Original Research Article

Applying Banach Algebra for Generalized $(\alpha - \Psi)$-Contractive Type Maps in Cone Metric Spaces

Abstract

Aims/ objectives: In this research, we initially present a novel mapping within the framework of cone metric spaces associated with Banach algebra, referred to as generalized $(\alpha - \Psi)$-contractive type mappings. Subsequently, a number of fixed point theorems related to $(\alpha - \Psi)$-contractive type mappings in these cone metric spaces over Banach algebra are generalized and expanded.

Keywords: α-admissible mapping, Banach algebras, normal cone, cone metric spaces, generalized $(\alpha - \Psi)$-contractive type mappings.

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1 Introduction

Cone metric spaces, which are a generalization of conventional metric spaces by replacing real numbers with ordered Banach spaces, were recently introduced by Huang and Zhang

(2). A fixed point theorem that applies in these spaces was established by them. These results have been further developed by later authors (6; 7; 12; 11). Interestingly, Abbas and Jungck (1) and Vetro (12) studied common fixed points for non-commuting mappings in the context of normal cone metric spaces.

It was shown by Rezapour and Hamlbarani (7) that normal cones with a normal constant. Furthermore, a number of writers have elaborated on specific definitions and findings in relation to cone metric spaces (7; 6; 12; 13). We refer the reader to (10; 1; 3) for newly developed fixed point theorems about cone metric spaces.

This study introduces various fixed point theorems for Generalized $(\alpha - \Psi)$-contractive mappings within cone metric spaces associated with Banach algebras, while also defining the concept of these mappings. Additionally, we offer example to support our conclusions.

2 Preliminaries

Definition 2.1. (See (9; 5)) Let us assume that $\mathcal{A}$ is consistently a real Banach algebra. This implies that $\mathcal{A}$ functions as a Banach space in which a multiplication operation is established, exhibiting the following properties for all elements $\zeta, \xi, \upsilon \in \mathcal{A}$ and for any α belonging to $\mathbb{R}$.

(1) $\zeta(\xi\upsilon) = (\zeta\xi)\upsilon$;

(2) $\zeta\xi + \zeta\upsilon = \zeta(\xi + \upsilon)$ and $\zeta\upsilon + \xi\upsilon = (\zeta + \xi)\upsilon$;

(3) $(\alpha\zeta)\xi = \alpha(\zeta\xi) = \zeta(\alpha\xi)$;

(4) $||\zeta\xi|| \leq ||\zeta||.||\xi||$.

In the context of a Banach algebra, we assume the existence of a unit, referred to as e, which satisfies the property that $e\zeta = \zeta e = \zeta$ for all elements $\zeta \in A$. An element ζ in the algebra A is considered invertible if there exists an element $\xi \in A$ that acts as its inverse, satisfying the equation $\zeta\xi = \xi\zeta = e$. The inverse of ζ is denoted by ζ^{-1}.

Proposition 2.1. *(See(9; 5)) Let $\mathcal{A}$ denote a Banach algebra equipped within a unit element e, and let ζ be an element of $\mathcal{A}$. If the spectral radius $\rho(\zeta)$ of the element $\zeta < 1$, that is,*

$$\rho(\zeta) = \lim_{n\to\infty} ||\zeta_n||^{\frac{1}{n}} = \inf ||\zeta^n||^{\frac{1}{n}} < 1$$

Consequently, the expression $(e - \zeta)$ is invertible, and its inverse is given by $(e - \zeta)^{-1} = \sum_{i=0}^{\infty} \zeta^i$.

A subset $\mathcal{P}$ of $\mathcal{A}$ is referred to as a cone if

(1) $\mathcal{P}$ is closed, non-empty, and $\{\theta, e\} \subset \mathcal{P}$, where θ is $\mathcal{A}$'s zero vector;

(2) $\mathcal{P}\mathcal{P} = \mathcal{P}^2 \subset \mathcal{P}$;

(3) For any non-negative real numbers β and α exists such that $\alpha\mathcal{P} + \beta\mathcal{P} \subset \mathcal{P}$,

(4) $(-\mathcal{P}) \cap (\mathcal{P}) = \{\theta\}$.

