# STABILITY OF GENERALIZED QUADRATIC FUNCTIONAL EQUATION IN 2-BANACH SPACES

ABSTRACT. In this paper, we investigate the general solution of functional equation

$$f(bx+y)+f(x-by)=(b^2+1)f(x)+(b^2+1)f(y)$$

for each $x, y \in X$, $b \in \mathbb{N}$ and we prove the generalized Hyers-Ulam stability of above equation in 2-Banach spaces.

## 1. INTRODUCTION

Stability of a functional equation for a function from a normed space to a Banach space has been studied by Hyers [2]. Stability of a functional equation for a function from a normed space to a 2-Banach space have been studied by B.M. Patel and A.B. Patel in [5], [6], [7]. Skof [9] has proved Hyers-Ulam stability of the functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y). \tag{1.1}$$

He proved that for a function $f : X \longrightarrow Y$, a function between normed space $X$ to Banach space $Y$ satisfying

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \leq \theta$$

for each $x, y \in X$ and $\theta > 0$, there exists a unique quadratic function $Q : X \longrightarrow Y$ such that

$$\|f(x)-Q(x)\| < \frac{\theta}{2}.$$

**Definition 1.1.** *Let $X$ and $Y$ be real vector spaces. A function $f : X \longrightarrow Y$ is said to be quadratic function if it satisfies* (1.1) *Every solution of function $f$ is said to be quadratic functional equation.*

*Consider the functional equation*

$$f(bx+y)+f(x-by)=(b^2+1)f(x)+(b^2+1)f(y) \tag{1.2}$$

*for each $x, y \in X, b \in \mathbb{N}$.*

Our aim is to study Hyers-Ulam stability of the functional equation (1.2).

**Definition 1.2.** *Let $X$ be a linear space over $\mathbb{R}$ with $\dim X > 1$ and let $\|\cdot,\cdot\| : X \times X \longrightarrow \mathbb{R}$ be a function satisfying the following properties:*

1. $\|x, y\| = 0$ *if and only if $x$ and $y$ are linearly dependent,*
2. $\|x, y\| = \|y, x\|$,

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(3) $\|ax, y\| = |a|\|x, y\|$,
(4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$

*for each* $x, y, z \in X$ *and* $a \in \mathbb{R}$. *Then the function* $\|\cdot, \cdot\|$ *is called a 2-norm on* $X$ *and* $(X, \|\cdot, \cdot\|)$ *is called a 2-normed space..*

We introduce a basic property of 2-normed spaces as follows. Let $(X, \|\cdot, \cdot\|)$ be a 2-normed space, $x \in X$ and $\|x, y\| = 0$ for each $y \in X$. Suppose $x \neq 0$, since $\dim X > 1$, choose $y \in X$ such that $\{x, y\}$ is linearly independent so we have $\|x, y\| \neq 0$, which is a contradiction. Therefore, we have the following lemma.

**Lemma 1.3.** *Let* $(X, \|\cdot, \cdot\|)$ *be a 2-normed space. If* $x \in X$ *and* $\|x, y\| = 0$ *for each* $y \in X$, *then* $x = 0$.

**Remark 1.4.** *Let* $(X, \|\cdot, \cdot\|)$ *be a 2-normed space. Note that the conditions* (2) *and* (4) *imply that*

$$\|x + y, z\| \leq \|x, z\| + \|y, z\|$$

*for each* $x, y, z \in X$. *Putting* $w = x + y$, *we get* $\|w, z\| \leq \|x, z\| + \|w - x, z\|$ *for each* $x, y, z \in X$. *So* $\|w, z\| - \|x, z\| \leq \|w - x, z\|$ *for each* $x, z, w \in X$. *Replacing* $w$ *by* $x$ *and* $x$ *by* $w$ *in the above inequality, we get* $\|x, z\| - \|w, z\| \leq \|x - w, z\|$ *for each* $x, z, w \in X$. *Thus, we have*

$$|\|x, z\| - \|y, z\|| \leq \|x - y, z\| \tag{1.3}$$

*for each* $x, y, z \in X$. *Hence the function* $x \to \|x, y\|$ *is continuous from* $X$ *into* $\mathbb{R}$ *for* $y \in X$.

In the 1960$s$, S. Gähler and A. White introduced the concept of 2-Banach spaces.

**Definition 1.5.** *A sequence* $\{z_n\}$ *in a 2-normed space* $X$ *is called a 2-Cauchy sequence if*

$$\lim_{m,n\to\infty} \|z_n - z_m, y\| = 0$$

*for each* $z \in X$.

**Definition 1.6.** *A sequence* $\{z_n\}$ *in a 2-normed space* $X$ *is called a 2-convergent sequence if there is a* $z \in X$ *such that*

$$\lim_{n\to\infty} \|z_n - z, x\| = 0$$

*for each* $x \in X$. *If* $\{z_n\}$ *converges to* $z$, *we write* $\lim_{n\to\infty} z_n = z$.

**Definition 1.7.** *A 2-normed space* $(X, \|\cdot, \cdot\|)$ *is a 2-Banach space if every 2-Cauchy sequence in* $X$ *is 2-convergent in* $X$.

Following shows that $\|\cdot, \cdot\|$ is continuous in each component.

