

Value distribution of differential polynomials generated by monomials of two distinct meromorphic functions

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Abstract

We study the value distribution of differential polynomials generated by monomials of two distinct transcendental meromorphic functions to extend a result of I. Lahiri [8] for these kind of differential polynomials.

1 Introduction, Definitions and Notations

First we consider the following definition.

Definition 1.1.[1] Let $f(z)$ be a transcendental meromorphic function and $m(r, f)$, $N(r, f)$, $T(r, f)$ denote respectively the average of the positive logarithm of $|f(z)|$, the counting function of poles of $f(z)$ and the characteristic function of $f(z)$ on the circle $|z| = r$ as it is usually done in Nevanlinna Theory. Then for any $j(= 1, 2, \dots)$ we define by

$$M_j[f] = [f(z)]^{n_{0j}} [f^{(1)}(z)]^{n_{1j}} \dots [f^{(m)}(z)]^{n_{mj}},$$

a monomial in $f(z)$, where $n_{0j}, n_{1j}, \dots, n_{mj}$ are non-negative integers. Further the degree and weight of the monomial $M_j[f]$ are defined by $d(M_j) = \sum_{i=0}^k n_{ij}$ and $\Gamma_{M_j} = \sum_{i=0}^k (i+1)n_{ij}$ respectively.

Next, let $a_j(z)$ be small functions with respect to f i.e., $T(r, a_j) = S(r, f)$ for $j = 1, 2, \dots, n$. Then we define a differential polynomial in $f(z)$ by

$$P[f] = \sum_{k=1}^n a_k(z) M_k[f].$$

Here, we define the degree, the lower degree and the weight of the differential polynomial $P[f]$ respectively by the following

$$\bar{d}(P) = \max\{d(M_k) : 1 \leq k \leq n\},$$

$$\underline{d}(P) = \min\{d(M_k) : 1 \leq k \leq n\}$$

and

$$\Gamma_P = \max\{\Gamma_{M_k} : 1 \leq k \leq n\}.$$

Also, the order of $P[f]$ is the highest order derivative m of f involved in $P[f]$. $P[f]$ is called homogeneous if $\bar{d}(P) = \underline{d}(P)$, otherwise we call it non-homogeneous.

Various works [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14] have already been done by several mathematicians so far in this prominent field of research. In [14], Yi proved the following.

Theorem 1.1. Let $Q_1[f], Q_2[f]$ be differential polynomials in transcendental meromorphic function f where both of them are not identically zero and

$$P[f] = a_n f^n + a_{n-1} f^{n-1} + \dots + a_0 \quad (a_n \neq 0). \quad (1.1)$$

Now if

$$F = P[f]Q_1[f] + Q_2[f] \quad (1.2)$$

then

$$[n - \bar{d}(Q_2)]T(r, f) \leq \bar{N}(r, \frac{1}{P[f]}) + [\Gamma_{Q_2} - \bar{d}(Q_2) + 1]\bar{N}(r, f) + \bar{N}(r, \frac{1}{F}) + S(r, f).$$

Here it can be noted that Yi ignored the influence of $Q_1[f]$ on the value distribution of F . Later, in [8], I. Lahiri improved the Theorem 1.1. further by taking the influence of $Q_1[f]$ into consideration and proved the following.

Theorem 1.2. Let $Q_1[f], Q_2[f]$ be differential polynomials in transcendental meromorphic function f where both of them are not identically zero with k and $\underline{d}(Q_1)$ being the order and lower degree of $Q_1[f]$ respectively and $P[f]$ is given by (1.1) where $a_0, a_1, \dots, a_{n-1}, a_n (\neq 0)$ are small functions of f . Now if F is given by (1.2), then

$$\begin{aligned} [n - \bar{d}(Q_2)]T(r, f) \leq \bar{N}(r, \frac{1}{P[f]}) + [\Gamma_{Q_2} - \bar{d}(Q_2) + 1]\bar{N}(r, f) - d[N(r, \frac{1}{f}) \\ - N_{k+1}(r, \frac{1}{f})] + \bar{N}(r, \frac{1}{F}) + S(r, f), \end{aligned}$$

where

$$d = \begin{cases} \underline{d}(Q_1) & \text{if } n \geq \bar{d}(Q_2) \\ 0 & \text{if } n < \bar{d}(Q_2) \end{cases}$$

and $N_{k+1}(r, \frac{1}{f})$ counts poles of f , where a pole of f with multiplicity μ is counted μ times if $\mu \leq (k+1)$ and $(k+1)$ times if $\mu > (k+1)$.

To prove the Theorem 1.2 Lahiri used the following lemmas.

Lemma 1.1. [6] Let $Q^*[f], Q[f]$ be differential polynomials in f containing arbitrary meromorphic coefficients $q_1^*, q_2^*, \dots, q_n^*$ and q_1, q_2, \dots, q_k respectively. Let $P[f]$ be given by (1.1). If $P[f]Q^*[f] = Q[f]$ with $\gamma_Q \leq n$, then

$$m(r, Q^*[f]) \leq \sum_{i=1}^k m(r, q_i) + \sum_{i=1}^n m(r, q_i^*) + S(r, f).$$

Lemma 1.2. [8] Let $P[f]$ be a differential polynomial in f defined by

$$P[f] = \sum_{j=1}^n a_j(z)M_j[f]$$

with order k and lower degree $\underline{d}(P)$. Let z_0 be a zero of f with multiplicity $\mu (> k)$ and it is not a pole of any of the coefficients a_1, a_2, \dots, a_n . Then $P[f]$ has a zero at z_0 of order at least $(\mu - k)\underline{d}(P)$.

Lemma 1.3. [3] For any differential polynomial $P[f]$ in f ,

$$N(r, P[f]) \leq \bar{d}(P)N(r, f) + [\Gamma_P - \bar{d}(P)]\bar{N}(r, f) + S(r, f).$$

Lemma 1.4. [12] If $P[f]$ is defined as in (1.1) with its coefficients $a_0, a_1, \dots, a_{n-1}, a_n (\neq 0)$ being small functions with respect to f then

$$m(r, P[f]) = nm(r, f) + S(r, f).$$

Lemma 1.5. [6] Let $Q[f]$ be a differential polynomial in f containing n arbitrary meromorphic coefficients q_1, q_2, \dots, q_n . Then

$$m(r, Q[f]) \leq \bar{d}(Q)m(r, f) + \sum_{j=1}^n m(r, q_j) + S(r, f).$$

Lemma 1.6. [13] Let $P[f]$ be defined as in (1.1) with its coefficients $a_0, a_1, \dots, a_{n-1}, a_n (\neq 0)$ and $N(r, f)$ being small functions with respect to f . Then

$$T(r, P[f]) = nT(r, f) + S(r, f).$$

Now it is quite natural to prove Theorem 1.2 for differential polynomials generated by two transcendental meromorphic functions instead of one. And to do that we first introduce the following definition of differential polynomials generated by transcendental meromorphic functions f and g . Then we prove an analogous form of Theorem 1.2 for these newly defined polynomials.

Definition 1.2. Let $f(z), g(z)$ be transcendental meromorphic functions. Then for any $j (= 1, 2, \dots)$, a monomial in $f(z)$ and $g(z)$ is defined by the following expression

$$M_j[f, g] = [f(z)]^{l_{0j}} [f^{(1)}(z)]^{l_{1j}} \dots [f^{(k)}(z)]^{l_{kj}} [g(z)]^{m_{0j}} [g^{(1)}(z)]^{m_{1j}} \dots [g^{(h)}(z)]^{m_{hj}}$$

where $l_{0j}, l_{1j}, \dots, l_{kj}, m_{0j}, m_{1j}, \dots, m_{hj}$ are non-negative integers.

We also define the degree and weight of the monomial $M_j[f, g]$ respectively by

$$d(M_j) = \sum_{i=0}^k l_{ij} + \sum_{i=0}^h m_{ij}$$

and

$$\Gamma_{M_j} = \sum_{i=0}^k (i+1)l_{ij} + \sum_{i=0}^h (i+1)m_{ij}.$$

Again, let $a_j(z)$ be small functions of f and of g . Then we define a differential polynomial in $f(z)$ and $g(z)$ by the following

$$P[f, g] = \sum_{j=1}^n a_j(z) M_j[f, g]. \quad (1.3)$$

Here we define the degree, lower degree and weight of the differential polynomial $P[f, g]$ respectively by

$$\bar{d}(P) = \max\{d(M_j) : 1 \leq j \leq n\},$$

$$\underline{d}(P) = \min\{d(M_j) : 1 \leq j \leq n\}$$

and

$$\Gamma_P = \max\{\Gamma_{M_j} : 1 \leq j \leq n\}.$$

Also, $\max\{k, h\} = \nu$, the highest order derivative of f and g involved in $P[f, g]$ is called the order of $P[f, g]$.

$P[f, g]$ is called homogeneous if $\bar{d}(P) = \underline{d}(P)$. Otherwise we call it non-homogeneous.

Moreover, let

$$\bar{d}_f(P) = \max\left\{\sum_{i=0}^k l_{ij} : 1 \leq j \leq n\right\};$$

$$\underline{d}_f(P) = \min\left\{\sum_{i=0}^k l_{ij} : 1 \leq j \leq n\right\};$$

$$\bar{d}_g(P) = \max\left\{\sum_{i=0}^h m_{ij} : 1 \leq j \leq n\right\}$$

and

$$\underline{d}_g(P) = \min\left\{\sum_{i=0}^h m_{ij} : 1 \leq j \leq n\right\}.$$

Now if there exists atleast one $j(= j_1, \text{ say})$ such that

$$\bar{d}_f(P) = \sum_{i=0}^k l_{ij_1} \quad \text{and} \quad \bar{d}_g(P) = \sum_{i=0}^h m_{ij_1} \quad (1.4)$$

then

$$\bar{d}(P) = \bar{d}_f(P) + \bar{d}_g(P).$$

Similarly if for one $j(= j_2, \text{ say})$

$$\underline{d}_f(P) = \sum_{i=0}^k l_{ij_2} \quad \text{and} \quad \underline{d}_g(P) = \sum_{i=0}^h m_{ij_2}, \quad (1.5)$$

then

$$\underline{d}(P) = \underline{d}_f(P) + \underline{d}_g(P).$$

To clarify the conditions mentioned in (1.4) and (1.5) we refer [1].

We do not explain the standard notations and definitions used in the paper because they are available in [7].

2 Main Results

Now we present our main result of the paper. And to do that we first need some results in the form of lemmas.

