, B_1 = 1.$

5. Yılmaz and Soykan [74] introduced dual hyperbolic generalized Guglielmo numbers are

$$\widehat{T_0} = T_0 + jT_1 + \varepsilon T_2 + j\varepsilon T_3$$

where $T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, T_0 = 0, T_1 = 1, T_2 = 3.$

6. Dikmen [14] introduced dual hyperbolic generalised Leonardo numbers given by

$$\widehat{l_0} = l_0 + jl_1 + \varepsilon l_2 + j\varepsilon l_3$$

where $l_n = 2l_{n-1} - l_{n-3}, l_0 = 1, l_1 = 1, l_2 = 3.$

7. Kalça and Soykan [32] introduced dual hyperbolic generalized Pandita numbers are

$$\widehat{P_0} = P_0 + jP_1 + \varepsilon P_2 + j\varepsilon P_3,$$

where $P_n = 2P_{n-1} - P_{n-2} + P_{n-3} - P_{n-4}, P_0 = 0 P_1 = 1, P_2 = 2, P_3 = 3.$

8. Demirci and Soykan [**??**] introduced dual hyperbolic generalized Adrien numbers are

$$\widehat{A_0} = A_0 + jA_1 + \varepsilon A_2 + j\varepsilon A_3,$$

where $A_n = 3A_{n-1} - A_{n-2} - A_{n-4}$, $A_0 = 0, A_1 = 1, A_2 = 3, A_3 = 8.$

9. Eren and Soykan [16] introduced dual hyperbolic generalized Woodall numbers given by

$$\widehat{R_0} = R_0 + jR_1 + \varepsilon R_2 + j\varepsilon R_3$$

where $R_n = 5R_{n-1} - 8R_{n-2} + 4R_{n-3},\ R_0 = -1, R_1 = 1, R_2 = 7.$

10. In [17], the outhers introduce the dual generalized Fibonacci matrices.

Next, the hyperbolic Fibonacci sequence will be introduced, followed by an explanation of its relationship with Pandita numbers. Subsequently, the practical applications and significance of Pandita numbers in daily life will be discussed.

Hyperbolic Fibonacci Numbers

Hyperbolic extensions of classical recursive sequences provide deeper algebraic and geometric interpretations, particularly within the context of hypercomplex systems, combinatorics, and theoretical physics. Among these, the hyperbolic formulation of the Fibonacci sequence has attracted attention due to its analytical richness and structural elegance.

Hyperbolic Fibonacci numbers generalize the classical Fibonacci sequence using hyperbolic functions. One such formulation involves the hyperbolic sine function defined as:

$$\sinh_F(x) = \frac{\phi^x - \psi^x}{\phi - \psi}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$ are the golden ratio and its conjugate, respectively. This expression yields values closely related to the classical Fibonacci numbers for integer inputs:

$$\sinh_F(n) \approx F_n$$

Further generalizations include hyperbolic cosine and tangent functions that encode Fibonacci-related ratios. These constructions are useful in analytic number theory and combinatorial identities [71]

1.1. Applications and Relevance of Pandita Numbers. Pandita numbers, introduced as a generalization of classical recursive sequences such as Narayana's Cows, possess rich algebraic structures that make them suitable for both theoretical exploration and practical modeling. Although primarily studied within pure mathematics, their properties—such as fourth-order recurrence relations, matrix representations, and closed-form expressions—enable interdisciplinary applications.

Potential Applications in Daily Life. While Pandita numbers are not yet widely known in applied engineering or consumer technologies, their structural similarities to Fibonacci, Narayana, and Lucas numbers suggest promising future uses:

1. **Digital Signal Processing:** Recursive sequences like Pandita numbers can be used to model waveforms, filter designs, and compression algorithms, especially in systems requiring layered or fourth-order recurrence behavior.
2. **Cryptography and Coding Theory:** The algebraic and modular properties of Pandita numbers may contribute to the design of secure key generation schemes and error-correcting codes.
3. **Pattern Recognition and Image Processing:** Pandita-based matrices and transformations can be adapted for feature extraction in visual data, particularly in systems with periodic or recursive structures.

(4) **Biological and Fluid Modeling:** As shown in recent studies, generalized number systems—including Pandita-Narayana variants—can model cilia-driven flow, microorganism propulsion, and mucus dynamics in low Reynolds number environments.

Contributions to Research Fields. Pandita numbers support advanced mathematical modeling in:

1. **Hypercomplex Systems:** Their extension into Gaussian, hyperbolic, and Clifford algebras allows for simulations in non-Euclidean geometries and relativistic frameworks.
2. **Special Functions and Combinatorics:** Pandita sequences yield new identities, generating functions, and summation formulas that enrich analytic number theory.
3. **Numerical Methods:** Their recurrence structure is compatible with finite-difference and iterative schemes used in computational fluid dynamics and bioengineering simulations.

In the following section, we define the hyperbolic generalized Pandita numbers and present some of their fundamental properties.

2. Hyperbolic Generalized Pandita Numbers and their Generating Functions and Binet's Formulas

In this section, we define the hyperbolic generalized Pandita numbers and present their generating functions and Binet formulas.We now define hyperbolic generalized Pandita numbers over $\mathbb{H}_{\mathbb{D}}$. The nth hyperbolic generalized Pandita number is

$$HW_n = W_n + jW_{n+1}. \tag{2.1}$$

The sequence $\{HW_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$HW_{-n} = W_{-n} + jW_{-n+1},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.2) holds for all integer n.

Note that

$$\begin{aligned}
HW_0 &= W_0 + jW_1 \\
HW_1 &= W_1 + jW_2 = W_1 + jW_2 \\
HW_2 &= W_2 + jW_3 = W_2 + jW_3 \\
HW_3 &= W_3 + jW_4 = W_3 + j(W_1 - W_0 - W_2 + 2W_3)
\end{aligned}$$

It can be easily shown that

$$HW_n = 2HW_{n-1} - HW_{n-2} + HW_{n-3} - HW_{n-4}, \tag{2.2}$$

and

$$HW_{-n} = HW_{-(n-1)} - HW_{-(n-2)} + 2HW_{-(n-3)} - HW_{-(n-4)}$$

Table 2 lists the initial values of the hyperbolic generalized Pandita numbers for both positive and negative subscripts.

