Sixth Parameter Symmetry for a Second Order Ordinary Differential Equation

Abstract

Lie symmetry analysis is a powerful tool for solving ordinary differential equations whose results have been a huge driving force in the history of mathematics. Symmetry is an operation which leaves invariant that upon which it operates. A lie group or symmetry group is a group of transformations which maps any solution of the system to another solution of the same system. The symmetries of a given system of ordinary differential equations inform a lot about the closed form or the ability of the differential equation to solve. In this study, we use infinitesimal generators to obtain a sixth parameter symmetry for an ordinary differential equation. Symmetry properties and reduction of order differential equations yield solutions that are important in the field of science.

Keywords: Lie; symmetry; differential equations; infinitesimal generators; invariant.

1. Introduction

There exists various standard methods for solving linear second order ordinary differential equations. Equations with constant coefficients are solved using the roots of the characteristic equation. The Euler-Legendre type equations can be transformed into equations with constant coefficients. Some equations are exact and so can be directly integrated where as some can be directly integrated after multiplying with an integrating factor. For second-order equations with rational coefficients there is the well-known algorithm for finding solutions. Solutions to nonlinear second order or higher order ordinary differential involve computing the Lie symmetries and application of transformation rules that reduce the equations order systematically. In this paper, we use Lie's theory for solving our differential equation based on symmetries. In this paper, we shall consider the application of infinitesimal transformations to obtain symmetries of a second order differential equation below

$$y'' + 2yy' + y^2 = 0 \tag{1}$$

2. Symmetry

What qualifies something to be termed symmetric? Let's consider a square for that matter. Given its even number of equal length sides, a square appears symmetric. Nonetheless it's not. However, it is what we do to the square which qualifies it as symmetric. If we rotate it say 90 degrees, or even 180 degrees, or even turn it over its axis, the square seems like it has not been moved an inch. we look at these movements, singly and all together, of the square as transformations. These transformations retain the square in its position. Clearly this is a case of symmetry as a transformation that maintains the position of our square. The transformations demonstrated above, that keep the square in its exact position together form a group. These transformations must all preserve the defined property. Symmetry is an operation which leaves invariant that upon which it operates. Symmetry of a geometric object is a transformation which leaves the object apparently unchanged. For instance, consider the result of rotating an equilateral triangle anticlockwise about its center. After a rotation of $2\pi/3$, the triangle looks the same as it did before the rotation, so this transformation is symmetry. Rotations of $4\pi/3$ and 2π are also symmetries of the equilateral triangle. A symmetry of a given system of differential equations is basically a transformation of both the dependent and independent variables that maps solutions of the system into other similar solutions. The symmetries of a system can really be a nonlinear problem, unless we limit our operations to one-parameter groups of transformations. The transformation mapping each point to itself is a symmetry of any geometrical object: it is called trivial symmetry. Symmetry can be used to develop a concept that provides an algorithm for constructing physically important exact solutions. When dealing with symmetries of various systems of different differential equations, the main aim is to check when a particular differential equation is determined in a distinctive manner by use of symmetries. Nonlinear Phenomena have a plethora of applications in a myriad of areas in applied sciences. These nonlinear equations, are generally very difficult to solve explicitly. This now brings in the issue numerical methods, which with much success obtain approximate solutions to these equations. Nonetheless, there is great need currently to obtain exact solutions to our differential equations. This is where symmetry methods and techniques come in handy. They provide one method for solving these very important and life applicable differential equations. A variety of differential equations have symmetries that are clear and obvious like translations, rotations and several other symmetries that are geometric in nature. Most of these symmetries can be obtained easily. However, our interest will be in what we call hidden symmetries. These are symmetries that won't be obtained via elementary techniques. In principle, the symmetries of any given problem can be obtained by assuming a general form for the symmetry then followed by solving the invariance condition for that particular symmetry