For a designated cone $\mathcal{P}$ that is a subset of $\mathcal{A}$, it is possible to define a partial ordering $\preceq$ in relation to $\mathcal{P}$ such that the relation $\zeta \preceq \xi$ is satisfied if and only if $\xi - \zeta \in \mathcal{P}$. The notation $\zeta \ll \xi$ signifies that $\xi - \zeta \in \mathcal{P}^o$, where $\mathcal{P}^o$ denotes the interior of the cone $\mathcal{P}$.

The cone $\mathcal{P}$ is typically designated as such when there exists a constant $\mathcal{K} > 0$ such that for any elements $\alpha, \beta \in \mathcal{A}$, the relation $\alpha \preceq \beta$ implies that $||\alpha|| \leq \mathcal{K}||\beta||$.

The smallest positive value of $\mathcal{K}$ that satisfies the aforementioned inequality is referred to as the normal constant (see (2)). It is important to note that for any normal cone P, the condition $\mathcal{K} \geq 1$ holds true (refer to (7)). In the subsequent discussion, we will define $\mathcal{P}$ as a cone within a real Banach algebra $\mathcal{A}$, where $\mathcal{P}^o \neq \phi$ (indicating that the cone $\mathcal{P}$ is solid) and $\preceq$ denotes the partial ordering associated with $\mathcal{P}$.

Definition 2.2. (See (2)) Let $\mathcal{U}$ represent a non-empty set. Assume that a function $d_c : \mathcal{U} \times \mathcal{U} \to \mathcal{A}$ fulfills the following conditions:

(1) For all $\zeta, \xi \in \mathcal{X}$, $\theta \preceq d_c(\zeta, \xi)$ and $d_c(\zeta, \xi) = \theta$ only in the event that $\zeta = \xi$;

(2) $d_c(\zeta, \xi) = d_c(\xi, \zeta), \forall \zeta, \xi \in \mathcal{U}$;

(3) $d_c(\zeta, \xi) \preceq d_c(\zeta, \upsilon) + d_c(\upsilon, \xi)$ for each $\zeta, \xi, \upsilon \in \mathcal{X}$.

A cone metric defined on the set $\mathcal{U}$ is represented by d_c, and the combination $(\mathcal{U}, d_c)$ is termed a cone metric space over the Banach algebra $\mathcal{A}$ (abbreviated as **CMSBA**). It is noteworthy that for any two elements ζ and ξ in $\mathcal{U}$, the value $d_c(\zeta, \xi)$ is an element of the set $\mathcal{P}$.

Definition 2.3. (See (2; 5)) Let $(\mathcal{U}, d_c)$ represent a cone metric spaces, where $\zeta \in \mathcal{U}$ and $\{\zeta_n\}$ denotes a sequence within $\mathcal{U}$. Consequently:

(1) The sequence $\{\zeta_n\}$ is said to converge to ζ if, for every $c \in \mathcal{A}$ with $\theta \ll c$, there exists an integer $n_0 \in \mathbb{N}$ so that $c \gg d_c(\zeta_n, \zeta)$ holds for every $n_0 < n$. That is expressed as $\lim_{n\to\infty} \zeta_n = \zeta$ or $\zeta_n \to \zeta$ as $n \to \infty$.

(2) The sequence $\{\zeta_n\}$ is classified as a Cauchy sequence if, for every $c \in \mathcal{A}$ where $c \gg \theta$, there exists an integer $n_0 \in \mathbb{N}$ so that $d_c(\zeta_n, \zeta_m) \ll c$ holds true for all $n_0 < n, m$.

(3) A complete cone metric spaces is defined as $(\mathcal{U}, d_c)$ if every Cauchy sequence contained in $\mathcal{U}$ converges..

Lemma 2.1. *(See(2)) Let $(\mathcal{U}, d_c)$ represent a cone metric spaces, and let $\mathcal{P}$ denote a normal cone characterized by a usual constant $\mathcal{K}$. Consider the sequence ζ_n within the space $\mathcal{U}$. Then.*

(1) The sequence ζ_n is said to converge to ζ iff the distance $d_c(\zeta_n, \zeta)$ approaches 0 as n approaches infinity.