Lemma 2.1. Let $Q^*[f, g]$, $Q[f, g]$ be differential polynomials in f and g with arbitrary meromorphic coefficients $q_1^*, q_2^*, \dots, q_n^*$ and q_1, q_2, \dots, q_k respectively. Suppose that

$$P[f, g] = f^l g^m + a_{l-1} f^{l-1} g^m + b_{m-1} f^l g^{m-1} + a_{l-2} f^{l-2} g^m + b_{m-2} f^l g^{m-2} + \dots + a_0 g^m + b_0 f^l. \quad (2.1)$$

If

$$P[f, g]Q^*[f, g] = Q[f, g], \quad (2.2)$$

where $\bar{d}_f(Q) \leq l$, $\bar{d}_g(Q) \leq m$ (i.e., $\bar{d}(Q) \leq l + m$) and on the circle $|z| = r$, $|f(re^{i\theta})| = |g(re^{i\theta})|$, then

$$m(r, Q^*[f, g]) \leq \sum_{i=1}^n m(r, q_i^*) + \sum_{i=1}^k m(r, q_i) + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g).$$

Proof : First we consider the case when

$$a_{l-1} = a_{l-2} = \dots = a_0 = 0 = b_{m-1} = b_{m-2} = \dots = b_0.$$

In this case we can rewrite (2.2) as

$$f^l g^m Q^*[f, g] = Q[f, g]. \quad (2.3)$$

Now we suppose

$$Q[f, g] = \sum_{i=1}^k q_i M_i[f, g] \quad (2.4)$$

where

$$M_i[f, g] = f^{\alpha_{i0}} (f^{(1)})^{\alpha_{i1}} \dots (f^{(j)})^{\alpha_{ij}} g^{\beta_{i0}} (g^{(1)})^{\beta_{i1}} \dots (g^{(h)})^{\beta_{ih}} \quad \text{for } i = 1, 2, \dots, k;$$

with $\alpha_{i0}, \alpha_{i1}, \dots, \alpha_{ij}, \beta_{i0}, \beta_{i1}, \dots, \beta_{ih}, j, h$ being non-negative integers, and

$$Q^*[f, g] = \sum_{i=1}^n q_i^* M_i^*[f, g] \quad (2.5)$$

where

$$M_i^*[f, g] = f^{s_{i0}} (f^{(1)})^{s_{i1}} \dots (f^{(u)})^{s_{iu}} g^{t_{i0}} (g^{(1)})^{t_{i1}} \dots (g^{(v)})^{t_{iv}} \quad \text{for } i = 1, 2, \dots, n$$

with $s_{i0}, s_{i1}, \dots, s_{iu}, t_{i0}, t_{i1}, \dots, t_{iv}, u, v$ being non-negative integers.

First we consider $|f(re^{i\theta})| = |g(re^{i\theta})| > 1$.

Then from (2.3) and (2.4)

$$\begin{aligned}
|Q^*[f, g]| &= |f^{-l}g^{-m}Q[f, g]| \\
&= |f^{-l}g^{-m} \sum_{i=1}^k q_i M_i[f, g]| \\
&= \left| \sum_{i=1}^k q_i \left(\frac{f^{(1)}}{f}\right)^{\alpha_{i1}} \dots \left(\frac{f^{(j)}}{f}\right)^{\alpha_{ij}} \left(\frac{g^{(1)}}{g}\right)^{\beta_{i1}} \dots \left(\frac{g^{(h)}}{g}\right)^{\beta_{ih}} f^{(\sum_{w=0}^j \alpha_{iw}-l)} g^{(\sum_{w=0}^h \beta_{iw}-m)} \right| \\
&\leq \sum_{i=1}^k |q_i| \left| \frac{f^{(1)}}{f} \right|^{\alpha_{i1}} \dots \left| \frac{f^{(j)}}{f} \right|^{\alpha_{ij}} \left| \frac{g^{(1)}}{g} \right|^{\beta_{i1}} \dots \left| \frac{g^{(h)}}{g} \right|^{\beta_{ih}} |f|^{(\sum_{w=0}^j \alpha_{iw}-l)} |g|^{(\sum_{w=0}^h \beta_{iw}-m)} \\
&\leq \sum_{i=1}^k |q_i| \left| \frac{f^{(1)}}{f} \right|^{\alpha_{i1}} \dots \left| \frac{f^{(j)}}{f} \right|^{\alpha_{ij}} \left| \frac{g^{(1)}}{g} \right|^{\beta_{i1}} \dots \left| \frac{g^{(h)}}{g} \right|^{\beta_{ih}} |f|^{(\bar{d}_f(Q)-l)} |g|^{(\bar{d}_g(Q)-m)} \\
&\leq \sum_{i=1}^k |q_i| \left| \frac{f^{(1)}}{f} \right|^{\alpha_{i1}} \dots \left| \frac{f^{(j)}}{f} \right|^{\alpha_{ij}} \left| \frac{g^{(1)}}{g} \right|^{\beta_{i1}} \dots \left| \frac{g^{(h)}}{g} \right|^{\beta_{ih}}, \\
&\quad \text{since } |f(re^{i\theta})| = |g(re^{i\theta})| > 1 \text{ and } \bar{d}_f(Q) - l \leq 0, \quad \bar{d}_g(Q) - m \leq 0.
\end{aligned}$$

Hence by using Milloux's Theorem,

$$\begin{aligned}
&\frac{1}{2\pi} \int_{|f|=|g|>1} \log^+ |Q^*[f, g]| d\theta \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left(\sum_{i=1}^k |q_i| \left| \frac{f^{(1)}}{f} \right|^{\alpha_{i1}} \dots \left| \frac{f^{(j)}}{f} \right|^{\alpha_{ij}} \left| \frac{g^{(1)}}{g} \right|^{\beta_{i1}} \dots \left| \frac{g^{(h)}}{g} \right|^{\beta_{ih}} \right) d\theta \\
&\leq \sum_{i=1}^k \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left(|q_i| \left| \frac{f^{(1)}}{f} \right|^{\alpha_{i1}} \dots \left| \frac{f^{(j)}}{f} \right|^{\alpha_{ij}} \left| \frac{g^{(1)}}{g} \right|^{\beta_{i1}} \dots \left| \frac{g^{(h)}}{g} \right|^{\beta_{ih}} \right) d\theta + O(1) \\
&= \sum_{i=1}^k \frac{1}{2\pi} \int_0^{2\pi} \log^+ |q_i| d\theta + \sum_{i=1}^k \left(\frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{f^{(1)}}{f} \right|^{\alpha_{i1}} d\theta + \dots \right. \\
&\quad \dots + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{f^{(j)}}{f} \right|^{\alpha_{ij}} d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{g^{(1)}}{g} \right|^{\beta_{i1}} d\theta + \dots \\
&\quad \left. \dots + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{g^{(h)}}{g} \right|^{\beta_{ih}} d\theta \right) + O(1) \\
&= \sum_{i=1}^k m(r, q_i) + \sum_{i=1}^k \left[\alpha_{i1} m\left(r, \frac{f^{(1)}}{f}\right) + \dots + \alpha_{ij} m\left(r, \frac{f^{(j)}}{f}\right) \right. \\
&\quad \left. + \beta_{i1} m\left(r, \frac{g^{(1)}}{g}\right) + \dots + \beta_{ih} m\left(r, \frac{g^{(h)}}{g}\right) \right] + O(1) \\
&= \sum_{i=1}^k m(r, q_i) + \sum_{i=1}^k \left[\alpha_{i1} S(r, f) + \dots + \alpha_{ij} S(r, f) \right. \\
&\quad \left. + \beta_{i1} S(r, g) + \dots + \beta_{ih} S(r, g) \right] + O(1), \\
&= \sum_{i=1}^k m(r, q_i) + S(r, f) + S(r, g). \tag{2.6}
\end{aligned}$$

Next we consider $|f(re^{i\theta})| = |g(re^{i\theta})| \leq 1$. From (2.5) we have

$$\begin{aligned}
|Q^*[f, g]| &= \left| \sum_{i=1}^n q_i^* M_i^*[f, g] \right| \\
&= \left| \sum_{i=1}^n q_i^* f^{s_{i0}} (f^{(1)})^{s_{i1}} \dots (f^{(u)})^{s_{iu}} g^{t_{i0}} (g^{(1)})^{t_{i1}} \dots (g^{(v)})^{t_{iv}} \right| \\
&\leq \sum_{i=1}^n |q_i^*| \left| \frac{f^{(1)}}{f} \right|^{s_{i1}} \dots \left| \frac{f^{(u)}}{f} \right|^{s_{iu}} \left| \frac{g^{(1)}}{g} \right|^{t_{i1}} \dots \left| \frac{g^{(v)}}{g} \right|^{t_{iv}}.
\end{aligned}$$

Hence again by using Milloux's Theorem,

$$\begin{aligned}
&\frac{1}{2\pi} \int_{|f|=|g|\leq 1} \log^+ |Q^*[f, g]| d\theta \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left(\sum_{i=1}^n |q_i^*| \left| \frac{f^{(1)}}{f} \right|^{s_{i1}} \dots \left| \frac{f^{(u)}}{f} \right|^{s_{iu}} \left| \frac{g^{(1)}}{g} \right|^{t_{i1}} \dots \left| \frac{g^{(v)}}{g} \right|^{t_{iv}} \right) d\theta \\
&\leq \sum_{i=1}^n \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left(|q_i^*| \left| \frac{f^{(1)}}{f} \right|^{s_{i1}} \dots \left| \frac{f^{(u)}}{f} \right|^{s_{iu}} \left| \frac{g^{(1)}}{g} \right|^{t_{i1}} \dots \left| \frac{g^{(v)}}{g} \right|^{t_{iv}} \right) d\theta + O(1) \\
&= \sum_{i=1}^n \frac{1}{2\pi} \int_0^{2\pi} \log^+ |q_i^*| d\theta + \sum_{i=1}^n \left(\frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{f^{(1)}}{f} \right|^{s_{i1}} d\theta + \dots \right. \\
&\quad \dots + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{f^{(u)}}{f} \right|^{s_{iu}} d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{g^{(1)}}{g} \right|^{t_{i1}} d\theta + \dots \\
&\quad \left. \dots + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{g^{(v)}}{g} \right|^{t_{iv}} d\theta \right) + O(1) \\
&= \sum_{i=1}^n m(r, q_i^*) + \sum_{i=1}^n \left(s_{i1} m\left(r, \frac{f^{(1)}}{f}\right) + \dots + s_{iu} m\left(r, \frac{f^{(u)}}{f}\right) \right. \\
&\quad \left. + t_{i1} m\left(r, \frac{g^{(1)}}{g}\right) + \dots + t_{iv} m\left(r, \frac{g^{(v)}}{g}\right) \right) + O(1) \\
&= \sum_{i=1}^n m(r, q_i^*) + \sum_{i=1}^n \left(s_{i1} S(r, f) + \dots + s_{iu} S(r, f) \right. \\
&\quad \left. + t_{i1} S(r, g) + \dots + t_{iv} S(r, g) \right) + O(1), \\
&= \sum_{i=1}^n m(r, q_i^*) + S(r, f) + S(r, g). \tag{2.7}
\end{aligned}$$