3. Symmetry of differential equations

The subject of symmetry of ordinary differential equations was developed by Lie. A symmetry of a differential equation is a transformation mapping any solution to another solution of the differential equation. The symmetries of a given system of ordinary differential equations inform a lot about the closed form or the ability of the differential equation to solve. If an ordinary differential equation is invariant under some symmetry, then it's possible to obtain similar or invariant solution. The solutions got are as a result of solving equation having a reduced order. For problems that are nonlinear in nature, solutions that are analytical are not easy to get. Analytical solutions as compared to numerical solutions, provide a better understanding of the problem under question. This is where symmetry of the equation is calculated, and from the symmetries, similarity solutions can be obtained. In principle, the symmetries of a system are obtained by assuming a general form for the symmetry and going ahead to acquire the condition for invariance for the symmetry. A differential equation whose symmetry can be obtained can be solved. The obtaining of symmetries and their applications in solving nonlinear ordinary differential equations constitutes the very essence of our work. We call a transformation a symmetry if it satisfies the following conditions

(i) The transformation is structure preserving

(ii) The transformation is a smooth invertible mapping whose inverse is also smooth

(iii) The transformation maps the object to itself

4. Mathematical Formulation

We shall consider reduction of order by use of differential invariants. To deal with the infinitesimal transformation and functions involving derivatives we require extensions of the generator T . For a generator of order n the extension is of form

$$T^{[n]} = T + \sum_{i=1}^{i=n} \{\chi^{(i)} - \sum_{j=1}^{i} \binom{i}{j} y^{(i+1-j)} \phi^{(j)}\} \frac{\partial}{\partial y^{(i)}} \quad (2)$$

$$T^{[1]} = T + (\chi' - y'\phi') \frac{\partial}{\partial y'} \quad (3)$$

$$T^{[2]} = T^{[1]} + (\chi'' - 2y''\phi' - y'\phi'') \frac{\partial}{\partial y''} \quad (4)$$

Where the generator T is given by

$$T = \phi \frac{\partial}{\partial x} + \chi \frac{\partial}{\partial y} \quad (5)$$

And T is a symmetry of the differential equation (1). Subjecting (1) to the second extension (4), we get

$$T^{[2]}[y'' + 2yy' + y^2] = 0 \quad (6)$$

From (4), we have

$$T^{[2]} = T^{[1]} + (\chi'' - 2y''\phi' - y'\phi'') \frac{\partial}{\partial y''}$$

Substituting the value of $T^{[1]}$ in (4) above, we get

$$T^{[2]} = T + (\chi' - y'\phi')\frac{\partial}{\partial y'} + (\chi'' - 2y''\phi' - y'\phi'')\frac{\partial}{\partial y''} \qquad (7)$$

Since T is defined in (5), then equation (7) becomes

$$T^{[2]} = \phi\frac{\partial}{\partial x} + \chi\frac{\partial}{\partial y} + (\chi' - y'\phi')\frac{\partial}{\partial y'} + (\chi'' - 2y''\phi' - y'\phi'')\frac{\partial}{\partial y''} \qquad (8)$$

And so our equation (6) becomes

$$\left[\phi\frac{\partial}{\partial x} + \chi\frac{\partial}{\partial y} + (\chi' - y'\phi')\frac{\partial}{\partial y'} + (\chi'' - 2y''\phi' - y'\phi'')\frac{\partial}{\partial y''}\right]\left[y'' + 2yy' + y^2\right] = 0 \qquad (9)$$

Expanding and simplifying (9) we get

$$\phi\left[y''' + 2yy'' + 2y'^2 + 2yy'\right] + \chi\left[2y' + 2y\right] + \left[\chi' - y'\phi'\right]\left[2y\right] + \left[\chi'' - 2y''\phi' - y'\phi'' = 0\right] \qquad (10)$$