(2) A sequence ζ_n is classified as a Cauchy sequence iff the distance $d_c(\zeta_n, \zeta_m)$ approaches zero as both m and n tend to infinity.

Definition 2.4. (See(2; 5)) Let $(\mathcal{U}, d_c)$ represent a cone metric spaces. If every Cauchy sequence within $\mathcal{U}$ converges, then $\mathcal{U}$ is referred to as a complete cone metric spaces.

Lemma 2.2. *(See(2))Let $(\mathcal{U}, d_c)$ represent a cone metric spaces, and let $\mathcal{P}$ denote a normal cone characterized by a normal constant $\mathcal{K}$. Consider the sequences ζ_n and ζ_n within the space $\mathcal{U}$.*

(1) If the sequence ζ_n approaches the value ζ and simultaneously converges to the value ξ, it follows that ζ must equal ξ. This indicates that the limit of the sequence ζ_n is unique, and it is evident that the limit of the sequence ξ_n is also unique.

(2) The sequences ζ_n and ξ_n converge to ζ and ξ, respectively as n approaches infinity, then the distance $d_c(\zeta_n, \zeta_m)$ converges to $d_c(\zeta, \xi)$ as n approaches infinity.

Lemma 2.3. *Let $(\mathcal{U}, d_c)$ represent a cone metric spaces, and let $\mathcal{P}$ be a normal cone with normal constant K. The sequences ζ_n and ξ_n in $\mathcal{U}$ with $\zeta_n \to \zeta$ and $\xi_n \to \xi$ as $n \to \infty$ then $d_c(\zeta_n, \xi_n) \to d_c(\zeta, \xi)$ as $n \to \infty$.*

Example 2.4. *(2) Consider a Banach space $E = R^2$, the cone $\mathcal{P} = \{(\zeta, \xi) \in E | \zeta, \xi \geq 0\} \subset \mathbb{R}^2, \mathcal{U} = R$ and $d_c : \mathcal{U} \times \mathcal{U} \to E$ for $\beta \geq 0$ a constant such that $d_c(\zeta, \xi) = (|\zeta - \xi|, \alpha|\zeta - \xi|)$, then $(\mathcal{U}, d_c)$ is a cone metric spaces.*

Recently, Samet and colleagues (10) presented the concept of $(\alpha - \psi)$-contractive mappings and α-admissible mappings within the context of metric spaces, defined as follows:

Definition 2.5. (10) We say that $\mathcal{J}$ is α-admissible if $\zeta, \xi \in \mathcal{U}, \alpha(\zeta, \xi) \geq 1$ implies $d_c(\mathcal{J}\zeta, \mathcal{J}\xi) \geq 1$. Let $\mathcal{J} : \mathcal{U} \to \mathcal{U}$ and $\alpha : \mathcal{U} \times \mathcal{U} \to [0, \infty]$,.

The non-decreasing function $\psi : [0, \infty) \to [0, \infty]$ is represented as Ψ. For any $t > 0$, $\sum_{n=1}^{\infty} \psi_n < +\infty$, where ψ^n is the nth iteration of ψ.

Lemma 2.5. *(10) For any function $\psi : [0, \infty) \to [0, \infty)$, the following statement is true: If ψ is non-decreasing, then for every $t > 0$, the condition $\lim_{n\to\infty} \psi^n(t) = 0$ necessitates that $\psi(t) = t$ and $\psi(0) = 0$.*

Definition 2.6. (10)Let $(\mathcal{U}, d_c)$ represent a metric space, and let $\mathcal{J} : \mathcal{U} \to \mathcal{U}$ denote a mapping. The mapping $\mathcal{J}$ is classified as an $\alpha - \psi$-contractive mapping if there exist two functions, $\alpha : \mathcal{U} \times \mathcal{U} \to [0, \infty)$ and $\psi \in \Psi$, such that the following conditions hold.

$$\alpha(\zeta, \xi) d_c(\mathcal{J}\zeta, \mathcal{J}\xi) \leq \psi d_c(\zeta, \xi)$$

for all $\zeta, \xi \in \mathcal{U}$.