Adding (2.6) and (2.7) we get

$$\begin{aligned}
&\frac{1}{2\pi} \int_{|f|=|g|>1} \log^+ |Q^*[f, g]| d\theta + \frac{1}{2\pi} \int_{|f|=|g|\leq 1} \log^+ |Q^*[f, g]| d\theta \\
&\leq \sum_{i=1}^n m(r, q_i^*) + \sum_{i=1}^k m(r, q_i) + S(r, f) + S(r, g)
\end{aligned}$$

i.e.,

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ |Q^*[f, g]| d\theta \leq \sum_{i=1}^n m(r, q_i^*) + \sum_{i=1}^k m(r, q_i) + S(r, f) + S(r, g)$$

i.e.,

$$m(r, Q^*[f, g]) \leq \sum_{i=1}^n m(r, q_i^*) + \sum_{i=1}^k m(r, q_i) + S(r, f) + S(r, g).$$

Now we consider the general case when $a_{l-1}, a_{l-2}, \dots, a_0; b_{m-1}, b_{m-2}, \dots, b_0$ are any arbitrary meromorphic functions with smaller growth than f and g respectively. Then from (2.2) we can write

$$f^l g^m [(f^l g^m + a_{l-1} f^{l-1} g^m + b_{m-1} f^l g^{m-1} + \dots + a_0 g^m + b_0 f^l) Q^*[f, g]] = f^l g^m Q[f, g]$$

or, $f^l g^m R^*[f, g] = R[f, g]$, say where $R^*[f, g]$ and $R[f, g]$ are differential polynomials in f and g with meromorphic coefficients $q_1^*(1 + a_{l-1} + b_{m-1} + \dots + a_0 + b_0)$, $q_2^*(1 + a_{l-1} + b_{m-1} + \dots + a_0 + b_0)$, \dots , $q_n^*(1 + a_{l-1} + b_{m-1} + \dots + a_0 + b_0)$ and q_1, q_2, \dots, q_k respectively.

Hence by first case

$$\begin{aligned} m(r, R^*[f, g]) &\leq \sum_{i=1}^n m(r, q_i^*(1 + a_{l-1} + b_{m-1} + \dots + a_0 + b_0)) + \sum_{i=1}^k m(r, q_i) \\ &\quad + S(r, f) + S(r, g) \\ &\leq \sum_{i=1}^n m(r, q_i^*) + \sum_{i=1}^k m(r, q_i) + S(r, f) + S(r, g). \end{aligned}$$

So,

$$\begin{aligned} m(r, Q^*[f, g]) &\leq m(r, R^*[f, g]) + m(r, \frac{1}{P[f, g]}) \\ &\leq \sum_{i=1}^n m(r, q_i^*) + \sum_{i=1}^k m(r, q_i) + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g). \end{aligned}$$

Hence the proof is complete.

Lemma 2.2 Let $P[f, g] = \sum_{j=1}^n a_j(z) M_j[f, g]$ be of order and lower degree ν and $\underline{d}(P)$ respectively and let $P[f, g]$ satisfies the condition (1.5). If z_0 is a zero of f and g with multiplicities $\mu_1(> \nu)$ and $\mu_2(> \nu)$ respectively and it is not a pole of any of the coefficients a_1, a_2, \dots, a_n ; then $P[f, g]$ has a zero at z_0 with multiplicity $\geq (\mu - \nu)\underline{d}(P)$ where $\mu = \min\{\mu_1, \mu_2\}$.

Proof : Since f and g have zeros at z_0 with multiplicities $\mu_1(> \nu)$ and $\mu_2(> \nu)$ respectively, from Definition 1.2 we can clearly say that z_0 will be a zero of

$M_j[f, g]$ with multiplicity

$$\begin{aligned}
&= [\mu_1 l_{0j} + (\mu_1 - 1)l_{1j} + \dots + (\mu_1 - k)l_{kj}] + [\mu_2 m_{0j} + (\mu_2 - 1)m_{1j} + \dots \\
&\quad \dots + (\mu_1 - h)m_{hj}] \\
&= \mu_1 \sum_{i=0}^k l_{ij} - \sum_{i=1}^k i l_{ij} + \mu_2 \sum_{i=0}^h m_{ij} - \sum_{i=1}^h i m_{ij} \\
&= (\mu_1 + 1) \sum_{i=0}^k l_{ij} - \sum_{i=0}^k (i+1)l_{ij} + (\mu_2 + 1) \sum_{i=0}^h m_{ij} - \sum_{i=0}^h (i+1)m_{ij} \\
&= (\mu_1 + 1)\bar{d}_f(M_j) + (\mu_2 + 1)\bar{d}_g(M_j) - \Gamma_{M_j} \\
&= \mu_1 \bar{d}_f(M_j) + \mu_2 \bar{d}_g(M_j) + \bar{d}(M_j) - \Gamma_{M_j} \\
&= (\mu_1 - \nu)\bar{d}_f(M_j) + (\mu_2 - \nu)\bar{d}_g(M_j) + (\nu + 1)\bar{d}(M_j) - \Gamma_{M_j} \\
&\geq (\mu_1 - \nu)\bar{d}_f(M_j) + (\mu_2 - \nu)\bar{d}_g(M_j) \\
&\geq (\mu_1 - \nu)\underline{d}_f(P) + (\mu_2 - \nu)\underline{d}_g(P) \\
&\geq (\mu - \nu)[\underline{d}_f(P) + \underline{d}_g(P)] \\
&= (\mu - \nu)\underline{d}(P).
\end{aligned}$$

Since z_0 is not a pole of any of the coefficients a_1, a_2, \dots, a_n , therefore z_0 is a zero of $P[f, g]$ with multiplicity $\geq (\mu - \nu)\underline{d}(P)$ where $\mu = \min\{\mu_1, \mu_2\}$.

Lemma 2.3. For $P[f, g] = \sum_{j=1}^n a_j(z)M_j[f, g]$,

$$\begin{aligned}
N(r, P[f, g]) &\leq n[\bar{d}(P)\{N(r, f) + N(r, g)\} + \{\Gamma_P - \underline{d}(P)\}\{\bar{N}(r, f) + \bar{N}(r, g)\}] \\
&\quad + S(r, f) + S(r, g).
\end{aligned}$$

Proof : Since $N(r, f^{(k)}) \leq N(r, f) + k\bar{N}(r, f)$, we have

$$\begin{aligned}
N(r, P[f, g]) &= N(r, \sum_{j=1}^n a_j(z)M_j[f, g]) \\
&\leq \sum_{j=1}^n N(r, a_j(z)M_j[f, g]) \\
&\leq \sum_{j=1}^n [N(r, a_j(z)) + N(r, M_j[f, g])] \\
&= \sum_{j=1}^n N(r, M_j[f, g]) + S(r, f) + S(r, g) \\
&= \sum_{j=1}^n N(r, [f]^{l_{0j}} [f^{(1)}]^{l_{1j}} \dots [f^{(k)}]^{l_{kj}} [g]^{m_{0j}} [g^{(1)}]^{m_{1j}} \dots [g^{(h)}]^{m_{hj}} \\
&\quad + S(r, f) + S(r, g)
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=1}^n [N(r, [f]^{l_{0j}}) + N(r, [f^{(1)}]^{l_{1j}}) + \dots + N(r, [f^{(k)}]^{l_{kj}}) + N(r, [g]^{m_{0j}}) \\
&\quad + N(r, [g^{(1)}]^{m_{1j}}) + \dots + N(r, [g^{(h)}]^{m_{hj}})] + S(r, f) + S(r, g) \\
&\leq \sum_{j=1}^n [l_{0j}N(r, f) + l_{1j}N(r, f^{(1)}) + \dots + l_{kj}N(r, f^{(k)}) + m_{0j}N(r, g) \\
&\quad + m_{1j}N(r, g^{(1)}) + \dots + m_{hj}N(r, g^{(h)})] + S(r, f) + S(r, g) \\
&\leq \sum_{j=1}^n [l_{0j}N(r, f) + l_{1j}\{N(r, f) + \bar{N}(r, f)\} + \dots + l_{kj}\{N(r, f) + k\bar{N}(r, f)\}] \\
&\quad + m_{0j}N(r, g) + m_{1j}\{N(r, g) + \bar{N}(r, g)\} + \dots + m_{hj}\{N(r, g) + h\bar{N}(r, g)\}] \\
&\quad + S(r, f) + S(r, g) \\
&\leq \sum_{j=1}^n [(l_{0j} + l_{1j} + \dots + l_{kj})N(r, f) + (l_{1j} + 2l_{2j} + \dots + kl_{kj})\bar{N}(r, f) \\
&\quad + (m_{0j} + m_{1j} + \dots + m_{hj})N(r, g) + (m_{1j} + 2m_{2j} + \dots + hm_{hj})\bar{N}(r, g)] \\
&\quad + S(r, f) + S(r, g) \\
&= \sum_{j=1}^n [\bar{d}_f(M_j)N(r, f) + \{(l_{0j} + 2l_{1j} + \dots + (k+1)l_{kj}) - (l_{0j} + l_{1j} + \dots + l_{kj})\}\bar{N}(r, f) \\
&\quad + \bar{d}_g(M_j)N(r, g) + \{(m_{0j} + 2m_{1j} + \dots + (h+1)m_{hj}) - (m_{0j} + m_{1j} \\
&\quad + \dots + m_{hj})\}\bar{N}(r, g)] + S(r, f) + S(r, g) \\
&\leq \sum_{j=1}^n [\bar{d}_f(M_j)N(r, f) + \bar{d}_g(M_j)N(r, g) + \{(l_{0j} + 2l_{1j} + \dots + (k+1)l_{kj} \\
&\quad + m_{0j} + 2m_{1j} + \dots + (h+1)m_{hj}) - (l_{0j} + l_{1j} + \dots + l_{kj} + m_{0j} + m_{1j} \\
&\quad + \dots + m_{hj})\}\{\bar{N}(r, f) + \bar{N}(r, g)\}] + S(r, f) + S(r, g) \\
&\leq \sum_{j=1}^n [\bar{d}(M_j)\{N(r, f) + N(r, g)\} + \{\Gamma_{M_j} - \bar{d}(M_j)\}\{\bar{N}(r, f) + \bar{N}(r, g)\}] \\
&\quad + S(r, f) + S(r, g) \\
&\leq n[\bar{d}(P)\{N(r, f) + N(r, g)\} + \{\Gamma_P - \bar{d}(P)\}\{\bar{N}(r, f) + \bar{N}(r, g)\}] \\
&\quad + S(r, f) + S(r, g).
\end{aligned}$$