Equation (10) can be rewritten as

$$\phi\left[\frac{d}{dx}\left[y''\right] + 2yy'' + 2y'^2 + 2yy'\right] + 2y'\chi + 2y\chi + 2y\chi' - 2yy'\phi' + \chi'' - 2y''\phi' - y'\phi'' = 0 \qquad (11)$$

From (1), we can say that

$$y'' = -2yy' - y^2 \qquad (12)$$

$$\phi\left[\frac{d}{dx}\left[-2yy' - y^2\right] + 2yy'' + 2y'^2 + 2yy'\right] + 2y'\chi + 2y\chi + 2y\chi' - 2yy'\phi' + \chi'' - 2y''\phi' - y'\phi'' = 0$$

(13)

On expanding (13)

$$\phi\left[-2yy'' - 2y'^2 - 2yy' + 2yy'' + 2y'^2 + 2yy'\right] + 2y'\chi + 2y\chi + 2y\chi' - 2yy'\phi' + \chi'' - 2y''\phi' - y'\phi'' = 0$$

(14)

Simplifying (14) we obtain

$$-2yy''\phi - 2y'^2\phi - 2yy'\phi + 2yy''\phi + 2y'^2\phi + 2yy'\phi + 2y'\chi + 2y\chi + 2y\chi' - 2yy'\phi' + \chi'' - 2y''\phi' - y'\phi'' = 0$$

(15)

On further simplification, we get

$$2y'\chi + 2y\chi + 2y\chi' - 2yy'\phi' + \chi'' - 2y''\phi' - y'\phi'' = 0 \tag{16}$$

We know that $y'' = -2yy' - y^2$

We also know that

$$\chi' = \frac{\partial \chi}{\partial x} + y'\frac{\partial \chi}{\partial y} \tag{17}$$

And

$$\phi' = \frac{\partial \phi}{\partial x} + y'\frac{\partial \phi}{\partial y} \tag{18}$$

Using (12), (17) and (18), equation (16) becomes

$$2y'\chi + 2y\chi + 2y\left(\frac{\partial \chi}{\partial x} + y'\frac{\partial \chi}{\partial y}\right) - 2yy'\left(\frac{\partial \phi}{\partial x} + y'\frac{\partial \phi}{\partial y}\right) + \frac{\partial^2 \chi}{\partial x^2} + 2y'\frac{\partial^2 \chi}{\partial x \partial y} + y'^2\frac{\partial^2 \chi}{\partial y^2} + y''\frac{\partial \chi}{\partial y}$$

$$-2\left(-2yy' - y^2\right)\left(\frac{\partial \phi}{\partial x} + y'\frac{\partial \phi}{\partial y}\right) - y'\left(\frac{\partial^2 \phi}{\partial x^2} + 2y'\frac{\partial^2 \phi}{\partial x \partial y} + y'^2\frac{\partial^2 \phi}{\partial y^2} + y''\frac{\partial \phi}{\partial y}\right) = 0 \tag{19}$$

Equation (19) simplifies to

$$2y'\chi + 2y\chi + 2y\frac{\partial \chi}{\partial x} + 2yy'\frac{\partial \chi}{\partial y} - 2yy'\frac{\partial \phi}{\partial x} - 2yy'^2\frac{\partial \phi}{\partial y} + \frac{\partial^2 \chi}{\partial x^2} + 2y'\frac{\partial^2 \chi}{\partial x \partial y} + y'^2\frac{\partial^2 \chi}{\partial y^2}$$

$$+\left(-2yy'-y^{2}\right)\frac{\partial\chi}{\partial y}-2\left(-2yy'\frac{\partial\phi}{\partial x}-2yy'^{2}\frac{\partial\phi}{\partial y}-y^{2}\frac{\partial\phi}{\partial y}-y'y^{2}\frac{\partial\phi}{\partial y}\right)$$

$$-y'\frac{\partial^{2}\phi}{\partial x^{2}}-2y'^{2}\frac{\partial^{2}\phi}{\partial x\partial y}-y'^{3}\frac{\partial^{2}\phi}{\partial y^{2}}-y'y''\frac{\partial\phi}{\partial y}=0 \qquad (20)$$