Kang et al. (4) subsequently present the concept of this mapping within the context of cone metric spaces as follows:

Definition 2.7. (4) Let $(\mathcal{U}, d_c)$ represent a cone metric space, and let P denote a normal cone characterized by a normal constant K. Consider a mapping $\mathcal{J} : \mathcal{U} \to \mathcal{U}$. The mapping $\mathcal{J}$ is classified as an $\alpha - \psi$-contractive mapping if there exist two functions, $\alpha : \mathcal{U} \times \mathcal{U} \to [0, \infty)$ and $\psi \in \Phi$, that satisfy certain conditions.

$$\alpha(\zeta, \xi) d_c(\mathcal{J}\zeta, \mathcal{J}\xi) \leq \psi d_c(\zeta, \xi)$$

for all $\zeta, \xi \in \mathcal{U}$.

We now introduce a novel concept of generalized $(\alpha - \Psi)$-contractive mappings within cone metric spaces associated with Banach algebras, and we establish fixed point theorems for these mappings in the context of cone metric spaces.

Definition 2.8. Let $(\mathcal{U}, d_c)$ represent a cone metric space, and let P denote a normal cone characterized by a normal constant K. Consider a mapping $\mathcal{J} : \mathcal{U} \to \mathcal{U}$. The mapping $\mathcal{J}$ is classified as an $(\alpha - \psi)$-contractive mapping if there exist two functions, $\alpha : \mathcal{U} \times \mathcal{U} \to [0, \infty)$ and $\psi \in \Psi$, such that the following condition holds.

$$\alpha(\zeta, \xi) d_c(\mathcal{J}\zeta, \mathcal{J}\xi) \leq \psi \mathcal{N}(\zeta, \xi) \tag{2.1}$$

for all $\zeta, \xi \in \mathcal{U}$. Where $\mathcal{N}(\zeta, \xi) = max\left[d_c(\zeta, \xi), \frac{d_c(\zeta, \mathcal{J}\zeta) + d_c(\xi, \mathcal{J}\xi)}{2}, \frac{d_c(\zeta, \mathcal{J}\xi) + d_c(\xi, \mathcal{J}\zeta)}{2}\right]$

3 Main Result

Samet (10) established the subsequent theorem.

Theorem 3.1. *Let $(\mathcal{U}, d_c)$ represent a complete metric space, and let $\mathcal{J} : \mathcal{U} \to \mathcal{U}$ denote an $\alpha - \psi$ contractive mapping that fulfills the subsequent conditions:*

1. *$\mathcal{J}$ is α-admissible,*
2. *There exists $\zeta_0 \in \mathcal{U}$ such that $\alpha(\zeta_0, \mathcal{J}\zeta_0) \geq 1$,*
3. *$\mathcal{J}$ is continuous*

If the sequence $\{\zeta_n\}$ is contained within $\mathcal{U}$ and satisfies the condition that $\alpha(\zeta_n, \zeta_{n+1}) \geq 1$ for every integer n, along with the convergence $\zeta_n \to \zeta$ as n approaches infinity, it follows that $\alpha(\zeta_n, \zeta) \geq 1$ for all n. Consequently, this implies that the set $\mathcal{J}$ possesses a fixed point.

Recently, Kang et al.(4) formulated the following theorem in the framework of Cone metric space.

Theorem 3.2. *Let $(\mathcal{U}, d_c)$ represent a complete cone metric space, and let P denote a normal cone characterized by a normal constant K. Consider the mapping $\mathcal{J} : \mathcal{U} \to \mathcal{U}$, which is an $(\alpha - \Psi)$-contractive mapping that fulfills the subsequent conditions:*

1. *$\mathcal{J}$ is α-admissible,*
2. *There exists $\zeta_0 \in \mathcal{U}$ such that $\alpha(\zeta_0, \mathcal{J}\zeta_0) \geq 1$,*
3. *$\mathcal{J}$ is continuous*

If the sequence $\{\zeta_n\}$ is contained within $\mathcal{U}$ and satisfies the condition that $\alpha(\zeta_n, \zeta_{n+1}) \geq 1$ for every integer n, along with the convergence $\zeta_n \to \zeta$ as n approaches infinity, it follows that $\alpha(\zeta_n, \zeta) \geq 1$ for all n. Consequently, this implies that the set $\mathcal{J}$ possesses a fixed point.