Table 2. A few hyperbolic generalized Pandita numbers

n	HW_n	HW_{-n}
0	HW_0	HW_0
1	HW_1	$HW_0 - HW_1 + 2HW_2 - HW_3$
2	HW_2	$HW_1 + HW_2 - HW_3$
3	HW_3	$HW_0 + HW_1 - HW_2$
4	$HW_1 - HW_0 - HW_2 + 2HW_3$	$2HW_0 - 2HW_1 + 2HW_2 - HW_3$
5	$HW_1 - 2HW_0 - HW_2 + 3HW_3$	$3HW_2 - 2HW_3$
6	$HW_1 - 3HW_0 - 2HW_2 + 5HW_3$	$3HW_1 - 2HW_2$
7	$2HW_1 - 5HW_0 - 4HW_2 + 8HW_3$	$3HW_0 - 2HW_1$
8	$3HW_1 - 8HW_0 - 6HW_2 + 12HW_3$	$HW_0 - 3HW_1 + 6HW_2 - 3HW_3$
9	$4HW_1 - 12HW_0 - 9HW_2 + 18HW_3$	$5HW_1 - 2HW_0 - HW_2 - HW_3$
10	$6HW_1 - 18HW_0 - 14HW_2 + 27HW_3$	$3HW_0 + HW_1 - 5HW_2 + 2HW_3$
11	$9HW_1 - 27HW_0 - 21HW_2 + 40HW_3$	$4HW_0 - 8HW_1 + 8HW_2 - 3HW_3$
12	$13HW_1 - 40HW_0 - 31HW_2 + 59HW_3$	$4HW_1 - 4HW_0 + 5HW_2 - 4HW_3$
13	$19HW_1 - 59HW_0 - 46HW_2 + 87HW_3$	$9HW_1 - 12HW_2 + 4HW_3$

As special cases, the nth hyperbolic Pandita numbers and the nth hyperbolic Pandita Lucas numbers are given as

$$HP_n = P_n + jP_{n+1}, \tag{2.3}$$

and

$$HS_n = S_n + jS_{n+1} \tag{2.4}$$

respectively. The sequences $\{HP_n\}_{n\geq 0}$ and $\{HS_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$HP_{-n} = P_{-(n-1)} - P_{-(n-2)} + 2P_{-(n-3)} - P_{-(n-4)},$$

and

$$HS_{-n} = S_{-(n-1)} - S_{-(n-2)} + 2S_{-(n-3)} - S_{-(n-4)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.3) and (2.4) holds for all integer n

For hyperbolic Pandita numbers (taking $W_n = P_n,\ \ P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 3,$) we get

$$\begin{aligned} HP_0 &= j, \\ HP_1 &= 2j+1, \\ HP_2 &= 3j+2, \end{aligned}$$

and for hyperbolic Pandita-Lucas numbers (taking $W_n = S_n,\ \ S_0 = 4, S_1 = 2, S_2 = 2, S_3 = 5,$) we get

$$\begin{aligned} HS_0 &= 2j+4, \\ HS_1 &= 2j+2. \\ HS_2 &= 5j+2 \end{aligned}$$

A few hyperbolic Pandita numbers and hyperbolic Pandita Lucas numbers with positive subscript and negative subscript are given in the following Table 3 and Table 4.

Table 3. hyperbolic Pandita numbers

n	HP_n	HP_{-n}
0	j	j
1	$2j+1$	0
2	$3j+2$	0
3	$5j+3$	-1
4	$8j+5$	$-j-1$
5	$12j+8$	$-j$

Table 4. hyperbolicPandita- Lucas numbers

n	HS_n	HS_{-n}
0	$2j+4$	$2j+4$
1	$2j+2$	$4j+1$
2	$5j+2$	$j-1$
3	$6j+5$	$-j+4$
4	$7j+6$	$4j+3$
5	$11j++7$	$-3j-4$

We now present Binet's formula for the hyperbolic generalized Pandita numbers. Throughout the remainder of the paper, the following notations will be used.

$$\widehat{\alpha} = 1 + j\alpha, \tag{2.5}$$

$$\widehat{\beta} = 1 + j\beta, \tag{2.6}$$

$$\widehat{\gamma} = 1 + j\gamma \tag{2.7}$$

$$\widehat{\delta} = \widehat{1} = 1 + j, \tag{2.8}$$

Note that we have the following identities:

$$\begin{aligned}
\widehat{\alpha}^2 &= 1+\alpha^2+2\alpha j, \\
\widehat{\beta}^2 &= 1+\beta^2+2j\beta, \\
\widehat{\alpha}\widehat{\beta} &= 1+\alpha\beta+(\alpha+\beta)\,j, \\
\widehat{\gamma}^2 &= 1+\gamma^2+2j\gamma, \\
\widehat{\delta}^2 &= \widehat{1}^2=2+2j, \\
\widehat{\gamma}\widehat{\delta} &= 1+\gamma+j+j\gamma
\end{aligned}$$

THEOREM 3. *(Binet's Formula) For any integer n, the nth hyperbolic generalized Pandita number is*

$$HW_n=A_1\alpha^n\widehat{\alpha}+A_2\beta^n\widehat{\beta}+A_3\gamma^n\widehat{\gamma}+\widehat{1}A_4. \tag{2.9}$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}, \widehat{\delta}$ are given as (2.5)-(2.8)

Proof. Using Binet's formula of the generalized Pandita numbers given below

$$W_n=A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4.$$

where A_1, A_2, A_3, A_4 are given in (1.4) we get

$$\begin{aligned}
HW_n &= W_n+jW_{n+1} \\
&= A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4+j(A_1\alpha^{n+1}+A_2\beta^{n+1}+A_3\gamma^{n+1}+A_4) \\
&= A_1\alpha^n(1+j\alpha)+A_2\beta^n(1+j\beta)+A_3\gamma^n(1+j\gamma)+A_4(1+j) \\
&= A_1\alpha^n\widehat{\alpha}+A_2\beta^n\widehat{\beta}+A_3\gamma^n\widehat{\gamma}+\widehat{1}A_4.
\end{aligned}$$

This proves (2.9). □

As special cases, for any integer n, the Binet's Formula of nth hyperbolic Pandita number is

$$HP_n=\frac{\alpha^{n+3}\widehat{\alpha}}{3\alpha-2}+\frac{\beta^{n+3}\widehat{\beta}}{3\beta-2}+\frac{\gamma^{n+3}\widehat{\gamma}}{3\gamma-2}-\widehat{1} \tag{2.10}$$

and the Binet's Formula of nth hyperbolic Pandita-Lucas number is

$$HS_n=\widehat{\alpha}\alpha^n+\widehat{\beta}\beta^n+\widehat{\gamma}\gamma^n+\widehat{1}, \tag{2.11}$$

Next, we present generating function.