This in turn gives us

$$2y'\chi+2y\chi+2y\frac{\partial\chi}{\partial x}+2yy'\frac{\partial\chi}{\partial y}+\frac{\partial^{2}\chi}{\partial x^{2}}+2y'\frac{\partial^{2}\chi}{\partial x\partial y}+y'^{2}\frac{\partial^{2}\chi}{\partial y^{2}}-y^{2}\frac{\partial\chi}{\partial y}+2y^{2}\frac{\partial\phi}{\partial x}+3y'y^{2}\frac{\partial\phi}{\partial y}-y'\frac{\partial^{2}\phi}{\partial x^{2}}$$

$$-2y'^{2}\frac{\partial^{2}\phi}{\partial x\partial y}-y'^{3}\frac{\partial^{2}\phi}{\partial y^{2}}=0 \qquad (21)$$

From (21) we come up with differential equations that are partial in nature. These partial differential are called determining equations

$$y'^{3}:\frac{\partial^{2}\phi}{\partial y^{2}}=0 \qquad (22)$$

$$y'^{2}:4y\frac{\partial\phi}{\partial y}+\frac{\partial^{2}\chi}{\partial y^{2}}-2\frac{\partial^{2}\phi}{\partial x\partial y}=0 \qquad (23)$$

$$y'^{1}:2\chi+2y\frac{\partial\phi}{\partial x}+2\frac{\partial^{2}\chi}{\partial x\partial y}+3y^{2}\frac{\partial\phi}{\partial y}-\frac{\partial^{2}\phi}{\partial x^{2}}=0 \qquad (24)$$

$$y'^{0}:2y\chi+2y\frac{\partial\chi}{\partial x}+2\frac{\partial^{2}\chi}{\partial x^{2}}-y^{2}\frac{\partial\chi}{\partial y}+2y^{2}\frac{\partial\phi}{\partial x}=0 \qquad (25)$$

From (22) we have

$$\frac{\partial^{2}\phi}{\partial y^{2}}=0$$

Or

$$\frac{\partial \phi}{\partial y} = 0$$

$$\phi = A_1 y + A_2 \qquad (26)$$

From (23) we deduce that

$$4y\frac{\partial \phi}{\partial y} + \frac{\partial^2 \chi}{\partial y^2} - 2\frac{\partial^2 \phi}{\partial x \partial y} = 0$$

Or $4yA_1 + \frac{\partial^2 \chi}{\partial y^2} - 2\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = 0$

Or $4yA_1 + \frac{\partial^2 \chi}{\partial y^2} - 2\frac{\partial}{\partial x}A_1 = 0$

Or $4yA_1 + \frac{\partial^2 \chi}{\partial y^2} - 2A_1' = 0$

Or $\frac{\partial^2 \chi}{\partial y^2} = 2A_1' - 4yA_1$

Or $\frac{\partial \chi}{\partial y} = 2A_1' y - 2y^2 A_1 + A_3$

And hence $\chi = A_1' y^2 - \frac{2}{3}A_1 y^3 + A_3 y + A_4 \qquad (27)$

From (24) we have

$$2\chi + 2y\frac{\partial \phi}{\partial x} + 2\frac{\partial^2 \chi}{\partial x \partial y} + 3y^2\frac{\partial \phi}{\partial y} - \frac{\partial^2 \phi}{\partial x^2} = 0$$

Or

$$2\chi + 2y\frac{\partial}{\partial x}\left(A_1 y + A_2\right) + 2\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right) + 3y^2 A_1 - \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) = 0$$

Or

$$2\chi + 2y\left(A_1' y + A_2'\right) + 2\frac{\partial}{\partial x}\left(2A_1' y - 2y^2 A_1 + A_3\right) + 3y^2 A_1 - \frac{\partial}{\partial x}\left(A_1' y + A_2'\right) = 0$$