Lemma 2.4. Let $P[f, g]$ be given by (2.1) where $a_{l-1}, a_{l-2}, \dots, a_0$ are small functions with respect to f and $b_{m-1}, b_{m-2}, \dots, b_0$ are with respect to g . Then

$$m(r, P[f, g]) \leq 2l \times m(r, f) + 2m \times m(r, g) + S(r, f) + S(r, g).$$

Proof : We have

$$\begin{aligned}
m(r, P[f, g]) &= m(r, f^l g^m + a_{l-1} f^{l-1} g^m + b_{m-1} f^l g^{m-1} + a_{l-2} f^{l-2} g^m \\
&\quad + b_{m-2} f^l g^{m-2} + \dots + a_0 g^m + b_0 f^l) \\
&\leq m(r, f^l g^m + a_{l-1} f^{l-1} g^m + a_{l-2} f^{l-2} g^m + \dots + a_0 g^m) \\
&\quad + m(r, b_{m-1} f^l g^{m-1} + b_{m-2} f^l g^{m-2} + \dots + b_0 f^l) + O(1)
\end{aligned}$$

$$\begin{aligned}
&= m(r, g^m \{f^l + a_{l-1}f^{l-1} + a_{l-2}f^{l-2} + \dots + a_0\}) \\
&\quad + m(r, f^l \{b_{m-1}g^{m-1} + b_{m-2}g^{m-2} + \dots + b_0\}) + O(1) \\
&\leq m(r, g^m) + m(r, f^l + a_{l-1}f^{l-1} + a_{l-2}f^{l-2} + \dots + a_0) \\
&\quad + m(r, f^l) + m(r, b_{m-1}g^{m-1} + b_{m-2}g^{m-2} + \dots + b_0) + O(1) \\
&\leq m \times m(r, g) + m(r, f^l + a_{l-1}f^{l-1} + a_{l-2}f^{l-2} + \dots + a_1f) + m(r, a_0) \\
&\quad + l \times m(r, f) + m(r, b_{m-1}g^{m-1} + b_{m-2}g^{m-2} + \dots + b_1g) + m(r, b_0) + O(1) \\
&\leq m \times m(r, g) + m(r, f \{f^{l-1} + a_{l-1}f^{l-2} + \dots + a_2f + a_1\}) + l \times m(r, f) \\
&\quad + m(r, g \{b_{m-1}g^{m-2} + b_{m-2}g^{m-3} + \dots + b_2g + b_1\}) + S(r, f) + S(r, g) \\
&\leq (l+1) \times m(r, f) + (m+1) \times m(r, g) + m(r, f^{l-1} + a_{l-1}f^{l-2} + \dots \\
&\quad + a_2f + a_1) + m(r, b_{m-1}g^{m-2} + b_{m-2}g^{m-3} + \dots + b_2g + b_1) \\
&\quad + S(r, f) + S(r, g) \\
&\leq (l+1) \times m(r, f) + (m+1) \times m(r, g) + m(r, f \{f^{l-2} + a_{l-1}f^{l-3} + \dots \\
&\quad + a_3f + a_2\}) + m(r, g \{b_{m-1}g^{m-3} + b_{m-2}g^{m-4} + \dots + b_3g + b_2\}) \\
&\quad + S(r, f) + S(r, g) \\
&\leq (l+2) \times m(r, f) + (m+2) \times m(r, g) + m(r, f^{l-2} + a_{l-1}f^{l-3} + \dots \\
&\quad + a_3f + a_2) + m(r, b_{m-1}g^{m-3} + b_{m-2}g^{m-4} + \dots + b_3g + b_2) \\
&\quad + S(r, f) + S(r, g).
\end{aligned}$$

Proceeding similarly, we ultimately get

$$\begin{aligned}
m(r, P[f, g]) &\leq 2l \times m(r, f) + (2m-1) \times m(r, g) + S(r, f) + S(r, g) \\
&\leq 2l \times m(r, f) + 2m \times m(r, g) + S(r, f) + S(r, g).
\end{aligned}$$

Lemma 2.5. If $Q[f, g]$ be a differential polynomial in f and g with n arbitrary meromorphic coefficients q_1, q_2, \dots, q_n , then

$$m(r, Q[f, g]) \leq n\bar{d}(Q)[m(r, f) + m(r, g)] + \sum_{i=1}^n m(r, q_i) + S(r, f) + S(r, g).$$

Proof : Let

$$Q[f, g] = \sum_{j=1}^n q_j M_j[f, g] \tag{2.8}$$

where each

$$M_j[f, g] = [f(z)]^{l_{0j}} [f^{(1)}(z)]^{l_{1j}} \dots [f^{(k)}(z)]^{l_{kj}} [g(z)]^{m_{0j}} [g^{(1)}(z)]^{m_{1j}} \dots [g^{(h)}(z)]^{m_{hj}}$$

is a monomial in f and g of degree

$$d(M_j) = \sum_{i=0}^k l_{ij} + \sum_{i=0}^h m_{ij}$$

Here the degree of the differential polynomial $Q[f, g]$ is given by

$$\bar{d}(Q) = \max\{d(M_j) : 1 \leq j \leq n\}.$$

From (2.8) and using Milloux's Theorem we can have

$$\begin{aligned}
m(r, Q[f, g]) &= m(r, \sum_{j=1}^n q_j M_j[f, g]) \\
&\leq \sum_{j=1}^n m(r, q_j M_j[f, g]) + O(1) \\
&\leq \sum_{j=1}^n m(r, q_j) + \sum_{j=1}^n m(r, M_j[f, g]) + O(1) \\
&= \sum_{j=1}^n m(r, q_j) + \sum_{j=1}^n m(r, [f]^{l_{0j}} [f^{(1)}]^{l_{1j}} \dots [f^{(k)}]^{l_{kj}} [g]^{m_{0j}} [g^{(1)}]^{m_{1j}} \dots [g^{(h)}]^{m_{hj}}) \\
&\quad + O(1) \\
&\leq \sum_{j=1}^n m(r, q_j) + \sum_{j=1}^n [m(r, [f]^{\sum_{i=0}^k l_{ij}}) + m(r, [\frac{f^{(1)}}{f}]^{l_{1j}}) + \dots + m(r, [\frac{f^{(k)}}{f}]^{l_{kj}}) \\
&\quad + m(r, [g]^{\sum_{i=0}^h m_{ij}}) + m(r, [\frac{g^{(1)}}{g}]^{m_{1j}}) + \dots + m(r, [\frac{g^{(h)}}{g}]^{m_{hj}})] + O(1) \\
&\leq \sum_{j=1}^n m(r, q_j) + \sum_{j=1}^n [\sum_{i=0}^k l_{ij} \times m(r, f) + l_{1j} m(r, \frac{f^{(1)}}{f}) + \dots + l_{kj} m(r, \frac{f^{(k)}}{f}) \\
&\quad + \sum_{i=0}^h m_{ij} \times m(r, g) + m_{1j} m(r, \frac{g^{(1)}}{g}) + \dots + m_{hj} m(r, \frac{g^{(h)}}{g})] + O(1) \\
&\leq \sum_{j=1}^n m(r, q_j) + \sum_{j=1}^n [\sum_{i=0}^k l_{ij} \times m(r, f) + \sum_{i=0}^h m_{ij} \times m(r, g)] \\
&\quad + S(r, f) + S(r, g) \\
&\leq \sum_{j=1}^n m(r, q_j) + \sum_{j=1}^n [\{\sum_{i=0}^k l_{ij} + \sum_{i=0}^h m_{ij}\} \times \{m(r, f) + m(r, g)\}] \\
&\quad + S(r, f) + S(r, g) \\
&= \sum_{j=1}^n m(r, q_j) + \sum_{j=1}^n [d(M_j) \times \{m(r, f) + m(r, g)\}] + S(r, f) + S(r, g) \\
&\leq \sum_{j=1}^n m(r, q_j) + n\bar{d}(Q) \times \{m(r, f) + m(r, g)\} + S(r, f) + S(r, g).
\end{aligned}$$

Hence the proof is complete.

Lemma 2.6. Let $f(z)$, $g(z)$ be transcendental meromorphic functions with $N(r, f) = S(r, f)$, $N(r, g) = S(r, g)$. Also let $P[f, g]$ be given by (2.1) where each a_0, a_1, \dots, a_{l-1} being small functions of f and b_0, b_1, \dots, b_{m-1} being small functions of g . Now if $|f(re^{i\theta})| = |g(re^{i\theta})| \geq 1$ on the circle $|z| = r$, then

$$\begin{aligned}
\frac{1}{2}[lT(r, f) + mT(r, g)] &\leq T(r, P[f, g]) + S(r, f) + S(r, g) \\
&\leq 2[lT(r, f) + mT(r, g)]. \tag{2.9}
\end{aligned}$$

Proof : From (2.1)

$$P[f, g] = f^l g^m \left\{ 1 + \frac{a_{l-1}}{f} + \frac{b_{m-1}}{g} + \frac{a_{l-2}}{f^2} + \frac{b_{m-2}}{g^2} + \dots + \frac{a_0}{f^l} + \frac{b_0}{g^m} \right\}. \quad (2.10)$$

Now on the circle $|z| = r$, let

$$A(re^{i\theta}) = \max\{|a_{l-1}(re^{i\theta})|, |a_{l-2}(re^{i\theta})|^{\frac{1}{2}}, \dots, |a_0(re^{i\theta})|^{\frac{1}{l}}\}$$

and

$$B(re^{i\theta}) = \max\{|b_{m-1}(re^{i\theta})|, |b_{m-2}(re^{i\theta})|^{\frac{1}{2}}, \dots, |b_0(re^{i\theta})|^{\frac{1}{m}}\}.$$