THEOREM 4. *Let $f_{HW_n}(x)=\sum\limits_{n=0}^{\infty}HW_nx^n$ denote the generating function of hyperbolic generalized Pandita numbers is given as follows:*

$f_{HW_n}(x)=\sum\limits_{n=0}^{\infty}HW_nx^n=\frac{1}{1-2x+x^2-x^3+x^4}(HW_0+(HW_1-2HW_0)x+(HW_2-2HW_1+HW_0)x^2+(HW_3-2HW_2+HW_1-HW_0)x^3).$

Proof. Using the definition of hyperbolic Pandita numbers, and substracting $xf(x)$, $x^2f(x)$ and $x^3f(x)$ from $f(x)$ we obtain $(1-2x+x^2-x^3+x^4)f_{HW_n}(x)$

$$\begin{aligned}
& (1-2x+x^2-x^3+x^4)f_{HW_n}(x) \\
= & \sum_{n=0}^{\infty} HW_n x^n - 2x\sum_{n=0}^{\infty} HW_n x^n + x^2\sum_{n=0}^{\infty} HW_n x^n - x^3\sum_{n=0}^{\infty} HW_n x^n + x^4\sum_{n=0}^{\infty} HW_n x^n, \\
= & \sum_{n=0}^{\infty} HW_n x^n - 2\sum_{n=0}^{\infty} HW_n x^{n+1} + \sum_{n=0}^{\infty} HW_n x^{n+2} - \sum_{n=0}^{\infty} HW_n x^{n+3} + \sum_{n=0}^{\infty} HW_n x^{n+4}, \\
= & \sum_{n=0}^{\infty} HW_n x^n - 2\sum_{n=1}^{\infty} HW_{(n-1)} x^{n} + \sum_{n=2}^{\infty} HW_{(n-2)} x^{n} - \sum_{n=3}^{\infty} HW_{(n-3)} x^{n} + \sum_{n=4}^{\infty} HW_{(n-4)} x^{n}, \\
= & (HW_0 + HW_1x + HW_2x^2 + HW_3x^3) - 2(HW_0x + HW_1x^2 + HW_2x^3) + (HW_0x^2 + HW_1x^3) - HW_0x^3 \\
& + \sum_{n=4}^{\infty} (HW_n - 2HW_{n-1} - HW_{n-2} - HW_{n-3} + HW_{n-4})x^n, \\
= & HW_0 + (HW_1 - 2HW_0)x + (HW_2 - 2HW_1 + HW_0)x^2 + (HW_3 - 2HW_2 + HW_1 - HW_0)x^3.
\end{aligned}$$

And rearranging above equation, we get (4). □

The following results are immediate consequences of the preceding Theorem.

Corollary 5. *For all integers n, we have following identities:*

- **a):** $\sum_{n=0}^{\infty} HP_n x^n = \dfrac{j+x}{1-2x+x^2-x^3+x^4}$.
- **b):** $\sum_{n=0}^{\infty} HS_n x^n = \dfrac{(2j+4)+(-2j-6)x+(3j+2)x^2+(-4j+7)x^3}{1-2x+x^2-x^3+x^4}$.

Theorem (4) gives the following results as special cases,

$(1-2x+x^2-x^3+x^4)f_{HP_n}(x) = HP_0 + (HP_1 - 2HP_0)x + (HP_2 - 2HP_1 + HP_0)x^2 + (HP_3 - 2HP_2 + HP_1 - HP_0)x^3 = j + x$,

$(1-2x+x^2-x^3+x^4)f_{HS_n}(x) = HS_0 + (HS_1 - 2HS_0)x + (HS_2 - 2HS_1 + HS_0)x^2 + (HS_3 - 2HS_2 + HS_1 - HS_0)x^3 = (2j+4) + (-2j-6)x + (3j+2)x^2 + (-4j+7)x^3$.

Next, we give the exponential hyperbolic generating function of $\sum\limits_{n=0}^{\infty} HW_n \frac{x^n}{n!}$ of the sequence HW_n.

Lemma 6. *Suppose that $f_{HW_n}(x) = \sum\limits_{n=0}^{\infty} HW_n \frac{x^n}{n!}$ is the exponential hyperbolic generating function of the generalized Pandita sequence $\{HW_n\}$.*

Then $\sum\limits_{n=0}^{\infty} HW_n \frac{x^n}{n!}$ is given by

$$\sum_{n=0}^{\infty} HW_n \frac{x^n}{n!} == A_1 e^{\alpha x}\widehat{\alpha} + A_2 e^{\beta x}\widehat{\beta} + A_3 e^{\gamma x}\widehat{\gamma} + A_4 e^{x}\widehat{1}.$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}, \widehat{\delta}$ are given as (2.5)-(2.8)

Proof. Using Binet's formula

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4.$$

where A_1, A_2, A_3, A_4 are given in (1.4) we get

$$\begin{aligned}
\sum_{n=0}^{\infty} HW_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty} W_n \frac{x^n}{n!} + j\sum_{n=0}^{\infty} W_{n+1}\frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty}(A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4)\frac{x^n}{n!} + j\sum_{n=0}^{\infty}(A_1\alpha^{n+1} + A_2\beta^{n+1} + A_3\gamma^{n+1} + A_4)\frac{x^n}{n!} \\
&= (A_1e^{\alpha x} + A_2e^{\beta x} + A_3e^{\gamma x} + A_4e^{x}) + j(A_1\alpha e^{\alpha x} + A_2\beta e^{\beta x} + A_3\gamma e^{\gamma x} + A_4e^{x}) \\
&= A_1e^{\alpha x}(1+j\alpha) + A_2e^{\beta x}(1+j\beta) + A_3e^{\gamma x}(1+j\gamma) + A_4e^{x}(1+j) \\
&= A_1e^{\alpha x}\widehat{\alpha} + A_2e^{\beta x}\widehat{\beta} + A_3e^{\gamma x}\widehat{\gamma} + A_4e^{x}\widehat{1}
\end{aligned}$$

This proves (6). □

The previous Lemma 6 gives the following results as particular examples.

Corollary 7. *Exponential hyperbolic generating function of Pandita and Pandita-Lucas numbers are*

a): $\sum\limits_{n=0}^{\infty} HP_n \frac{x^n}{n!} = \dfrac{\alpha^3 e^{\alpha x}\widehat{\alpha}}{3\alpha-2} + \dfrac{\beta^3 e^{\beta x}\widehat{\beta}}{3\beta-2} + \dfrac{\gamma^3 e^{\gamma x}\widehat{\gamma}}{3\gamma-2} - e^{x}\widehat{1}.$

b): $\sum\limits_{n=0}^{\infty} HS_n \frac{x^n}{n!} = e^{\alpha x}\widehat{\alpha} + e^{\beta x}\widehat{\beta} + e^{\gamma x}\widehat{\gamma} + e^{x}\widehat{1}.$

3. Obtaining Binet Formula From Generating Function

We next find Binet's formula generalized hyperbolic Pandita number $\{HW_n\}$ by the use of generating function for HW_n.