Or

$$2\chi + 2y^2 A_1' + 2yA_2' + 2\left(2A_1'' y - 2A_1' y^2 + A_3\right) + 3y^2 A_1 - \left(A_1'' y + A_2''\right) = 0$$

Or

$$2A_1' y^2 - \frac{4}{3} y^3 A_1 + 2A_3 y + 2A_4 + 2y^2 A_1' + 2yA_2' + 4A_1'' y - 4A_1' y^2 + 2A_3' + 3y^2 A_1 - A_1'' y - A_2'' = 0$$

Which solves to

$$-\frac{4}{3} A_1 y^3 + 3A_1 y^2 + 2A_3 y + 2A_2' y + 4A_1'' y - A_1'' y + 2A_4 + 2A_3' - A_2'' = 0 \qquad (28)$$

From (28) we have the following determining equations

$$y^3: \quad -\frac{4}{3} A_1 = 0 \qquad (29)$$

$$y^2: 3A_1 = 0 \qquad (30)$$

$$y^1: 2A_3 + 2A_2' + 4A_1'' = 0 \qquad (31)$$

$$y^0: 2A_4 + 2A_3' - A_2'' = 0 \qquad (32)$$

From (25) we have

$$2y\left(A_1' y^2 - \frac{2}{3}A_1 y^3 + A_3 y + A_4\right) + 2y\left(A_1'' y^2 - \frac{2}{3}A_1' y^3 + A_3' y + A_4'\right)$$
$$+\left(A_1''' y^2 - \frac{2}{3}A_1'' y^3 + A_3'' y + A_4''\right) - y^2\left(2A_1' y - 2y^2 A_1 + A_3\right) \quad (33)$$
$$+2y^2\left(A_1' y + A_2'\right) = 0$$

Which simplifies to

$$2y^3 A_1' - \frac{4}{3}y^4 A_1 + 2A_3 y^2 + 2A_4 y + 2y^3 A_1''' - \frac{4}{3}y^4 A_1' + 2A_3' y^2$$

$$+2yA_4' + A_1''' y^2 - \frac{2}{3}A_1'' y^3 + A_3'' y + A_4'' - 2A_1' y^3 + 2y^2 A_1 - A_3 y^2 \quad (34)$$

$$+2A_1' y^3 + 2A_2' y^2 = 0$$

From (34) we obtain the determining equations

$$y^4 : -\frac{4}{3}A_1 - \frac{4}{3}A_1' + 2A_1 = 0 \quad (35)$$

$$y^3 : 2A_1' + 2A_1'' - \frac{2}{3}A_1'' - 2A_1' + 2A_1' = 0 \quad (36)$$

$$y^2 : A_3 + 2A_3' + A_1''' = 0 \quad (37)$$

$$y^1 : 2A_4 + 2A_4' + A_3'' = 0 \quad (38)$$

$$y^0 : A_4'' = 0 \quad (39)$$

From (39) we have

$$A_4' = P_1$$

Or

$$A_4 = P_1 x + P_2 \tag{40}$$

From equations (38) we have

$$2A_4 + 2A_4' + A_3'' = 0$$

$$A_3'' = -2A_4 - 2A_4'$$

$$A_3'' = -2(P_1 x + P_2) - 2(P_1)$$

$$A_3'' = -2P_1 x - 2P_2 - 2P_1$$

$$A_3' = -P_1 x^2 - 2P_2 x - 2P_1 x + P_3$$

$$A_3 = -\frac{1}{3}P_1 x^3 - P_1 x^2 + 2P_2 x^2 + P_3 x + P_4 \tag{41}$$

From equations (38) we have

$$2A_4 + 2A_3' - A_2'' = 0$$

$$A_2'' = 2A_3' + 2A_4$$

$$A_2'' = -2P_1 x^2 - 4P_2 x - 4P_1 x + 2P_3 + 2P_1 x + 2P_2$$

$$A_2' = -\frac{2}{3}P_1 x^3 - 2P_2 x^2 - 2P_1 x^2 + 2P_3 x + P_1 x^2 + 2P_2 x + P_5$$