Again if we suppose

$$E_1 = \{\theta \in [0, 2\pi] : |f(re^{i\theta})| > 4A(re^{i\theta})\}$$

and

$$E_2 = \{\theta \in [0, 2\pi] : |g(re^{i\theta})| > 4B(re^{i\theta})\},$$

then on $E_1 \cap E_2$, we have from (2.10)

$$\begin{aligned} |P[f, g]| &= |f|^l |g|^m \left| 1 + \frac{a_{l-1}}{f} + \frac{b_{m-1}}{g} + \frac{a_{l-2}}{f^2} + \frac{b_{m-2}}{g^2} + \dots + \frac{a_0}{f^l} + \frac{b_0}{g^m} \right| \\ &\geq |f|^l |g|^m \left[1 - \left| \frac{a_{l-1}}{f} \right| - \left| \frac{a_{l-2}}{f^2} \right| - \dots - \left| \frac{a_0}{f^l} \right| - \left| \frac{b_{m-1}}{g} \right| - \left| \frac{b_{m-2}}{g^2} \right| - \dots - \left| \frac{b_0}{g^m} \right| \right] \\ &\geq |f|^l |g|^m \left[1 - \left| \frac{A}{f} \right| - \left| \frac{A}{f} \right|^2 - \dots - \left| \frac{A}{f} \right|^l - \left| \frac{B}{g} \right| - \left| \frac{B}{g} \right|^2 - \dots - \left| \frac{B}{g} \right|^m \right] \\ &\geq |f|^l |g|^m \left[1 - \frac{1}{4} - \left(\frac{1}{4} \right)^2 - \dots - \left(\frac{1}{4} \right)^l - \frac{1}{4} - \left(\frac{1}{4} \right)^2 - \dots - \left(\frac{1}{4} \right)^m \right], \\ &= |f|^l |g|^m \left[1 - \left\{ \frac{1}{4} + \left(\frac{1}{4} \right)^2 + \dots + \left(\frac{1}{4} \right)^l \right\} - \left\{ \frac{1}{4} + \left(\frac{1}{4} \right)^2 + \dots + \left(\frac{1}{4} \right)^m \right\} \right] \\ &\geq |f|^l |g|^m \left[1 - \left\{ \frac{\frac{1}{4}}{1 - \frac{1}{4}} \right\} - \left\{ \frac{\frac{1}{4}}{1 - \frac{1}{4}} \right\} \right] \\ &= |f|^l |g|^m \left[1 - \frac{1}{3} - \frac{1}{3} \right] \\ &= \frac{1}{3} |f|^l |g|^m. \end{aligned}$$

Hence on $E_1 \cap E_2$

$$\begin{aligned} 3|P[f, g]| &\geq |f|^l |g|^m \\ \text{i.e., } \log^+ 3|P[f, g]| &\geq \log^+ |f|^l |g|^m. \end{aligned}$$

So,

$$l \log^+ |f| \leq \log 3 + \log^+ |P[f, g]| \quad (2.11)$$

and

$$m \log^+ |g| \leq \log 3 + \log^+ |P[f, g]|. \quad (2.12)$$

Therefore using (2.11) and by our hypothesis that on the circle $|z| = r$, $|f(re^{i\theta})| = |g(re^{i\theta})| \geq 1$, on E_1^c , $|f(re^{i\theta})| \leq 4A(re^{i\theta})$ and on $E_1 \cap E_2^c$, $|g(re^{i\theta})| \leq 4B(re^{i\theta})$, we have

$$\begin{aligned}
l \times m(r, f) &= l \times \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta \\
&= \frac{1}{2\pi} \int_{E_1 \cap E_2} l \log^+ |f(re^{i\theta})| d\theta + \frac{l}{2\pi} \int_{E_1 \cap E_2^c} \log^+ |f(re^{i\theta})| d\theta \\
&\quad + \frac{l}{2\pi} \int_{E_1^c} \log^+ |f(re^{i\theta})| d\theta \\
&\leq \frac{1}{2\pi} \int_{E_1 \cap E_2} (\log 3 + \log^+ |P[f, g]|) d\theta + \frac{l}{2\pi} \int_{E_1 \cap E_2^c} \log^+ |g(re^{i\theta})| d\theta \\
&\quad + \frac{l}{2\pi} \int_{E_1^c} \log^+ 4A d\theta \\
&\leq \frac{1}{2\pi} \int_{E_1 \cap E_2} \log 3 d\theta + \frac{1}{2\pi} \int_{E_1 \cap E_2} \log^+ |P[f, g]| d\theta \\
&\quad + \frac{l}{2\pi} \int_{E_1 \cap E_2^c} \log^+ 4B d\theta + \frac{l}{2\pi} \int_{E_1^c} \log^+ 4A d\theta \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \log 3 d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log^+ |P[f, g]| d\theta + \frac{l}{2\pi} \int_0^{2\pi} \log^+ 4B d\theta \\
&\quad + \frac{l}{2\pi} \int_0^{2\pi} \log^+ 4A d\theta \\
&= \log 3 + \frac{1}{2\pi} \int_0^{2\pi} \log^+ |P[f, g]| d\theta + \frac{l}{\pi} \int_0^{2\pi} \log^+ 4d\theta \\
&\quad + \frac{l}{2\pi} \int_0^{2\pi} \log^+ B d\theta + \frac{l}{2\pi} \int_0^{2\pi} \log^+ A d\theta \\
&= \log 3 + m(r, P[f, g]) + 2l \log 4 + l \times m(r, B) + l \times m(r, A) \\
&= m(r, P[f, g]) + S(r, f) + S(r, g). \tag{2.13}
\end{aligned}$$

Adding $l \times N(r, f)$ on both sides and recalling that $N(r, f) = S(r, f)$ we get

$$\begin{aligned}
l \times T(r, f) &\leq m(r, P[f, g]) + S(r, f) + S(r, g) \\
&\leq T(r, P[f, g]) + S(r, f) + S(r, g). \tag{2.14}
\end{aligned}$$

Similarly, using (2.12) and by our hypothesis that on the circle $|z| = r$, $|f(re^{i\theta})| = |g(re^{i\theta})| \geq 1$, on E_2^c , $|g(re^{i\theta})| \leq 4B(re^{i\theta})$ and on $E_1^c \cap E_2$, $|f(re^{i\theta})| \leq 4A(re^{i\theta})$, we have

$$\begin{aligned}
m \times m(r, g) &= m \times \frac{1}{2\pi} \int_0^{2\pi} \log^+ |g(re^{i\theta})| d\theta \\
&= \frac{1}{2\pi} \int_{E_1 \cap E_2} m \log^+ |g(re^{i\theta})| d\theta + \frac{m}{2\pi} \int_{E_1^c \cap E_2} \log^+ |g(re^{i\theta})| d\theta \\
&\quad + \frac{m}{2\pi} \int_{E_2^c} \log^+ |g(re^{i\theta})| d\theta
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2\pi} \int_{E_1 \cap E_2} (\log 3 + \log^+ |P[f, g]|) d\theta + \frac{m}{2\pi} \int_{E_1^c \cap E_2} \log^+ |f(re^{i\theta})| d\theta \\
&\quad + \frac{m}{2\pi} \int_{E_2^c} \log^+ 4B d\theta \\
&\leq \frac{1}{2\pi} \int_{E_1 \cap E_2} \log 3 d\theta + \frac{1}{2\pi} \int_{E_1 \cap E_2} \log^+ |P[f, g]| d\theta \\
&\quad + \frac{m}{2\pi} \int_{E_1^c \cap E_2} \log^+ 4A d\theta + \frac{m}{2\pi} \int_{E_2^c} \log^+ 4B d\theta \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \log 3 d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log^+ |P[f, g]| d\theta + \frac{m}{2\pi} \int_0^{2\pi} \log^+ 4A d\theta \\
&\quad + \frac{m}{2\pi} \int_0^{2\pi} \log^+ 4B d\theta \\
&= \log 3 + \frac{1}{2\pi} \int_0^{2\pi} \log^+ |P[f, g]| d\theta + \frac{m}{\pi} \int_0^{2\pi} \log^+ 4 d\theta \\
&\quad + \frac{m}{2\pi} \int_0^{2\pi} \log^+ A d\theta + \frac{m}{2\pi} \int_0^{2\pi} \log^+ B d\theta \\
&= \log 3 + m(r, P[f, g]) + 2m \log 4 + m \times m(r, A) + m \times m(r, B) \\
&= m(r, P[f, g]) + S(r, f) + S(r, g). \tag{2.15}
\end{aligned}$$

Adding $m \times N(r, g)$ on both sides and recalling that $N(r, g) = S(r, g)$ we get

$$\begin{aligned}
m \times T(r, g) &\leq m(r, P[f, g]) + S(r, f) + S(r, g) \\
&\leq T(r, P[f, g]) + S(r, f) + S(r, g). \tag{2.16}
\end{aligned}$$

Now adding (2.14) and (2.16), we get

$$\begin{aligned}
l \times T(r, f) + m \times T(r, g) &\leq 2T(r, P[f, g]) + S(r, f) + S(r, g) \\
\text{i.e., } \frac{1}{2}[l \times T(r, f) + m \times T(r, g)] &\leq T(r, P[f, g]) + S(r, f) + S(r, g). \tag{2.17}
\end{aligned}$$

Next by Lemma 2.4,

$$m(r, P[f, g]) \leq 2l \times m(r, f) + 2m \times m(r, g) + S(r, f) + S(r, g). \tag{2.18}$$

Finally using (2.18) and by our hypotheses that $N(r, f) = S(r, f)$ and $N(r, g) = S(r, g)$, we get

$$\begin{aligned}
T(r, P[f, g]) &= m(r, P[f, g]) + N(r, P[f, g]) \\
&\leq 2l \times m(r, f) + 2m \times m(r, g) + S(r, f) + S(r, g) \\
&\leq 2[l \times T(r, f) + m \times T(r, g)] + S(r, f) + S(r, g). \tag{2.19}
\end{aligned}$$

Combining (2.17) and (2.19) we obtain the desired result.

Lemma 2.7. Let $M_j[f, g]$ be a monomial in f and g . If z_0 is a pole of f and g of order p and q respectively, then $M_j[f, g]$ has a pole at z_0 of order $\leq (p + q - 2)d(M_j) + \Gamma_{M_j}$.

Proof : Let

$$M_j[f, g] = (f(z))^{l_{oj}} (f^{(1)}(z))^{l_{1j}} \dots (f^{(k)}(z))^{l_{kj}} (g(z))^{m_{oj}} (g^{(1)}(z))^{m_{1j}} \dots (g^{(h)}(z))^{m_{hj}}$$

be of degree

$$d(M_j) = \sum_{i=0}^k l_{ij} + \sum_{i=0}^h m_{ij}$$

and of weight

$$\Gamma_{M_j} = \sum_{i=0}^k (i+1)l_{ij} + \sum_{i=0}^h (i+1)m_{ij}.$$

where $l_{0j}, l_{1j}, \dots, l_{kj}, m_{0j}, m_{1j}, \dots, m_{hj}$ are non-negative integers. By hypothesis z_0 will be a pole of $M_j[f, g]$ of order

$$\begin{aligned} &= \{pl_{0j} + (p+1)l_{1j} + \dots + (p+k)l_{kj}\} + \{qm_{0j} + (q+1)m_{1j} + \dots + (q+h)m_{hj}\} \\ &= p \sum_{i=0}^k l_{ij} + \sum_{i=1}^k il_{ij} + q \sum_{i=0}^h m_{ij} + \sum_{i=1}^h im_{ij} \\ &= (p-1) \sum_{i=0}^k l_{ij} + \sum_{i=0}^k (i+1)l_{ij} + (q-1) \sum_{i=0}^h m_{ij} + \sum_{i=0}^h (i+1)m_{ij} \\ &\leq (p-1)d(M_j) + (q-1)d(M_j) + \sum_{i=0}^k (i+1)l_{ij} + \sum_{i=0}^h (i+1)m_{ij} \\ &= (p+q-2)d(M_j) + \Gamma_{M_j}. \end{aligned}$$

Hence the proof is complete.