Theorem 8. *Binet's formula of generalized hyperbolic Pandita numbers:*

$$HW_n = \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}. \tag{3.1}$$

where

$$\begin{aligned}
q_1 &= HW_0\alpha^3 + (HW_1 - 2HW_0)\,\alpha^2 + (HW_0 - 2HW_1 + HW_2)\,\alpha - HW_0 + HW_1 - 2HW_2 + HW_3, \\
q_2 &= HW_0\beta^3 + (HW_1 - 2HW_0)\,\beta^2 + (HW_0 - 2HW_1 + HW_2)\,\beta - HW_0 + HW_1 - 2HW_2 + HW_3, \\
q_3 &= HW_0\gamma^3 + (HW_1 - 2HW_0)\,\gamma^2 + (HW_0 - 2HW_1 + HW_2)\,\gamma - HW_0 + HW_1 - 2HW_2 + HW_3, \\
q_4 &= HW_0\delta^3 + (HW_1 - 2HW_0)\,\delta^2 + (HW_0 - 2HW_1 + HW_2)\,\delta - HW_0 + HW_1 - 2HW_2 + HW_3.
\end{aligned}$$

Proof. Let

$$h(x) = x^4 - x^3 + x^2 - 2x + 1.$$

Then for some α, β, γ and δ we write

$$h(x) = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x).$$

i.e.,

$$x^4 - x^3 + x^2 - 2x + 1 = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x). \tag{3.2}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ and $\frac{1}{\delta}$ are the roots of $h(x)$. This gives α, β, γ and δ as the roots of

$$h(\frac{1}{x}) = \frac{1}{x^2} - \frac{2}{x} - \frac{1}{x^3} + \frac{1}{x^4} + 1 = 0.$$

This implies $x^4 - x^3 + x^2 - 2x + 1 = 0$. Now, by it follows that

$$\sum_{n=0}^{\infty} HW_n x^n = \frac{\left(HW_1 - HW_0 - 2HW_2 + HW_3\right) x^3 + \left(HW_0 - 2HW_1 + HW_2\right) x^2 + \left(HW_1 - 2HW_0\right) x + HW_0}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}.$$

Then we write

$$\frac{\left(HW_1 - HW_0 - 2HW_2 + HW_3\right) x^3 + \left(HW_0 - 2HW_1 + HW_2\right) x^2 + \left(HW_1 - 2HW_0\right) x + HW_0}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)} \tag{3.3}$$

$$= \frac{B_1}{(1-\alpha x)} + \frac{B_2}{(1-\beta x)} + \frac{B_3}{(1-\gamma x)} + \frac{B_4}{(1-\delta x)}. \tag{3.4}$$

So

$$\begin{aligned} & \left(HW_1 - HW_0 - 2HW_2 + HW_3\right) x^3 + \left(HW_0 - 2HW_1 + HW_2\right) x^2 + \left(HW_1 - 2HW_0\right) x + HW_0 \\ = \; & B_1(1-\beta x)(1-\gamma x)(1-\delta x) + B_2(1-\alpha x)(1-\gamma x)(1-\delta x) \\ & + B_3(1-\alpha x)(1-\beta x)(1-\delta x) + B_3(1-\alpha x)(1-\beta x)(1-\gamma x). \end{aligned}$$

If we consider $x = \frac{1}{\alpha}$, we get $HW_0 + \frac{1}{\alpha^2}\left(HW_0 - 2HW_1 + HW_2\right) - \frac{1}{\alpha^3}\left(HW_0 - HW_1 + 2HW_2 - HW_3\right) + \frac{1}{\alpha}\left(HW_1 - 2HW_0\right) = -B_1\left(\frac{1}{\alpha}\beta - 1\right)\left(\frac{1}{\alpha}\gamma - 1\right)\left(\frac{1}{\alpha}\delta - 1\right).$

This gives

$$\begin{aligned} B_1 &= \alpha^3 (HW_0 + \frac{1}{\alpha^2}\left(HW_0 - 2HW_1 + HW_2\right) + \frac{1}{\alpha^3}\left(HW_1 - 5HW_0 - 4HW_2 + HW_3\right) + \frac{1}{\alpha}\left(HW_1 - 2HW_0\right)) \\ &= \frac{HW_0\alpha^3 + \left(HW_1 - 2HW_0\right)\alpha^2 + \left(HW_0 - 2HW_1 + HW_2\right)\alpha - HW_0 + HW_1 - 2HW_2 + HW_3}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} B_2 &= \frac{HW_0\beta^3 + \left(HW_1 - 2HW_0\right)\beta^2 + \left(HW_0 - 2HW_1 + HW_2\right)\beta - HW_0 + HW_1 - 2HW_2 + HW_3}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\ B_3 &= \frac{HW_0\gamma^3 + \left(HW_1 - 2HW_0\right)\gamma^2 + \left(HW_0 - 2HW_1 + HW_2\right)\gamma - HW_0 + HW_1 - 2HW_2 + HW_3}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\ B_4 &= \frac{HW_0\delta^3 + \left(HW_1 - 2HW_0\right)\delta^2 + \left(HW_0 - 2HW_1 + HW_2\right)\delta - HW_0 + HW_1 - 2HW_2 + HW_3}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}. \end{aligned}$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty} HW_n x^n = B_1(1-\alpha x)^{-1} + B_2(1-\beta x)^{-1} + B_3(1-\gamma x)^{-1} + B_4(1-\delta x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} HW_n x^n = B_1 \sum_{n=0}^{\infty} \alpha^n x^n + B_2 \sum_{n=0}^{\infty} \beta^n x^n + B_3 \sum_{n=0}^{\infty} \gamma^n x^n + B_4 \sum_{n=0}^{\infty} \delta^n x^n = \sum_{n=0}^{\infty} (B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n) x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$HW_n = B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n.$$

The following identity reveals a connection between the hyperbolic Pandita numbers and the Pandita–Lucas numbers.

Corollary 9. *For all integers m, n the following identities holds:*

$$HW_{m+n} = P_{m-2}HW_{n+3} + (P_{m-4} - P_{m-3} - P_{m-5})HW_{n+2} + (P_{m-3} - P_{m-4})HW_{n+1} - HW_n P_{m-3}.$$

Proof. First we assume that $m, n \geq 0$.The Theorem (9) can be proved by mathematical induction on m. If $m = 0$ we get

$$HW_n = P_{-2}HW_{n+3} + (P_{-4} - P_{-3} - P_{-5})HW_{n+2} + (P_{-3} - P_{-4})HW_{n+1} - HW_n P_{-3}.$$

which is true since $P_{-2} = 0, P = -1, P_{-4} = -1, P_{-5} = 0$. Assume that the equality holds for $m \leq k$. For $m = k+1$, we get

$$\begin{aligned} HW_{k+1+n} &= 2HW_{n+k} - HW_{n+k-1} + HW_{n+k-2} - HW_{n+k-3}, \\ & 2(P_{m-2}HW_{n+3} + (P_{m-4} - P_{m-3} - P_{m-5})HW_{n+2} + (P_{m-3} - P_{m-4})HW_{n+1} - HW_n P_{m-3}) \\ & -(P_{m-3}HW_{n+3} + (P_{m-5} - P_{m-4} - P_{m-6})HW_{n+2} + (P_{m-4} - P_{m-5})HW_{n+1} - HW_n P_{m-4}) \\ & +(P_{m-4}HW_{n+3} + (P_{m-6} - P_{m-5} - P_{m-7})HW_{n+2} + (P_{m-5} - P_{m-6})HW_{n+1} - HW_n P_{m-5}) \\ & -(P_{m-5}HW_{n+3} + (P_{m-7} - P_{m-6} - P_{m-8})HW_{n+2} + (P_{m-6} - P_{m-7})HW_{n+1} - HW_n P_{m-6}). \end{aligned}$$

Consequently, by mathematical induction on m, this proves Theorem 9.