We shall now illustrate Theorem 3.2 in the context of generalized $(\alpha - \Psi)$-contractive mappings within cone metric spaces, as detailed below.

Theorem 3.3. *Let $(\mathcal{U}, d_c)$ represent a complete cone metric space associated with a Banach algebra, and let P denote a normal cone characterized by a normal constant K. Consider $\mathcal{J} : \mathcal{U} \to \mathcal{U}$ as any generalized $\alpha - \psi$ contractive mapping that fulfills the subsequent conditions:*

1. *$\mathcal{J}$ is α-admissible,*
2. *There exists $\zeta_0 \in \mathcal{U}$ such that $\alpha(\zeta_0, \mathcal{J}\zeta_0) \geq 1$,*
3. *$\mathcal{J}$ is continuous*

If the sequence $\{\zeta_n\}$ is contained within $\mathcal{U}$ and satisfies the condition that $\alpha(\zeta_n, \zeta_{n+1}) \geq 1$ for every integer n, along with the convergence $\zeta_n \to \zeta$ as n approaches infinity, it follows that $\alpha(\zeta_n, \zeta) \geq 1$ for all n. Consequently, this implies that the set $\mathcal{J}$ possesses a fixed point.

Proof. Let $\zeta_0 \in \mathcal{U}$ such that $\alpha(\zeta_0, \mathcal{J}\zeta_0) \geq 1$. We will define a sequence $\{\zeta_n\}$ within the set $\mathcal{U}$ such that

$$\mathcal{J}\zeta_n = \zeta_{n+1} \tag{3.1}$$

For certain values of n in the set of natural numbers $\mathbb{N}$, it is noteworthy that if $\zeta_n = \zeta_{n+1}$ for some $n \in \mathbb{N}$, then the sequence $\{x_n\}$ represents a fixed point for the operator $\mathcal{J}$. However, if we assume that $\zeta_n \neq \zeta_{n+1}$ for all $n \in \mathbb{N}$, it follows that, given the α-admissibility of $\mathcal{J}$, we can conclude the following.

$$\alpha(\zeta_0, \zeta_1) = \alpha(\zeta_0, \mathcal{J}\zeta_1) \geq 1$$

$$\alpha(\mathcal{J}\zeta_0, \mathcal{J}\zeta_1) = \alpha(\zeta_1, \zeta_2) \geq 1$$

By induction, we get

$$\alpha(\zeta_n, \zeta_{n+1} \geq 1, \ \forall n \in N \tag{3.2}$$

By now utilizing inequalities **??** and 3.1, we derive

$$\begin{aligned} d_c(\zeta_n, \zeta_{n+1}) &= d_c(\mathcal{J}\zeta_{n-1}, \mathcal{J}\zeta_n) \\ &\preceq \alpha(\zeta_{n-1}, \zeta_n) d_c(\mathcal{J}\zeta_{n-1}, \mathcal{J}\zeta_n) \\ &\preceq \psi\mathcal{N}(\zeta_{n-1}, \zeta_n) \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \mathcal{N}(\zeta_{n-1}, \zeta_n) = \max\Big\{ & d_c(\zeta_{n-1}, \mathcal{J}\zeta_{n-1}), \frac{d_c(\zeta_{n-1}, \mathcal{J}\zeta_{n-1}) + d_c(\zeta_n, \mathcal{J}\zeta_n)}{2}, \\ & \frac{d_c(\zeta_{n-1}, \mathcal{J}\zeta_n) + d_c(\zeta_n, \mathcal{J}\zeta_{n-1})}{2} \Big\} \\ = \max & \left\{ d_c(\zeta_{n-1}, \zeta_n), \frac{d_c(\zeta_{n-1}, \zeta_n) + d_c(\zeta_n, \zeta_{n+1})}{2}, \frac{d_c(\zeta_{n-1}, \zeta_{n+1})}{2} \right\} \\ \preceq \max & \{ d_c(\zeta_{n-1}, \zeta_n), d_c(\zeta_n, \zeta_{n+1}) \} \end{aligned}$$

Due to the monotonic properties of the function ψ and by applying the inequalities 3.1 and 3.3, it can be inferred that this holds true for all $n \geq 1$.