Lemma 2.8. Suppose that $Q[f, g]$ is a differential polynomial. Let f and g have poles at z_0 of respective order p and q and z_0 is not a zero or a pole of coefficients of $Q[f, g]$. Then $Q[f, g]$ has a pole at z_0 of order atmost $(p+q-1)\gamma_Q + (\Gamma_Q - \gamma_Q)$.

Proof : Let

$$Q[f, g] = \sum_{j=1}^n a_j M_j[f, g]$$

where each

$$M_j[f, g] = (f(z))^{l_{0j}} (f^{(1)}(z))^{l_{1j}} \dots (f^{(k)}(z))^{l_{kj}} (g(z))^{m_{0j}} (g^{(1)}(z))^{m_{1j}} \dots (g^{(h)}(z))^{m_{hj}}$$

is a monomial in f and g of degree

$$d(M_j) = \sum_{i=0}^k l_{ij} + \sum_{i=0}^h m_{ij}$$

and of weight

$$\Gamma_{M_j} = \sum_{i=0}^k (i+1)l_{ij} + \sum_{i=0}^h (i+1)m_{ij}.$$

Here the degree and weight of the differential polynomial $Q[f, g]$ is given by

$$\bar{d}(Q) = \max\{d(M_j) : 1 \leq j \leq n\}$$

and

$$\Gamma_Q = \max\{\Gamma_{M_j} : 1 \leq j \leq n\}$$

respectively.

Now since f and g have a pole at z_0 of respective order p and q and it is not a zero or a pole of the coefficients of $Q[f, g]$, by using the Lemma 2.7 we can say that $Q[f, g]$ has a pole at z_0 of order

$$\begin{aligned} &\leq \max_{1 \leq j \leq n} \{(p+q-2)d(M_j) + \Gamma_{M_j}\} \\ &\leq \max_{1 \leq j \leq n} \{(p+q-2)d(M_j)\} + \max_{1 \leq j \leq n} \{\Gamma_{M_j}\} \\ &= (p+q-2)\bar{d}(Q) + \Gamma_Q \\ &= (p+q-1)\bar{d}(Q) + [\Gamma_Q - \bar{d}(Q)]. \end{aligned}$$

Hence the proof is complete.

We now present the main result of the paper.

Theorem 2.1 Suppose that f, g are transcendental and $Q_1[f, g], Q_2[f, g]$ are two differential polynomials in f and g where both of $Q_1[f, g]$ and $Q_2[f, g]$ are not identically zero. Let the order and lower degree of $Q_1[f, g]$ be ν and $\underline{d}(Q_1)$ respectively and $Q_2[f, g]$ consists of n arbitrary meromorphic coefficients which are small functions of f and g . Also let $P[f, g]$ be given by (2.1) where each a_0, a_1, \dots, a_{l-1} are small functions of f and b_0, b_1, \dots, b_{m-1} are of g . If

$$F = P[f, g]Q_1[f, g] + Q_2[f, g] \quad (2.20)$$

and on the circle $|z| = r$, $|f(re^{i\theta})| = |g(re^{i\theta})| \geq 1$, then

$$\begin{aligned} &[\frac{l}{2} - n\bar{d}(Q_2)]T(r, f) + [\frac{m}{2} - n\bar{d}(Q_2)]T(r, g) \\ &\leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{P[f, g]}) + [\Gamma_{Q_2} - 2\bar{d}(Q_2)][\bar{N}(r, f) + \bar{N}(r, g)] \\ &\quad - d[\{N(r, \frac{1}{f}) - N_{\nu+1}(r, \frac{1}{f})\} + \{N(r, \frac{1}{g}) - N_{\nu+1}(r, \frac{1}{g})\}] + m(r, \frac{1}{P[f, g]}) \\ &\quad + S(r, f) + S(r, g), \end{aligned}$$

where

$$d = \begin{cases} \underline{d}(Q_1) & \text{if } l \geq 2n\bar{d}(Q_2) \text{ and } m \geq 2n\bar{d}(Q_2) \\ 0 & \text{if } l < 2n\bar{d}(Q_2) \text{ and } m < 2n\bar{d}(Q_2) \end{cases}$$

and $N_{\nu+1}(r, \frac{1}{f})$ denotes the counting function of poles of f where a pole of f with multiplicity μ is counted μ times if $\mu \leq \nu+1$ and $(\nu+1)$ times if $\mu > (\nu+1)$.

Proof : We assume $l > 2n\bar{d}(Q_2)$ and $m > 2n\bar{d}(Q_2)$.

From (2.20) we have

$$1 = \frac{1}{F}P[f, g]Q_1[f, g] + \frac{1}{F}Q_2[f, g]$$

i.e.,

$$F' = \frac{F'}{F} P[f, g] Q_1[f, g] + \frac{F'}{F} Q_2[f, g]. \quad (2.21)$$

Again differentiating (2.20)

$$F' = (P[f, g])' Q_1[f, g] + P[f, g] (Q_1[f, g])' + (Q_2[f, g])'. \quad (2.22)$$

Now comparing (2.21) and (2.22) we have

$$\begin{aligned} \frac{F'}{F} P[f, g] Q_1[f, g] + \frac{F'}{F} Q_2[f, g] &= (P[f, g])' Q_1[f, g] + P[f, g] (Q_1[f, g])' + (Q_2[f, g])' \\ \text{i.e., } \frac{F'}{F} P[f, g] Q_1[f, g] - (P[f, g])' Q_1[f, g] - P[f, g] (Q_1[f, g])' &= (Q_2[f, g])' - \frac{F'}{F} Q_2[f, g] \\ \text{i.e., } P[f, g] \left\{ \frac{F'}{F} Q_1[f, g] - \frac{(P[f, g])'}{P[f, g]} Q_1[f, g] - (Q_1[f, g])' \right\} &= (Q_2[f, g])' - \frac{F'}{F} Q_2[f, g] \end{aligned}$$

i.e.,

$$P[f, g] Q^*[f, g] = Q[f, g] \quad (2.23)$$

where

$$Q^*[f, g] = \frac{F'}{F} Q_1[f, g] - \frac{(P[f, g])'}{P[f, g]} Q_1[f, g] - (Q_1[f, g])' \quad (2.24)$$

and

$$Q[f, g] = (Q_2[f, g])' - \frac{F'}{F} Q_2[f, g]. \quad (2.25)$$

First we suppose that $Q^*[f, g] \neq 0$. Then from (2.23) we have $Q[f, g] \neq 0$.

Now for any $j (= 1, 2, \dots)$ a monomial is usually defined by

$$M_j[f, g] = [f(z)]^{l_{0j}} [f^{(1)}(z)]^{l_{1j}} \dots [f^{(k)}(z)]^{l_{kj}} [g(z)]^{m_{0j}} [g^{(1)}(z)]^{m_{1j}} \dots [g^{(h)}(z)]^{m_{hj}}$$

with degree

$$d(M_j) = \sum_{i=0}^k l_{ij} + \sum_{i=0}^h m_{ij},$$

where $l_{0j}, l_{1j}, \dots, l_{kj}, m_{0j}, m_{1j}, \dots, m_{hj}$ are non-negative integers.

On differentiation, we get

$$\begin{aligned} (M_j[f, g])' &= l_{0j} (f)^{l_{0j}-1} (f^{(1)})^{l_{1j}+1} \dots (f^{(k)})^{l_{kj}} (g)^{m_{0j}} (g^{(1)})^{m_{1j}} \dots (g^{(h)})^{m_{hj}} \\ &\quad + l_{1j} (f)^{l_{0j}} (f^{(1)})^{l_{1j}-1} \dots (f^{(k)})^{l_{kj}} (g)^{m_{0j}} (g^{(1)})^{m_{1j}} \dots (g^{(h)})^{m_{hj}} + \dots \\ &\quad \dots + l_{kj} (f)^{l_{0j}} (f^{(1)})^{l_{1j}} \dots (f^{(k)})^{l_{kj}-1} f^{(k+1)} (g)^{m_{0j}} (g^{(1)})^{m_{1j}} \dots (g^{(h)})^{m_{hj}} \\ &\quad + m_{0j} (f)^{l_{0j}} (f^{(1)})^{l_{1j}} \dots (f^{(k)})^{l_{kj}} (g)^{m_{0j}-1} (g^{(1)})^{m_{1j}+1} \dots (g^{(h)})^{m_{hj}} \\ &\quad + m_{1j} (f)^{l_{0j}} (f^{(1)})^{l_{1j}} \dots (f^{(k)})^{l_{kj}} (g)^{m_{0j}} (g^{(1)})^{m_{1j}-1} \dots (g^{(h)})^{m_{hj}} + \dots \\ &\quad \dots + m_{hj} (f)^{l_{0j}} (f^{(1)})^{l_{1j}} \dots (f^{(k)})^{l_{kj}} (g)^{m_{0j}} (g^{(1)})^{m_{1j}} \dots (g^{(h)})^{m_{hj}-1} g^{(h+1)}. \end{aligned}$$

Hence

$$\begin{aligned}
d(M'_j) &= \max\{[(l_{0j} - 1) + (l_{1j} + 1) + \dots + l_{kj} + m_{0j} + m_{1j} + \dots + m_{hj}], \\
&\quad [l_{0j} + (l_{1j} - 1) + (l_{2j} + 1) + \dots + l_{kj} + m_{0j} + m_{1j} + \dots + m_{hj}], \dots \\
&\quad \dots [l_{0j} + l_{1j} + \dots + (l_{kj} - 1) + 1 + m_{0j} + m_{1j} + \dots + m_{hj}], \\
&\quad [l_{0j} + l_{1j} + \dots + l_{kj} + (m_{0j} - 1) + (m_{1j} + 1) + \dots + m_{hj}], \\
&\quad [l_{0j} + l_{1j} + \dots + l_{kj} + m_{0j} + (m_{1j} - 1) + (m_{2j} + 1) \dots + m_{hj}], \dots \\
&\quad \dots [l_{0j} + l_{1j} + \dots + l_{kj} + m_{0j} + m_{1j} + \dots + (m_{hj} - 1) + 1]\} \\
&= \max\{d(M_j), d(M_j), \dots, d(M_j)\} \\
&= d(M_j).
\end{aligned}$$

Thus we can write $\bar{d}(Q_2) = \bar{d}(Q'_2)$. Hence from (2.25) we have $\bar{d}(Q) = \bar{d}(Q_2)$.