And hence

$$A_2 = -\frac{1}{6}P_1 x^4 - \frac{2}{3}P_2 x^3 - \frac{2}{3}P_1 x^3 + P_3 x^2 + \frac{1}{3}P_1 x^3 + P_2 x^2 + P_5 x + P_6 \tag{42}$$

From (29) and (30) we have

$$A_1 = 0 \tag{43}$$

Using equations (26) and (27) we can deduce that

$$\phi = -\frac{1}{6}P_1x^4 - \frac{2}{3}P_2x^3 - \frac{2}{3}P_1x^3 + P_3x^2 + \frac{1}{3}P_1x^3 + P_2x^2 + P_5x + P_6$$

And

$$\chi = -\frac{1}{3}P_1x^3y - P_1x^2y - P_2x^2y + P_3xy + P_4y + P_1x + P_2$$

Our generator is given by

$$T = \phi\frac{\partial}{\partial x} + \chi\frac{\partial}{\partial y}$$

This expands to

$$T = \left(-\frac{1}{6}P_1x^4 - \frac{2}{3}P_2x^3 - \frac{2}{3}P_1x^3 + P_3x^2 + \frac{1}{3}P_1x^3 + P_2x^2 + P_5x + P_6\right)\frac{\partial}{\partial x}$$

$$+\chi\left(-\frac{1}{3}P_1x^3y - P_1x^2y - P_2x^2y + P_3xy + P_4y + P_1x + P_2\right)\frac{\partial}{\partial y} \tag{44}$$

This simplifies to

$$T = -\frac{1}{6}P_1x^4\frac{\partial}{\partial x} - \frac{2}{3}P_2x^3\frac{\partial}{\partial x} - \frac{2}{3}P_1x^3\frac{\partial}{\partial x} + P_3x^2\frac{\partial}{\partial x} + \frac{1}{3}P_1x^3\frac{\partial}{\partial x}$$

$$+P_2x^2\frac{\partial}{\partial x} + P_5x\frac{\partial}{\partial x} + P_6\frac{\partial}{\partial x} - \frac{1}{3}P_1x^3y\frac{\partial}{\partial y} - P_1x^2y\frac{\partial}{\partial y}$$

$$-P_2x^2y\frac{\partial}{\partial y} + P_3xy\frac{\partial}{\partial y} + P_4y\frac{\partial}{\partial y} + P_1x\frac{\partial}{\partial y} + P_2\frac{\partial}{\partial y} \tag{45}$$

Or

$$T = P_1\left(-\frac{1}{6}x^4\frac{\partial}{\partial x}-\frac{2}{3}x^3\frac{\partial}{\partial x}+\frac{1}{3}P_1x^3\frac{\partial}{\partial x}-\frac{1}{3}x^3y\frac{\partial}{\partial y}-x^2y\frac{\partial}{\partial y}+x\frac{\partial}{\partial y}\right)$$

$$+P_2\left(-\frac{2}{3}x^3\frac{\partial}{\partial x}+x^2\frac{\partial}{\partial x}-x^2y\frac{\partial}{\partial y}+\frac{\partial}{\partial y}\right)$$

$$+P_3\left(x^2\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}\right)+P_4\left(y\frac{\partial}{\partial y}\right)+P_5\left(x\frac{\partial}{\partial x}\right)+P_6\left(\frac{\partial}{\partial x}\right) \quad (46)$$