$$d_c(\zeta_n, \zeta_{n+1}) \preceq \psi \max\{d_c(\zeta_{n-1}, \zeta_n), d_c(\zeta_n, \zeta_{n+1})\} \quad (3.4)$$

For any integer n such that $n \geq 1$, it holds that.

$$d_c(\zeta_{n-1}, \zeta_n) \prec d_c(\zeta_n, \zeta_{n+1})$$

Subsequently, 3.4 is transformed into.

$$d_c(\zeta_n, \zeta_{n+1}) \preceq \psi d_c(\zeta_n, \zeta_{n+1})$$

Which implies

$$||d_c(\zeta_n, \zeta_{n+1})|| \preceq ||\psi d_c(\zeta_n, \zeta_{n+1})|| \prec ||d_c(\zeta_n, \zeta_{n+1})||$$

This presents a contradiction. Therefore, for every $n \geq 1$, we conclude.

$$\max\{d_c(\zeta_{n-1}, \zeta_n), d_c(\zeta_n, \zeta_{n+1})\} = d_c(\zeta_{n-1}, \zeta_n) \quad (3.5)$$

Considering equqtion 3.5 and 3.4, we obtain the result for all $n \geq 1$.

$$d_c(\zeta_n, \zeta_{n+1}) \preceq \psi d_c(\zeta_{n-1}, \zeta_n) \quad (3.6)$$

By extending this process inductively, we derive.

$$d_c(\zeta_n, \zeta_{n+1}) \preceq ...\psi^n d_c(\zeta_0, \zeta_1) \qquad \forall n \in N \quad (3.7)$$

For values of n greater than m, we can derive the result by applying equation 3.7 along with the triangle inequality.

$$\begin{aligned} d_c(\zeta_n, \zeta_m) &\preceq d_c(\zeta_n, \zeta_{n-1}) + d_c(\zeta_{n-1}, \zeta_{n-2}) + ... + d_c(\zeta_m, \zeta_{m+1}) \\ &\preceq \psi^{n-1} d_c(\zeta_0, \zeta_1) + \psi^{n-2} d_c(\zeta_0, \zeta_1) + ... + \psi^m d_c(\zeta_0, \zeta_1) \\ &\preceq (\psi^{n-1} + \psi^{n-2} + ... + \psi^m) d_c(\zeta_0, \zeta_1) \\ &\preceq \psi(e - \psi)^{-1} d_c(\zeta_0, \zeta_1) \end{aligned}$$

Considering that P is a normal cone characterized by a normal constant K, we deduce that

$$||d_c(\zeta_n, \zeta_m)|| \preceq K||\psi(e - \psi)^{-1} d_c(\zeta_0, \zeta_1)||$$

As n and m approach infinity, it follows that $d_c(\zeta_n, \zeta_m)$ converges to θ. Therefore, the sequence $\{\zeta_n\}$ qualifies as a Cauchy sequence within the cone metric space $(\mathcal{U}, d_c)$. Given that $(\mathcal{U}, d_c)$ is a complete space, there exists an element $\zeta^* \in \mathcal{U}$ such that ζ_n converges to ζ^* as n approaches infinity.

case-1: If the operator $\mathcal{J}$ is continuous, it follows that $\zeta_{n-1} = \mathcal{J}\zeta_n$ converges to $\mathcal{J}\zeta^*$ as n approaches infinity. Due to the uniqueness of limits, we have $\mathcal{J}\zeta^* = \zeta^*$. Therefore, ζ^* is identified as a fixed point of the operator $\mathcal{J}$.

case-2:If the sequence $\{\zeta_n\}$ is contained within $\mathcal{J}$ and satisfies the condition that $\alpha(\zeta_n, \zeta_{n+1}) \geq 1$ for every integer n, along with the convergence $\zeta_n \to \zeta$ as n approaches infinity, it follows that $\alpha(\zeta_n, \zeta) \geq 1$ for all integers n.