Now using Lemma 2.1 on (2.23) and assuming that the coefficients of $Q^*[f, g]$ and $Q[f, g]$ are small functions of f and g , we get

$$m(r, Q^*[f, g]) = m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g). \quad (2.26)$$

Again from (2.23),

$$\begin{aligned}
m(r, P[f, g]) &= m(r, \frac{Q[f, g]}{Q^*[f, g]}) \\
&\leq m(r, Q[f, g]) + m(r, \frac{1}{Q^*[f, g]}). \quad (2.27)
\end{aligned}$$

Using (2.25), Lemma 2.5, Milloux's theorem and the fact that the coefficients of $Q_2[f, g]$ are small functions of f and g we have

$$\begin{aligned}
m(r, Q[f, g]) &= m(r, (Q_2[f, g])' - \frac{F'}{F} Q_2[f, g]) \\
&= m(r, Q_2[f, g]) \{ \frac{(Q_2[f, g])'}{Q_2[f, g]} - \frac{F'}{F} \} \\
&\leq m(r, Q_2[f, g]) + m(r, \frac{(Q_2[f, g])'}{Q_2[f, g]}) + m(r, -\frac{F'}{F}) + O(1) \\
&= m(r, Q_2[f, g]) + S(r, f) + S(r, g) \\
&\leq n\bar{d}(Q_2)[m(r, f) + m(r, g)] + S(r, f) + S(r, g). \quad (2.28)
\end{aligned}$$

Also, by the First Fundamental Theorem and by using (2.26)

$$\begin{aligned}
T(r, \frac{1}{Q^*[f, g]}) &= T(r, Q^*[f, g]) + O(1) \\
i.e., \quad m(r, \frac{1}{Q^*[f, g]}) + N(r, \frac{1}{Q^*[f, g]}) &= m(r, Q^*[f, g]) + N(r, Q^*[f, g]) + O(1) \\
i.e., \quad m(r, \frac{1}{Q^*[f, g]}) + N(r, \frac{1}{Q^*[f, g]}) &= m(r, \frac{1}{P[f, g]}) + N(r, Q^*[f, g]) + S(r, f) + S(r, g) \\
i.e., \quad m(r, \frac{1}{Q^*[f, g]}) &= N(r, Q^*[f, g]) - N(r, \frac{1}{Q^*[f, g]}) + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g). \quad (2.29)
\end{aligned}$$

Using (2.28) and (2.29) in (2.27) we can write

$$\begin{aligned} m(r, P[f, g]) &\leq n\bar{d}(Q_2)[m(r, f) + m(r, g)] + N(r, Q^*[f, g]) - N(r, \frac{1}{Q^*[f, g]}) \\ &\quad + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g). \end{aligned} \quad (2.30)$$

Now using (2.13) and (2.15) in (2.30) we get

$$\begin{aligned} \frac{l}{2} \times m(r, f) + \frac{m}{2} \times m(r, g) &\leq n\bar{d}(Q_2)[m(r, f) + m(r, g)] + N(r, Q^*[f, g]) \\ &\quad - N(r, \frac{1}{Q^*[f, g]}) + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g). \end{aligned}$$

$$\begin{aligned} i.e., [\frac{l}{2} - n\bar{d}(Q_2)]m(r, f) + [\frac{m}{2} - n\bar{d}(Q_2)]m(r, g) &\leq N(r, Q^*[f, g]) - N(r, \frac{1}{Q^*[f, g]}) \\ &\quad + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g). \end{aligned} \quad (2.31)$$

Obviously, from (2.24) we can say that the poles of $Q^*[f, g]$ occurs possibly only from the zeros of F and $P[f, g]$, the poles of f and g and the zeros and poles of the coefficients. Also it can be noted that the zeros of F and $P[f, g]$ are poles of $Q^*[f, g]$ of order atmost one.

Now if f and g have a pole at z_0 of respective order p and q and it is not a zero or a pole of the coefficients of $P[f, g]$, $Q_1[f, g]$ and $Q_2[f, g]$ then from (2.1) we can say that $P[f, g]$ has a pole at z_0 of order $pl + qm$.

Also by using Lemma 2.8 on (2.25) we can say that $Q[f, g]$ has a pole at z_0 of order atmost $(p + q - 2)\bar{d}(Q_2) + \Gamma_{Q_2}$.

Now from (2.23),

$$Q^*[f, g] = \frac{Q[f, g]}{P[f, g]}$$

and hence we can say that if z_0 is a pole of $Q^*[f, g]$, it will be a pole of $Q^*[f, g]$ of order atmost

$$\begin{aligned} &(p + q - 2)\bar{d}(Q_2) + \Gamma_{Q_2} - (pl + qm) \\ &= [\Gamma_{Q_2} - 2\bar{d}(Q_2)] - p[l - \bar{d}(Q_2)] - q[m - \bar{d}(Q_2)]. \end{aligned}$$

Thus we obtain

$$\begin{aligned} N(r, Q^*[f, g]) &\leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{P[f, g]}) + [\Gamma_{Q_2} - 2\bar{d}(Q_2)][\bar{N}(r, f) + \bar{N}(r, g)] \\ &\quad - [l - \bar{d}(Q_2)]N(r, f) - [m - \bar{d}(Q_2)]N(r, g) + S(r, f) + S(r, g). \end{aligned} \quad (2.32)$$

Again $(Q_1[f, g])'$ is a differential polynomial of order $\nu + 1$. Now if f and g have zero at z_0 with respective multiplicities $\mu_1 (> \nu + 1)$ and $\mu_2 (> \nu + 1)$ then by Lemma 2.2, we can say that it is a zero of $Q_1[f, g]$ with multiplicity $\geq (\mu - \nu)\underline{d}(Q_1)$ where $\mu = \min\{\mu_1, \mu_2\}$. Also z_0 may be a pole of $[\frac{F'}{F} - \frac{(P[f, g])'}{P[f, g]}]$

of order atmost 1. Hence z_0 is a zero of $[\frac{F'}{F} - \frac{(P[f,g])'}{P[f,g]}]Q_1[f, g]$ with multiplicity $\geq (\mu - \nu)\underline{d}(Q_1) - 1$.

Again, since $\underline{d}(Q_1') = \underline{d}(Q_1)$, from Lemma 2.2 it follows that z_0 is a zero of $(Q_1[f, g])'$ with multiplicity $\geq (\mu - \nu - 1)\underline{d}(Q_1)$.

Hence, assuming $\underline{d}(Q_1) \geq 1$, from (2.24) we can say that z_0 is a zero of $Q^*[f, g]$ with multiplicity $\geq (\mu - \nu - 1)\underline{d}(Q_1)$.

Thus if $N(r, \frac{1}{Q^*[f, g]} | f, g = 0, > \nu + 1)$ denotes the counting function of those poles of $Q^*[f, g]$, counted with proper order, which are zeros of f and g with multiplicities greater than $\nu + 1$; $N(r, \frac{1}{f} | > \nu + 1)$ and $\bar{N}(r, \frac{1}{f} | > \nu + 1)$ denote the counting functions of those zeros of f whose multiplicities are greater than $\nu + 1$, counted according to its multiplicities and ignoring its multiplicities respectively, then

$$\begin{aligned}
N(r, \frac{1}{Q^*[f, g]}) &\geq N(r, \frac{1}{Q^*[f, g]} | f, g = 0, > \nu + 1) \\
&\geq \underline{d}(Q_1)[N(r, \frac{1}{f} | > \nu + 1) + N(r, \frac{1}{g} | > \nu + 1)] \\
&\quad - \underline{d}(Q_1)(\nu + 1)[\bar{N}(r, \frac{1}{f} | > \nu + 1) + \bar{N}(r, \frac{1}{g} | > \nu + 1)] \\
&\quad + S(r, f) + S(r, g) \\
&= \underline{d}(Q_1)[N(r, \frac{1}{f}) - \{N(r, \frac{1}{f} | \leq \nu + 1) + (\nu + 1)\bar{N}(r, \frac{1}{f} | > \nu + 1)\}] \\
&\quad + \underline{d}(Q_1)[N(r, \frac{1}{g}) - \{N(r, \frac{1}{g} | \leq \nu + 1) \\
&\quad + (\nu + 1)\bar{N}(r, \frac{1}{g} | > \nu + 1)\}] + S(r, f) + S(r, g) \\
&= \underline{d}(Q_1)[N(r, \frac{1}{f}) - N_{\nu+1}(r, \frac{1}{f})] + \underline{d}(Q_1)[N(r, \frac{1}{g}) - N_{\nu+1}(r, \frac{1}{g})] \\
&\quad + S(r, f) + S(r, g) \tag{2.33}
\end{aligned}$$

If $\underline{d}(Q_1) = 0$, (2.33) obviously holds.