The other cases of m, n can be proved smilarly for all integers m, n. □

Taking $HW_n = HP_n$ or $HW_n = HS_n$ in above Theorem, respectively, we get:

Corollary 10.

$$\begin{aligned} HP_{m+n} &= P_{m-2}HP_{n+3} + (P_{m-4} - P_{m-3} - P_{m-5})HP_{n+2} + (P_{m-3} - P_{m-4})HP_{n+1} - HP_n P_{m-3}, \\ HS_{m+n} &= P_{m-2}HS_{n+3} + (P_{m-4} - P_{m-3} - P_{m-5})HS_{n+2} + (P_{m-3} - P_{m-4})HS_{n+1} - HS_n P_{m-3}. \end{aligned}$$

4. SIMSON'S FORMULA

In this section, we present Simpson's formula for the hyperbolic generalized Pandita numbers . This is a special case of [47, Theorem 4.1].

Theorem 11. *(Simpson's formula for hyperbolic generalized Pandita numbers) For all integers n we have,*

$$\begin{vmatrix} HW_{n+3} & HW_{n+2} & HW_{n+1} & HW_{n} \\ HW_{n+2} & HW_{n+1} & HW_{n} & HW_{n-1} \\ HW_{n+1} & HW_{n} & HW_{n-1} & HW_{n-2} \\ HW_{n} & HW_{n-1} & HW_{n-2} & HW_{n-3} \end{vmatrix} = \begin{vmatrix} HW_{3} & HW_{2} & HW_{1} & HW_{0} \\ HW_{2} & HW_{1} & HW_{0} & HW_{-1} \\ HW_{1} & HW_{0} & HW_{-1} & HW_{-2} \\ HW_{0} & HW_{-1} & HW_{-2} & HW_{-3} \end{vmatrix}$$

$= (HW_0+HW_2-HW_3)(-HW_3^3+3HW_2^3-HW_1^3+HW_0^3+(5HW_2-2HW_1)HW_3^2+(4HW_0-5HW_1-8HW_3)HW_2^2+(4HW_0+4HW_2-5HW_3)HW_1^2$

$+(HW_2-3HW_1-HW_3)HW_0^2+9HW_1HW_2HW_3-3HW_0HW_2HW_3+5HW_0HW_1HW_3-7HW_0HW_1HW_2)$

.

Proof. Using Theorem 3 it can be proved by using induction use [47, Theorem 4.1]

From the Theorem 11 we get the following Corollary.

Corollary 12. *For all integers n, the Simson's formulas of hyperbolic Pandita numbers and hyperbolic Pandita Lucas numbers are given as,*

a):

$$\begin{aligned} n &= 0 \\ \begin{vmatrix} HP_{n+3} & HP_{n+2} & HP_{n+1} & HP_{n} \\ HP_{n+2} & HP_{n+1} & HP_{n} & HP_{n-1} \\ HP_{n+1} & HP_{n} & HP_{n-1} & HP_{n-2} \\ HP_{n} & HP_{n-1} & HP_{n-2} & HP_{n-3} \end{vmatrix} &= \begin{vmatrix} HP_{3} & HP_{2} & HP_{1} & HP_{0} \\ HP_{2} & HP_{1} & HP_{0} & HP_{-1} \\ HP_{1} & HP_{0} & HP_{-1} & HP_{-2} \\ HP_{0} & HP_{-1} & HP_{-2} & HP_{-3} \end{vmatrix} \\ &= \begin{vmatrix} 5j+3 & 3j+2 & 2j+1 & j \\ 3j+2 & 2j+1 & j & 0 \\ 2j+1 & j & 0 & 0 \\ j & 0 & 0 & -1 \end{vmatrix} = j^4+j^3+j^2+2j+1 \\ &= 1+j+1+2j+1 = 3+3j, \end{aligned}$$

b):

$$\begin{aligned}
n &= 0 \\
\begin{vmatrix} HS_{n+3} & HS_{n+2} & HS_{n+1} & HS_n \\ HS_{n+2} & HS_{n+1} & HS_n & HS_{n-1} \\ HS_{n+1} & HS_n & HS_{n-1} & HS_{n-2} \\ HS_n & HS_{n-1} & HS_{n-2} & HS_{n-3} \end{vmatrix} &= \begin{vmatrix} HS_3 & HS_2 & HS_1 & HS_0 \\ HS_2 & HS_1 & HS_0 & HS_{-1} \\ HS_1 & HS_0 & HS_{-1} & HS_{-2} \\ HS_0 & HS_{-1} & HS_{-2} & HS_{-3} \end{vmatrix} \\
&= \begin{vmatrix} 6j+5 & 5j+2 & 2j+2 & 2j+4 \\ 5j+2 & 2j+2 & 2j+4 & 4j+1 \\ 2j+2 & 2j+4 & 4j+1 & j-1 \\ 2j+4 & 4j+1 & j-1 & -j+4 \end{vmatrix} \\
&= -31j^4 - 31j^3 - 31j^2 - 62j - 31 \\
&= -93j - 93.
\end{aligned}$$

respectively.

5. Linear Sums

In this section, we give the summation formulas of the hyperbolic generalized Pandita numbers with positive and negatif subscripts.

Now, we present the summation formulas of the generalized Pandita numbers.

Theorem 13. *For the generalized Pandita numbers, we have the following formulas:*

(a): $\sum\limits_{k=0}^{n} W_k = -(n+3)W_{n+3} + (n+4)W_{n+2} + (n+4)W_n + 3W_3 - 4W_2 - 3W_0.$

(b): $\sum\limits_{k=0}^{n} W_{2k} = \frac{1}{3}(-3(n+2)W_{2n+2} + (3n+8)W_{2n+1} + 2W_{2n} + (3n+7)W_{2n-1} + 7W_3 - 8W_2 - W_1 - 6W_0).$

(c): $\sum\limits_{k=0}^{n} W_{2k+1} = \frac{1}{3}(-(3n+4)W_{2n+2} + (3n+8)W_{2n+1} + W_{2n} + 3(n+2)W_{2n-1} + 6W_3 - 8W_2 + W_1 - 7W_0).$

Proof. For the proof, see Soykan [63, Theorem 3.12]. □

Theorem 14. *For the hyperbolic Pandita numbers, we have the following formulas:*

(a): $\sum\limits_{k=0}^{n} HW_k = -(n+3)HW_{n+3} + (n+4)HW_{n+2} + (n+4)HW_n + 3HW_3 - 4HW_2 - 3HW_0.$

(b): $\sum\limits_{k=0}^{n} HW_{2k} = \frac{1}{3}(-3(n+2)HW_{2n+2} + (3n+8)HW_{2n+1} + 2HW_{2n} + (3n+7)HW_{2n-1} + 7HW_3 - 8HW_2 - HW_1 - 6HW_0).$

(c): $\sum\limits_{k=0}^{n} HW_{2k+1} = \frac{1}{3}(-(3n+4)HW_{2n+2} + (3n+8)HW_{2n+1} + HW_{2n} + 3(n+2)HW_{2n-1} + 6HW_3 - 8HW_2 + HW_1 - 7HW_0).$

Proof. Use Theorem 13 and the definition of HW_n. □

As a special case of the theorem 14, we present the following Corollary.