We will now demonstrate that $d_c(\mathcal{J}\zeta^*, \zeta^*) \geq 0$ as n approaches infinity. Conversely, let us assume that $d_c(\mathcal{J}\zeta^*, \zeta^*) > 0$. We proceed from this assumption.

$$\begin{aligned} d_c(\mathcal{J}\zeta^*, \zeta^*) &\preceq d_c(\mathcal{J}\zeta_n, \mathcal{J}\zeta^*) + d_c(\mathcal{J}\zeta_n, \zeta^*) \\ &C\alpha(\zeta_n, \zeta^*) d_c(\mathcal{J}\zeta_n, \mathcal{J}\zeta^*) + d_c(\mathcal{J}\zeta_n, \zeta^*) \\ &\preceq \psi\mathcal{N}(\zeta_n, \zeta^*) + d_c(\mathcal{J}\zeta_n, \zeta^*) \end{aligned}$$

Considering that P is a normal cone defined by a normal constant K, it can be concluded that

$$||d_c(\mathcal{J}\zeta^*, \zeta^*)|| \preceq K||\psi\mathcal{N}(\zeta_{n+1}, \zeta^*)|| + ||d_c(\zeta_n, \zeta^*)|| \tag{3.8}$$

where

$$\mathcal{N}(\zeta_n, \zeta^*) = \max\left\{ d_c(\zeta_n, \zeta^*), \frac{d_c(\zeta_n, \mathcal{J}\zeta_n) + d_c(\zeta^*, \mathcal{J}\zeta^*)}{2}, \frac{d_c(\zeta_n, \mathcal{J}\zeta^*) + d_c(\zeta^*, \mathcal{J}\zeta_n)}{2} \right\}$$

Letting $n \to \infty$, we have

$$\mathcal{N}(\zeta_n, \zeta^*) = \frac{d_c(\zeta^*, \mathcal{J}\zeta^*)}{2}$$

Using in (2.8) and taking $n \to \infty$, We have

$$\begin{aligned} ||d_c(\mathcal{J}\zeta^*, \zeta^*)|| &\preceq K||\psi \frac{d_c(\mathcal{J}\zeta^*, \zeta^*)}{2}|| \\ &\frac{K}{2}||d_c(\mathcal{J}\zeta^*, \zeta^*)|| \end{aligned}$$

This is not applicable for all values of $K > 0$. Consequently, we arrive at a contradiction. Thus, it follows that $||d_c(\mathcal{J}\zeta^*, \zeta^*)|| \to \theta$ as n approaches infinity. This indicates that $\mathcal{J}\zeta^* = \zeta^*$, confirming that ζ^* is a fixed point of $\mathcal{J}$. This concludes the proof. □

Corollary 3.4. *Let $(\mathcal{U}, d_c)$ represent a complete cone metric space, and let P denote a normal cone characterized by a normal constant K. Consider the mapping $\mathcal{J} : \mathcal{U} \to \mathcal{U}$, which is an $\alpha - \psi$ contractive mapping that fulfills the subsequent conditions:*

$$\alpha(\zeta, \xi) d_c(\mathcal{J}\zeta, \mathcal{J}\xi) \preceq \psi d_c(\zeta, \xi)$$

for all $\zeta, \xi \in \mathcal{U}$. Assume also that the following conditions hold:

1. *$\mathcal{J}$ is α-admissible,*
2. *There exists $\zeta_0 \in \mathcal{U}$ such that $\alpha(\zeta_0, \mathcal{J}\zeta_0) \geq 1$,*
3. *$\mathcal{J}$ is regular*
4. *Either $\alpha(\zeta, \xi) \geq 1$ or $\alpha(\xi, \zeta) \geq 1$ whenever $\mathcal{J}\zeta = \zeta$ and $\mathcal{J}\xi = \xi$*

If the sequence $\{\zeta_n\}$ is contained within $\mathcal{U}$ and satisfies the condition that $\alpha(\zeta_n, \zeta_{n+1}) \geq 1$ for every integer n, along with the convergence $\zeta_n \to \zeta$ as n approaches infinity, it follows that $\alpha(\zeta_n, \zeta) \geq 1$ for all n. Consequently, this implies that the set $\mathcal{J}$ possesses a fixed point.