Now using (2.32) and (2.33) in (2.31) we get

$$\begin{aligned}
&[\frac{l}{2} - n\bar{d}(Q_2)]m(r, f) + [\frac{m}{2} - n\bar{d}(Q_2)]m(r, g) \\
&\leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{P[f, g]}) + [\Gamma_{Q_2} - 2\bar{d}(Q_2)][\bar{N}(r, f) + \bar{N}(r, g)] \\
&\quad - [l - \bar{d}(Q_2)]N(r, f) - [m - \bar{d}(Q_2)]N(r, g) - \underline{d}(Q_1)[N(r, \frac{1}{f}) - N_{\nu+1}(r, \frac{1}{f})] \\
&\quad - \underline{d}(Q_1)[N(r, \frac{1}{g}) - N_{\nu+1}(r, \frac{1}{g})] + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g)
\end{aligned}$$

$$\begin{aligned}
& \text{i.e., } [\frac{l}{2} - n\bar{d}(Q_2)]m(r, f) + [\frac{m}{2} - n\bar{d}(Q_2)]m(r, g) \\
& \leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{P[f, g]}) + [\Gamma_{Q_2} - 2\bar{d}(Q_2)][\bar{N}(r, f) + \bar{N}(r, g)] \\
& \quad - [\frac{l}{2} - n\bar{d}(Q_2)]N(r, f) - [\frac{m}{2} - n\bar{d}(Q_2)]N(r, g) - \underline{d}(Q_1)[N(r, \frac{1}{f}) \\
& \quad - N_{\nu+1}(r, \frac{1}{f})] - \underline{d}(Q_1)[N(r, \frac{1}{g}) - N_{\nu+1}(r, \frac{1}{g})] + m(r, \frac{1}{P[f, g]}) \\
& \quad + S(r, f) + S(r, g)
\end{aligned}$$

$$\begin{aligned}
& \text{i.e., } [\frac{l}{2} - n\bar{d}(Q_2)]T(r, f) + [\frac{m}{2} - n\bar{d}(Q_2)]T(r, g) \\
& \leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{P[f, g]}) + [\Gamma_{Q_2} - 2\bar{d}(Q_2)][\bar{N}(r, f) + \bar{N}(r, g)] \\
& \quad - \underline{d}(Q_1)[N(r, \frac{1}{f}) - N_{\nu+1}(r, \frac{1}{f})] - \underline{d}(Q_1)[N(r, \frac{1}{g}) - N_{\nu+1}(r, \frac{1}{g})] \\
& \quad + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g).
\end{aligned}$$

Next let us suppose that $Q^*[f, g] \equiv 0$, then from (2.23) we have

$$Q[f, g] \equiv 0,$$

which implies from (2.25),

$$\begin{aligned}
& (Q_2[f, g])' - \frac{F'}{F}Q_2[f, g] = 0 \\
& \text{i.e., } \frac{(Q_2[f, g])'}{Q_2[f, g]} = \frac{F'}{F}.
\end{aligned}$$

Integrating we get

$$\begin{aligned}
& F = C_1 Q_2[f, g] \\
& \text{i.e., } P[f, g]Q_1[f, g] + Q_2[f, g] = C_1 Q_2[f, g], \quad \text{using (2.20)} \\
& \text{i.e., } P[f, g]Q_1[f, g] = C Q_2[f, g], \quad (2.34) \\
& \text{assuming } C_1 - 1 = C (\neq 0) \text{ is a finite complex number.}
\end{aligned}$$

Now using the Lemma 2.1 and assuming that the coefficients of $Q_1[f, g]$ and $Q_2[f, g]$ are small functions of f and g , we get

$$m(r, Q_1[f, g]) = m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g). \quad (2.35)$$

Again from (2.34) we have

$$\begin{aligned}
& m(r, P[f, g]) = m(r, C \frac{Q_2[f, g]}{Q_1[f, g]}) \\
& \leq m(r, Q_2[f, g]) + m(r, \frac{1}{Q_1[f, g]}) + O(1). \quad (2.36)
\end{aligned}$$

Using (2.13) and (2.15), we can rewrite (2.36) as following

$$\frac{l}{2} \times m(r, f) + \frac{m}{2} \times m(r, g) \leq m(r, Q_2[f, g]) + m(r, \frac{1}{Q_1[f, g]}) + S(r, f) + S(r, g). \quad (2.37)$$

Again if $Q_2[f, g]$ consists n arbitrary meromorphic coefficients then applying Lemma 2.5 and using the hypothesis that the coefficients of $Q_2[f, g]$ to be small functions, we get

$$m(r, Q_2[f, g]) \leq n\bar{d}(Q_2)[m(r, f) + m(r, g)] + S(r, f) + S(r, g). \quad (2.38)$$

Also, by the Nevanlinna's Theorem and using (2.35) we have

$$\begin{aligned} T(r, \frac{1}{Q_1[f, g]}) &= T(r, Q_1[f, g]) + O(1) \\ \text{i.e., } m(r, \frac{1}{Q_1[f, g]}) + N(r, \frac{1}{Q_1[f, g]}) &= m(r, Q_1[f, g]) + N(r, Q_1[f, g]) + O(1) \\ \text{i.e., } m(r, \frac{1}{Q_1[f, g]}) + N(r, \frac{1}{Q_1[f, g]}) &= N(r, Q_1[f, g]) + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g) \\ \text{i.e., } m(r, \frac{1}{Q_1[f, g]}) &= N(r, Q_1[f, g]) - N(r, \frac{1}{Q_1[f, g]}) + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g). \end{aligned} \quad (2.39)$$

Now using (2.38) and (2.39) in (2.37)

$$\begin{aligned} \frac{l}{2} \times m(r, f) + \frac{m}{2} \times m(r, g) &\leq n\bar{d}(Q_2)[m(r, f) + m(r, g)] + N(r, Q_1[f, g]) \\ &\quad - N(r, \frac{1}{Q_1[f, g]}) + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g) \\ \text{i.e., } [\frac{l}{2} - n\bar{d}(Q_2)]m(r, f) + [\frac{m}{2} - n\bar{d}(Q_2)]m(r, g) \\ &\leq N(r, Q_1[f, g]) - N(r, \frac{1}{Q_1[f, g]}) + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g) \end{aligned} \quad (2.40)$$

It is clear that a pole of $Q_1[f, g]$ is either a pole of f or g , or a pole of the coefficients of $Q_1[f, g]$.

Now let f and g have pole at z_0 of respective order p and q and it is not a zero or a pole of the coefficients of $P[f, g]$, $Q_1[f, g]$ and $Q_2[f, g]$. Then from (2.1) we can say that z_0 is a pole of $P[f, g]$ of order $pl + qm$.

Also from Lemma 2.8 we can say that z_0 is a pole of $Q_2[f, g]$ of order atmost $(p + q - 2)\bar{d}(Q_2) + \Gamma_{Q_2}$.

Now from (2.34) we have

$$Q_1[f, g] = C \frac{Q_2[f, g]}{P[f, g]}$$

and hence, we can say that z_0 is a pole of $Q_1[f, g]$ of order atmost

$$\begin{aligned} (p + q - 2)\bar{d}(Q_2) + \Gamma_{Q_2} - (pl + qm) \\ = [\Gamma_{Q_2} - 2\bar{d}(Q_2)] - p[l - \bar{d}(Q_2)] - q[m - \bar{d}(Q_2)]. \end{aligned}$$

Thus we obtain

$$\begin{aligned} N(r, Q_1[f, g]) &\leq [\Gamma_{Q_2} - 2\bar{d}(Q_2)][\bar{N}(r, f) + \bar{N}(r, g)] - [l - \bar{d}(Q_2)]N(r, f) \\ &\quad - [m - \bar{d}(Q_2)]N(r, g) + S(r, f) + S(r, g). \end{aligned} \quad (2.41)$$

Again if z_0 is a zero of f and g with respective multiplicities $\mu_1 (> \nu + 1)$ and $\mu_2 (> \nu + 1)$ then by Lemma 2.2, it follows that it is a zero of $Q_1[f, g]$ with multiplicity greater than $(\mu - \nu - 1)\underline{d}(Q_1)$ where $\mu = \min\{\mu_1, \mu_2\}$.

Thus we have

$$\begin{aligned} N(r, \frac{1}{Q_1[f, g]}) &\geq N(r, \frac{1}{Q_1[f, g]} | f, g = 0, > \nu + 1) \\ &\geq \underline{d}(Q_1)[N(r, \frac{1}{f} | > \nu + 1) + N(r, \frac{1}{g} | > \nu + 1)] \\ &\quad - \underline{d}(Q_1)(\nu + 1)[\bar{N}(r, \frac{1}{f} | > \nu + 1) + \bar{N}(r, \frac{1}{g} | > \nu + 1)] \\ &\quad + S(r, f) + S(r, g) \\ &= \underline{d}(Q_1)[N(r, \frac{1}{f}) - \{N(r, \frac{1}{f} | \leq \nu + 1) + (\nu + 1)\bar{N}(r, \frac{1}{f} | > \nu + 1)\}] \\ &\quad + \underline{d}(Q_1)[N(r, \frac{1}{g}) - \{N(r, \frac{1}{g} | \leq \nu + 1) \\ &\quad + (\nu + 1)\bar{N}(r, \frac{1}{g} | > \nu + 1)\}] + S(r, f) + S(r, g) \\ &= \underline{d}(Q_1)[N(r, \frac{1}{f}) - N_{\nu+1}(r, \frac{1}{f})] + \underline{d}(Q_1)[N(r, \frac{1}{g}) - N_{\nu+1}(r, \frac{1}{g})] \\ &\quad + S(r, f) + S(r, g) \end{aligned} \quad (2.42)$$

Now using (2.41) and (2.42) in (2.40) we get

$$\begin{aligned} &[\frac{l}{2} - n\bar{d}(Q_2)]m(r, f) + [\frac{m}{2} - n\bar{d}(Q_2)]m(r, g) \\ &\leq [\Gamma_{Q_2} - 2\bar{d}(Q_2)][\bar{N}(r, f) + \bar{N}(r, g)] \\ &\quad - [l - \bar{d}(Q_2)]N(r, f) - [m - \bar{d}(Q_2)]N(r, g) - \underline{d}(Q_1)[N(r, \frac{1}{f}) - N_{\nu+1}(r, \frac{1}{f})] \\ &\quad - \underline{d}(Q_1)[N(r, \frac{1}{g}) - N_{\nu+1}(r, \frac{1}{g})] + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g) \end{aligned}$$

$$\begin{aligned} i.e., &[\frac{l}{2} - n\bar{d}(Q_2)]m(r, f) + [\frac{m}{2} - n\bar{d}(Q_2)]m(r, g) \\ &\leq \bar{N}(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{P[f, g]}) + [\Gamma_{Q_2} - 2\bar{d}(Q_2)][\bar{N}(r, f) + \bar{N}(r, g)] \\ &\quad - [\frac{l}{2} - n\bar{d}(Q_2)]N(r, f) - [\frac{m}{2} - n\bar{d}(Q_2)]N(r, g) - \underline{d}(Q_1)[N(r, \frac{1}{f}) \\ &\quad - N_{\nu+1}(r, \frac{1}{f})] - \underline{d}(Q_1)[N(r, \frac{1}{g}) - N_{\nu+1}(r, \frac{1}{g})] + m(r, \frac{1}{P[f, g]}) \\ &\quad + S(r, f) + S(r, g) \end{aligned}$$

$$\begin{aligned}
& \text{i.e., } [\frac{l}{2} - n\bar{d}(Q_2)]T(r, f) + [\frac{m}{2} - n\bar{d}(Q_2)]T(r, g) \\
& \leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{P[f, g]}) + [\Gamma_{Q_2} - 2\bar{d}(Q_2)][\bar{N}(r, f) + \bar{N}(r, g)] \\
& \quad - \underline{d}(Q_1)[N(r, \frac{1}{f}) - N_{\nu+1}(r, \frac{1}{f})] - \underline{d}(Q_1)[N(r, \frac{1}{g}) - N_{\nu+1}(r, \frac{1}{g})] \\
& \quad + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g).
\end{aligned}$$

Hence the proof is complete.

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