5. Results and Discussion

Equation (46) is a sixth-parameter symmetry from which we can obtain the following symmetries

$$T_1 = \frac{\partial}{\partial x} \quad (47)$$

$$T_2 = x\frac{\partial}{\partial x} \quad (48)$$

$$T_3 = y\frac{\partial}{\partial y} \quad (49)$$

$$T_4 = x^2\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y} \quad (50)$$

$$T_5 = -\frac{2}{3}x^3\frac{\partial}{\partial x}+x^2\frac{\partial}{\partial x}-x^2\frac{\partial}{\partial x}-x^2y\frac{\partial}{\partial y}-\frac{\partial}{\partial y} \quad (51)$$

$$T_6 = -\frac{1}{6}x^4\frac{\partial}{\partial x}-\frac{2}{3}x^3\frac{\partial}{\partial x}+\frac{1}{3}x^3\frac{\partial}{\partial x}-\frac{1}{3}x^3y\frac{\partial}{\partial y}-x^2y\frac{\partial}{\partial y}+x\frac{\partial}{\partial y} \quad (52)$$

The Lie symmetry method plays a vital role in obtaining reductions of order of differential equations. Prolongations are used to develop symmetries that we later use to reduce the order of a differential equation. These symmetries will later be used to come up with exact solution to our problem. The biggest challenge in the symmetry methods is to obtain symmetries of differential equations. The work is tedious and voluminous. The greatest handicap to the application of symmetries is the huge volume of calculations required to solve even a modest problem.

6. Conclusion

In this paper, we have obtained a sixth-parameter symmetry to our ordinary differential equation. Mathematical symmetry patterns help to obtain analytic solutions to differential equations. The advantage of the symmetry method is that it can be applied successfully to non-linear differential equations.

7. References

Aminov A.V., & Aminov N.A. (2006). Projectile Geometry Theory of Systems of Second Order Differential Equations. Sb. Math. 197, 951-975.

Arrigo D.J. (2014). Symmetry Analysis of Differential Equations. Wiley, New Jersey.

Baikov V.A., Gazizov R.K., & Ibragimov N. H. (1989). Approximate Symmetries with a Smaller Parameter. Math. U.S.S.R-Sb 64, 427-411.

Bluman G.W., & Kumei S. (1989). Symmetries and Differential Equations. Springer-Verlag New York, NY, USA.

Bronstein M. (1992). Linear Ordinary Differential equations: breaking through the order 2 Barrier. Proc. ISSAC’92, 42-48.

Campoamor-Stursberg R. (2012). System of Second-Order Linear ODE’s With Constant Coefficients and Their Symmetries. Commun. Nonlinear Linear Sci. Numer. Simul. 16, 3015-3023.

Doudrov B., & Medvedev A. (2014). Fundamental Invariants of Systems of ODE's of Higher Order. Differential Geom. Appl. 35, 123-143.

Gungor F., & Torres P.J. (2017). Lie Symmetry Analysis of a Second Order Differential Equation with Singularity. J.Math. Anal. Appl. 451, 976-989.

Kovaic J. (1986). An algorithm for solving second-order linear homogeneous differential equations. J. Symbolic Computation 2, 3-43.

Mahomed F.M. Symmetry Group Classification of Ordinary Differential Equations. Math. Methods Appl. 151, 80-107.

Olver P.J. (1986). Applications of Lie Groups to Differential Equations. Springer: New York, NY, USA.

Sebert F. (2021). Symmetry Analysis for a Second-Order Ordinary Differential Equation. Electronic Journal of Differential Equations. 85, 1-12.

Stephani H. (1989). Differential Equations: Their Solution using Symmetries. Cambridge University Press.

Vyacheslav M. B., Roman O. P., & Nataliya M.S. (2013). Lie Symmetries of Systems of Second-order linear Ordinary Differential Equations with Constant Coefficients. J.Math. Anal. Appl. 397, 434-440.

Vyacheslav M. B., Oleksandra V.L., & Roman O. P. (2024). Admissible Transformations and Lie Symmetries of Linear Systems of Second-Order Ordinary Differential Equations. J.Math. Anal. Appl. 539(2): 128543.