Proof. The proof can be derived from theorem 3.3 by selecting $\mathcal{J}$ as I, where I represents the identity mapping on $\mathcal{U}$. □

Example 3.5. *Let us consider* $\mathcal{J} = [0, \infty]$ *and* $\mathcal{A} = R^2$ *and* $P = \{(\zeta, \xi) \in \mathcal{A} | \zeta, \xi > 0\} \subset R^2$ *and* $d_c : \mathcal{U} \times \mathcal{U} \to \mathcal{A}$ *such that* $d_c(\zeta, \xi) = (|\zeta - \xi|, b|\zeta - \xi|)$ *where* $b \geq 0$ *is a constant. Then* $(\mathcal{U}, d_c)$ *is cone metric space. Difine* $\mathcal{J} : \mathcal{U} \to \mathcal{U}$ *by*

$$\mathcal{J}\zeta = \begin{cases} 2\zeta - \frac{13}{7} & if \quad \zeta > 1, \\ \frac{\zeta}{7} & if \;\; 0 \leq \zeta \leq 1, \\ 0 & if \;\; \zeta < 0 \end{cases}$$

It is noted that in this context, $\mathcal{J}$ *is continuous, and thus the Banach contraction principle cannot be utilized within the framework of a cone metric space.*

We now establish a mapping $\alpha : \mathcal{U} \times \mathcal{U} \to [0, \infty]$ *as follows.*

$$\alpha(\zeta, \xi) = \begin{cases} 1 & if \quad (\zeta, \;\; \xi) \in [0, 1], \\ 0 & otherwise \end{cases}$$

Clearly T is generalized $\alpha - \psi$*-contractive mapping with* $\psi(t) = \frac{3t}{5}$ *for all* $t \geq 0$. *Infact for all* $\zeta, \xi \in \mathcal{U}$*, we have*

$$\begin{aligned} \alpha(\zeta, \xi) d_c(\mathcal{J}\zeta, \mathcal{J}\xi) &= 1\dot{(}|\mathcal{J}\zeta - \mathcal{J}\xi|, b|\mathcal{J}\zeta - \mathcal{J}\xi|) \\ &= \left(\left| \frac{\zeta}{7} - \frac{\xi}{7} \right|, b \left| \frac{\zeta}{7} - \frac{\xi}{7} \right| \right) \\ &= \frac{(|\zeta - \xi|, b|\zeta - \xi|)}{7} \\ &= \frac{1}{7} d_c(\zeta, \xi) \\ &\preceq \frac{3}{5} d_c(\zeta, \xi) \\ &\preceq \frac{3}{5} \mathcal{J}(\zeta, \xi) \\ &= \psi \mathcal{J}(\zeta, \xi) \end{aligned}$$

More over there exists $\zeta_0 \in \mathcal{U}$*, such that* $\alpha(\zeta_0, \mathcal{J}\zeta_0) \geq 1$ *for* $\zeta_0 = 1$*, we have*

$$\alpha(1, \mathcal{J}_1) = \alpha(1, \frac{1}{7}) = 1$$

It is now necessary to demonstrate that $\mathcal{J}$ *is* α*-admissible. Consider* $\zeta, \xi \in \mathcal{U}$*, with the condition that* $\alpha(\zeta, \xi) \geq 1$.

Consequently, we have $\zeta, \xi \in [0, 1]$. *By the definitions of* $\mathcal{J}$ *and* α*, it follows that we have.*

$\mathcal{J}\zeta = \frac{\zeta}{7} \in [0, 1]$ $\qquad$ $\mathcal{J}\xi = \frac{\xi}{7} \in [0, 1]$ *and* $(\mathcal{J}\zeta, \mathcal{J}\xi) = 1$

So, $\mathcal{J}$ *is* α*-admissible.*

The conditions outlined in Theorem 3.3 have now been fully met. As a result, the operator $\mathcal{J}$ *possesses a fixed point. In this scenario, both 0 and* $\frac{13}{7}$ *serve as fixed points of* $\mathcal{J}$*.*

4 Conclusion

In this research, we have developed particular fixed point theorems for generalized $(\alpha - \Psi)$-contractive type mappings within the framework of complete cone metric spaces associated with Banach algebra. The results presented in this paper build upon and refine several aspects of earlier studies recorded in the existing literature.

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