Corollary 15. *For $n \geq 0$, hyperbolic Pandita numbers have the following properties:*

(a): $\sum\limits_{k=0}^{n} HP_k = -(n+3)HP_{n+3} + (n+4)HP_{n+2} + (n+4)HP_n + 1.$

(b): $\sum\limits_{k=0}^{n} HP_{2k} = \frac{1}{3}(-3(n+2)HP_{2n+2} + (3n+8)HP_{2n+1} + 2HP_{2n} + (3n+7)HP_{2n-1} + 3j + 4).$

(c): $\sum\limits_{k=0}^{n} HP_{2k+1} = \frac{1}{3}(-(3n+4)HP_{2n+2} + (3n+8)HP_{2n+1} + HP_{2n} + 3(n+2)HP_{2n-1} + j + 3).$

Corollary 16. *For $n \geq 0$, hyperbolic Pandita Lucas numbers have the following properties.*

(a): $\sum\limits_{k=0}^{n} HS_k = -(n+3)HS_{n+3} + (n+4)HS_{n+2} + (n+4)HS_n - 8j - 5.$

(b): $\sum\limits_{k=0}^{n} HS_{2k} = \frac{1}{3}(-3(n+2)HS_{2n+2} + (3n+8)HS_{2n+1} + 2HS_{2n} + (3n+7)HS_{2n-1} - 12j - 7).$

(c): $\sum\limits_{k=0}^{n} HS_{2k+1} = \frac{1}{3}(-(3n+4)HS_{2n+2} + (3n+8)HS_{2n+1} + HS_{2n} + 3(n+2)HS_{2n-1} - 16j - 12).$

Next, we present the ordinary generating functions corresponding to selected special cases of hyperbolic generalized Pandita numbers.

Theorem 17. *The ordinary generating functions of the sequences HW_{2n}, HW_{2n+1} are given as follows:*

(a): $\sum_{n=0}^{\infty} HW_{2n}x^n = \dfrac{HW_2\left(x^3+3x^2-x\right) + HW_0\left(2x^2+2x-1\right) - HW_1\left(x^2-x^3\right) - HW_3\left(x^3+2x^2\right)}{-x^4-x^3+x^2+2x-1}.$

(b): $\sum_{n=0}^{\infty} HW_{2n+1}x^n = \dfrac{HW_0\left(x^3+2x^2\right) - HW_3\left(x^3+x^2+x\right) - HW_1\left(x^3-2x+1\right) + HW_2\left(2x^3+x^2\right)}{-x^4-x^3+x^2+2x-1}.$

Proof. The argument can be constructed analogously to that in [4, Theorem 4].

From the last Theorem, we have the following Corollary which gives sum formula of hyperbolic Pandita numbers (Take $HW_n = HP_n$ whit $HP_0 = j$,$HP_1 = 2j+1, HP_2 = 3j+2, HP_3 = 5j+3$)

Corollary 18. *For $n \geq 0$ hyperbolic Pandita numbers have the following properties.*

(a): $\sum_{n=0}^{\infty} HW_{2n}x^n = \dfrac{j+x}{1-2x+x^2-x^3+x^4},$

(b): $\sum_{n=0}^{\infty} HW_{2n+1}x^n = \dfrac{(2j+4)+(-2j-6)x+(3j+2)x^2+(-4j+7)x^3}{1-2x+x^2-x^3+x^4}.$

6. Matrices related with Hyperbolic Generalized Pandita Numbers

In this section, we construct several matrices associated with hyperbolic Pandita numbers.

We define the square matrix A of order 4 as

$$A = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

uch that $detA = 1$. Note that

$$A^n = \begin{pmatrix} P_{n+1} & -P_n + P_{n-1} - P_{n-2} & P_n - P_{n-1} & -P_n \\ P_n & -P_{n-1} + P_{n-2} - P_{n-3} & P_{n-1} - P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-2} + P_{n-3} - P_{n-4} & P_{n-2} - P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-3} + P_{n-4} - P_{n-5} & P_{n-3} - P_{n-4} & -P_{n-3} \end{pmatrix}$$

for the proof see [68].

Next, we state the following lemma.

Lemma 19. *For $n \geq 0$ the following identitiy is true:*

$$\begin{pmatrix} HW_{n+3} \\ HW_{n+2} \\ HW_{n+1} \\ HW_n \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}.$$

Proof. The identitiy (19) can be proved by mathematical induction on n. If $n = 0$ we obtain

$$\begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for $n = k$. Thus the following identitiy is true

$$\begin{pmatrix} HW_{k+3} \\ HW_{k+2} \\ HW_{k+1} \\ HW_k \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}.$$

For $n = k + 1$, we get

$$\begin{aligned}
\begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix} &= \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} HW_{k+3} \\ HW_{k+2} \\ HW_{k+1} \\ HW_k \end{pmatrix} \\
&= \begin{pmatrix} HW_{k+4} \\ HW_{k+3} \\ HW_{k+2} \\ HW_{k+1} \end{pmatrix}.
\end{aligned}$$

Consequently, by mathematical induction on n, the proof completed. □

We define

$$N_{HW} = \begin{pmatrix} HW_3 & HW_2 & HW_1 & HW_0 \\ HW_2 & HW_1 & HW_0 & HW_{-1} \\ HW_1 & HW_0 & HW_{-1} & HW_{-2} \\ HW_0 & HW_{-1} & HW_{-2} & HW_{-3} \end{pmatrix}, \tag{6.1}$$

$$E_{HW} = \begin{pmatrix} HW_{n+3} & HW_{n+2} & HW_{n+1} & HW_n \\ HW_{n+2} & HW_{n+1} & HW_n & HW_{n-1} \\ HW_{n+1} & HW_n & HW_{n-1} & HW_{n-2} \\ HW_n & HW_{n-1} & HW_{n-2} & HW_{n-3} \end{pmatrix}. \tag{6.2}$$

Now, we have the following theorem with N_{HW} and E_{HW}

THEOREM 20. *Using N_{HW} and E_{HW}, we get*

$$A^n N_{HW} = E_{HW}.$$

Proof. Note that we get

$$\begin{aligned}
A^n N_{HW} &= \begin{pmatrix} P_{n+1} & -P_n + P_{n-1} - P_{n-2} & P_n - P_{n-1} & -P_n \\ P_n & -P_{n-1} + P_{n-2} - P_{n-3} & P_{n-1} - P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-2} + P_{n-3} - P_{n-4} & P_{n-2} - P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-3} + P_{n-4} - P_{n-5} & P_{n-3} - P_{n-4} & -P_{n-3} \end{pmatrix} \begin{pmatrix} HW_3 & HW_2 & HW_1 & HW_0 \\ HW_2 & HW_1 & HW_0 & HW_{-1} \\ HW_1 & HW_0 & HW_{-1} & HW_{-2} \\ HW_0 & HW_{-1} & HW_{-2} & HW_{-3} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
a_{11} &= HW_1 \left(P_n - P_{n-1}\right) - HW_2 \left(P_n - P_{n-1} + P_{n-2}\right) - HW_0 P_n + W_3 P_{n+1} = HW_{n+3}, \\
a_{12} &= HW_0 \left(P_n - P_{n-1}\right) - HW_1 \left(P_n - P_{n-1} + P_{n-2}\right) - P_n HW_{-1} + HW_2 P_{n+1} = HW_{n+2}, \\
a_{13} &= HW_{-1} \left(P_n - P_{n-1}\right) - HW_0 \left(P_n - P_{n-1} + P_{n-2}\right) - P_n HW_{-2} + HW_1 P_{n+1} = HW_{n+1}, \\
a_{14} &= HW_{-2} \left(P_n - P_{n-1}\right) - HW_{-1} \left(P_n - P_{n-1} + P_{n-2}\right) - P_n HW_{-3} + HW_0 P_{n+1} = HW_n, \\
a_{21} &= HW_3 P_n - HW_2 \left(P_{n-1} - P_{n-2} + P_{n-3}\right) + HW \left(P_{n-1} - P_{n-2}\right) - HW_0 P_{n-1} = HW_{n+2}, \\
a_{22} &= HW_2 P_n - HW_{-1} P_{n-1} - HW_1 \left(P_{n-1} - P_{n-2} + P_{n-3}\right) + HW \left(P_{n-1} - P_{n-2}\right) = HW_{n+1}, \\
a_{23} &= HW_{-1} \left(P_{n-1} - P_{n-2}\right) - HW_{-2} P_{n-1} + HW_1 P_n - HW_0 \left(P_{n-1} - P_{n-2} + P_{n-3}\right) = HW_n, \\
a_{24} &= HW_{-2} \left(P_{n-1} - P_{n-2}\right) - HW_{-3} P_{n-1} + HW_0 P_n - HW_{-1} \left(P_{n-1} - P_{n-2} + P_{n-3}\right) = HW_{n-1}, \\
a_{31} &= HW_1 \left(P_{n-2} - P_{n-3}\right) - HW_2 \left(P_{n-2} - P_{n-3} + P_{n-4}\right) - HW_0 P_{n-2} + HW_3 P_{n-1} = HW_{n+1}, \\
a_{32} &= HW_0 \left(P_{n-2} - P_{n-3}\right) - HW_1 \left(P_{n-2} - P_{n-3} + P_{n-4}\right) - HW_{-1} P_{n-2} + HW_2 P_{n-1} = HW_n, \\
a_{33} &= HW_{-1} \left(P_{n-2} - P_{n-3}\right) - HW_{-2} P_{n-2} - HW_0 \left(P_{n-2} - P_{n-3} + P_{n-4}\right) + HW_1 P_{n-1} = HW_{n-1}, \\
a_{34} &= HW_{-2} \left(P_{n-2} - P_{n-3}\right) - HW_{-3} P_{n-2} - HW_{-1} \left(P_{n-2} - P_{n-3} + P_{n-4}\right) + HW_0 P_{n-1} = HW_{n-2}, \\
a_{41} &= HW_1 \left(P_{n-3} - P_{n-4}\right) - HW_2 \left(P_{n-3} - P_{n-4} + P_{n-5}\right) - HW_0 P_{n-3} + HW_3 P_{n-2} = HW_n, \\
a_{42} &= HW_0 \left(P_{n-3} - P_{n-4}\right) - HW_1 \left(P_{n-3} - P_{n-4} + P_{n-5}\right) - HW_{-1} P_{n-3} + HW_2 P_{n-2} = HW_{n-1}, \\
a_{43} &= HW_{-1} \left(P_{n-3} - P_{n-4}\right) - HW_{-2} P_{n-3} - HW_0 \left(P_{n-3} - P_{n-4} + P_{n-5}\right) + HW_1 P_{n-2} = HW_{n-2}, \\
a_{44} &= HW_{-2} \left(P_{n-3} - P_{n-4}\right) - HW_{-3} P_{n-3} - HW_{-1} \left(P_{n-3} - P_{n-4} + P_{n-5}\right) + HW_0 P_{n-2} = HW_{n-3}.
\end{aligned}$$

Using the theorem (9) the proof is done. □

By taking $HW_n = HP_n$ with HP_0, HP_1, HP_2, HP_3 in (6.1) and (6.2)

$HW_n = S_n$ with HS_0, HS_1, HS_2, HS_3 in (6.1) and (6.2) respectively, we get:

$$N_{HP} = \begin{pmatrix} 5j+3 & 3j+2 & 2j+1 & j \\ 3j+2 & 2j+1 & j & 0 \\ 2j+1 & j & 0 & 0 \\ j & 0 & 0 & -1 \end{pmatrix},$$

$$E_{HP} = \begin{pmatrix} HP_{n+3} & HP_{n+2} & HP_{n+1} & HP_{n} \\ HP_{n+2} & HP_{n+1} & HP_{n} & HP_{n-1} \\ HP_{n+1} & HP_{n} & HP_{n-1} & HP_{n-2} \\ HP_{n} & HP_{n-1} & HP_{n-2} & HP_{n-3} \end{pmatrix},$$

$$N_{HS} = \begin{pmatrix} 6j+5 & 5j+2 & 2j+2 & 2j+4 \\ 5j+2 & 2j+2 & 2j+4 & 4j+1 \\ 2j+2 & 2j+4 & -4j+1 & j-1 \\ 2j+4 & -4j+1 & j-1 & -j+4 \end{pmatrix},$$

$$E_{HS} = \begin{pmatrix} HS_{n+3} & HS_{n+2} & HS_{n+1} & HS_{n} \\ HS_{n+2} & HS_{n+1} & HS_{n} & HS_{n-1} \\ HS_{n+1} & S_{n} & HS_{n-1} & HS_{n-2} \\ HS_{n} & HS_{n-1} & HS_{n-2} & HS_{n-3} \end{pmatrix}.$$

From Theorem [20], we can write the following corollary.

Corollary 21. *The following identities are hold:*

- **a):** $A^{n}N_{HP} = E_{HP}$.
- **b):** $A^{n}N_{HS} = E_{HS}$.

7. Conclusions

Recurrence relations define sequences where each term depends on previous ones. These sequences such as Fibonacci, Pell, Jacobsthal, Tribonacci, Padovan, Narayana's Cows, Leonardo, Tetranacci, and Pentanacci arise across fields including engineering, biology, mathematics, and physics. Below, we present their definitions with initial conditions using A_n notation and outline their real-world relevance.

- **Fibonacci Sequence:**

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, \quad F_1 = 1$$

- **Pell Sequence:**

$$P_n = 2P_{n-1} + P_{n-2}, \quad P_0 = 0, \quad P_1 = 1$$

- **Jacobsthal Sequence:**

$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 0, \quad A_1 = 1$$

- **Tribonacci Sequence:**

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \quad T_0 = 0, \quad T_1 = 1, \quad T_2 = 1$$

- **Padovan Sequence:**

$$P_n = P_{n-2} + P_{n-3}, \quad P_0 = P_1 = P_2 = 1$$

- **Narayana's Cows Sequence:**

$$N_n = N_{n-1} + N_{n-3}, \quad N_0 = N_1 = N_2 = 1$$

- **Leonardo Sequence:**

$$L_n = L_{n-1} + L_{n-2} + 1, \quad L_0 = 1, \quad L_1 = 1$$

- **Tetranacci Sequence:**

$$M_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4}, \quad M_0 = M_1 = M_2 = 0, \quad M_3 = 1$$

- **Pentanacci Sequence:**

$$P_n = P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5}, \quad P_0 = P_1 = P_2 = P_3 = 0, \quad P_4 = 1$$

These sequences demonstrate how mathematical recursions extend into the fabric of our world whether designing structures, analyzing algorithms, modeling nature, or probing the quantum realm. Their recursive beauty continues to inspire both theoretical and practical exploration.

Next, we explore several real-world applications of recurrence relations across disciplines.

- Engineering
  - **Fibonacci:** Models recursive filters in control systems and signal processing.
  - **Padovan and Perrin:** Guide architectural proportions using the plastic number.
  - **Jacobsthal:** Applied in digital circuits for counting and encoding.
- Science
  - **Tribonacci and Tetranacci:** Simulate biological systems with delayed reproduction.
  - **Leonardo:** Reflect branching in plants and trees.
  - **Fibonacci and Narayana's Cows:** Describe phyllotaxis and seed arrangement in botany.
- Mathematics
  - **Recurrence Relations:** Analyze algorithms like mergesort and quicksort.
  - **Pell:** Solve Diophantine equations and approximate square roots with continued fractions.
  - **Jacobsthal and Padovan:** Used in tiling and combinatorics problems.
- Physics

  - **Fibonacci and Tribonacci:** Appear in wave interference and quantum systems.
  - **Pentanacci:** Used in recursive models of particle interactions and fractals.
  - **Padovan:** Linked to equilibrium modeling via the plastic constant.

In this study, we extend the classical framework to fourth-order recurrence systems by introducing the hyperbolic Pandita numbers, along with two distinguished subclasses. For these novel sequences, we derive Binet-type formulas, ordinary and exponential generating functions, and generalized Simson-type identities. Our analysis also encompasses closed-form summation formulas, algebraic properties, recurrence behaviors, and matrix-based representations.

Recognizing the theoretical depth and real-world utility of recurrence-based sequences, we first revisit the applications of second-order sequences to establish context. We then position our fourth-order generalizations as a natural progression within this broader mathematical landscape—offering new insights and powerful tools for modeling, analysis, and optimization in both pure and applied settings.

- For a detailed discussion on Gaussian Fibonacci and Gaussian Lucas numbers applied to Pauli Fibonacci and Pauli Lucas quaternions, see [1].
- For an in-depth analysis of Pell numbers in solving three-dimensional difference equation systems, refer to [5].
- For a study on Jacobsthal numbers and their role in special matrices, see [73].
- For a comprehensive analysis of generalized k-order Fibonacci numbers in hybrid quaternions, see [23].
- For the application of Fibonacci and Lucas numbers to Split Complex Bi-Periodic numbers, see [76].
- For the use of generalized bivariate Fibonacci and Lucas polynomials in matrix polynomials, see [75].
- For the application of generalized Fibonacci numbers to binomial sums, see [72].
- For the role of generalized Jacobsthal numbers in hyperbolic numbers, see [67].
- For the application of generalized Fibonacci numbers to dual hyperbolic numbers, see [66].
- For the use of Laplace transform and matrix operations in the characteristic polynomial of Fibonacci numbers, see [11].
- For the application of generalized Fibonacci matrices in cryptographic systems, see [44].
- For higher-order Jacobsthal numbers applied to quaternion structures, see [42].
- For Fibonacci and Lucas identities in Toeplitz-Hessenberg matrices, see [20].
- For Fibonacci numbers in lacunary statistical convergence, see [4].
- For lacunary statistical convergence in intuitionistic fuzzy normed linear spaces, see [37].
- For ideal convergence in intuitionistic fuzzy normed linear spaces, see [38].
- For the applications of k-Fibonacci and $k-$Lucas numbers to spinors, see [36].

- For dual-generalized complex Fibonacci and Lucas numbers in quaternion structures, see [35].
- For special cases of Horadam numbers in Neutrosophic analysis, see [21].
- For hyperbolic Fibonacci numbers applied to quaternions, see [9].
- For Pell numbers in Gaussian hyperbolic number systems, see [19].

In the following, we explore several applications of third-order recurrence sequences across various mathematical and applied contexts.

- For the applications of third order Jacobsthal numbers and Tribonacci numbers to quaternions, see [7] and [6], respectively.
- For the application of Tribonacci numbers to special matrices, see [77].
- For the applications of Padovan numbers and Tribonacci numbers to coding theory, see [49] and [2], respectively.
- For the application of Pell-Padovan numbers to groups, see [10].
- For the application of adjusted Jacobsthal-Padovan numbers to the exact solutions of some difference equations, see [18].
- For the application of Gaussian Tribonacci numbers to various graphs, see [64].
- For the application of third-order Jacobsthal numbers to hyperbolic numbers, see [13].For the application of Narayan numbers to finite groups see [34].
- For the application of generalized third-order Jacobsthal sequence to binomial transform, see [57].
- For the application of generalized Generalized Padovan numbers to Binomial Transform, see [56].
- For the application of generalized Tribonacci numbers to Gaussian numbers, see [55].
- For the application of generalized Tribonacci numbers to Sedenions, see [54].
- For the application of Tribonacci and Tribonacci-Lucas numbers to matrices, see [52].
- For the application of generalized Tribonacci numbers to circulant matrix, see [53].
- For the application of Tribonacci and Tribonacci-Lucas numbers to hybrinomials, see [70].
- For the application of hyperbolic Leonardo and hyperbolic Francois numbers to quaternions, see [12].

In the following lists, we outline several applications of fourth-order recurrence sequences across theoretical and applied domains.

- For the application of Tetranacci and Tetranacci-Lucas numbers to quaternions, see [59].
- For the application of generalized Tetranacci numbers to Gaussian numbers, see [60].
- For the application of Tetranacci and Tetranacci-Lucas numbers to matrices, see [61].
- For the application of generalized Tetranacci numbers to binomial transform, see [62].

We now explore several applications of fifth-order sequences.

- For the application of Pentanacci numbers to matrices, see [58].
- For the application of generalized Pentanacci numbers to quaternions, see [51].

- For the application of generalized Pentanacci numbers to binomial transform, see [50].

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