**Lemma 1.8.** [5] *For a convergent sequence* $\{z_n\}$ *in a 2-normed space* $X$,

$$\lim_{n\to\infty} \|z_n, x\| = \|\lim_{n\to\infty} z_n, x\|$$

*for each* $x \in X$.

## 2. **General Solution of Equation** (1.2)

**Theorem 2.1.** *Let $X$ and $Y$ be vector spaces over $\mathbb{R}$. A mapping $f : X \longrightarrow Y$ satisfies equation* (1.1) *if and only if $f : X \longrightarrow Y$ satisfies equation* (1.2)*.*

*Proof.* First assume that $f : X \longrightarrow Y$ satisfies equation (1.2), that is

$$f(bx+y)+f(x-by)=(b^2+1)f(x)+(b^2+1)f(y) \tag{2.1}$$

Take $b = 1$ in (2.1), we get

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

Therefore $f$ satisfies equation (1.1).
Conversely assume that $f$ satisfies (1.1). Therefore $f(x+y)+f(x-y)=2f(x)+2f(y)$. We show that $f$ satisfies (1.2).We shall prove the result by principle of mathematical induction on $b$.
Let $b = 1$ then (1.2) reduces to (1.1).
Let $b = 2$ then

$$\begin{aligned}
f(2x+y)+f(x-2y) &= f(x+y+x)+f(x-y-y)\\
&= 2f(x+y)+2f(x)-f(y)+2f(x-y)+2f(y)-f(x)\\
&= 2[f(x+y)+f(x-y)]+f(x)+f(y)\\
&= 4f(x)+4f(y)+f(x)+f(y)\\
&= 5f(x)+5f(y)
\end{aligned}$$

for each $x, y \in X$. Therefore $f$ satisfies (1.2), for $b = 2$.
Next assume that $f$ satisfies equation (1.2), for all $b < m$. For $m \in \mathbb{N}$

$$\begin{aligned}
f(mx+y)+f(x-my) &= f[(m-1)x+y+x]+f[x-(k-1)y-y]\\
&= 2f((m-1)x+y)+2f(x)-f((m-2)x+y)\\
&\quad +2f(x-(m-1)y)+2f(y)-f(x-(m-2)y)\\
&= 2[f((m-1)x+y)+f(x-(m-1)y)]\\
&\quad +2f(x)+2f(y)-[f((m-2)x+y)+f(x-(m-2)y)]\\
&= 2\Big[\big((m-1)^2+1\big)f(x)+\big((m-1)^2+1\big)f(y)\Big]+2f(x)+2f(y)\\
&\quad -\Big[\big((m-2)^2+1\big)f(x)+\big((m-2)^2+1\big)f(y)\Big]\\
&= \big[2(m-1)^2+2+2-(m-2)^2-1\big]f(x)\\
&\quad +\big[2(m-1)^2+2+2-(m-2)^2-1\big]f(y)\\
&= (m^2+1)f(x)+(m^2+1)f(y)
\end{aligned}$$

for each $x, y \in X$. Therefore by the principle of mathematical induction, $f$ satisfies (1.2), for each $b \in \mathbb{N}$, for each $x, y \in X$. ☐

## 3. **Stability of Functional Equations for Functions** $f : (X, \|\cdot\|) \longrightarrow (X, \|\cdot,\cdot\|)$

Throughout this section, consider $X$ a real normed space, $Y$ a 2-Banach space. For function $f : X \longrightarrow Y$, define $D_f : X \times X \longrightarrow Y$ by

$$D_f(x,y)=f(ax+y)+f(x-ay)-(a^2+1)f(x)-(a^2+1)f(y)$$

for each $x, y \in X, a \in \mathbb{N}$.

**Theorem 3.1.** *Let* $G : X \times X \times Y \longrightarrow [0, \infty)$ *be a function such that*

$$H(x, y, z) := \sum_{n=0}^{\infty} \frac{1}{a^{2n}} G\big(a^n x, a^n y, z\big) < \infty \tag{3.1}$$

*for each* $x, y \in X, z \in Y, a \in \mathbb{N}$*. Let* $f : X \longrightarrow Y$ *be a function satisfying*

$$\|D_f(x, y), z\| \le G(x, y, z) \tag{3.2}$$

*for each* $x, y \in X, z \in Y$*. Then there exists a unique quadratic function* $Q : X \longrightarrow Y$ *such that*

$$\|f(x) - Q(x), z\| \le \frac{1}{a^2} H(x, 0, z) \tag{3.3}$$

*for each* $x \in X, z \in Y$*.*

*Proof.* Letting $y = 0$ in (3.2), we get

$$\|f(ax) - a^2 f(x), z\| \le G(x, 0, z) \tag{3.4}$$

for each $x \in X, z \in Y$. Therefore

$$\left\|\frac{f(ax)}{a^2} - f(x), z\right\| \le \frac{1}{a^2} G(x, 0, z) \tag{3.5}$$

for each $x \in X, z \in Y$. Replacing $x$ by $ax$ in (3.5), we get

$$\left\|\frac{f(a^2 x)}{a^2} - f(ax), z\right\| \le \frac{1}{a^2} G(ax, 0, z) \tag{3.6}$$

for each $x \in X, z \in Y$. By (3.5) and (3.6), we get

$$\begin{aligned}
\left\|\frac{f(a^2 x)}{a^4} - f(x), z\right\| &= \left\|\frac{f(a^2 x)}{a^4} - \frac{f(ax)}{a^2} + \frac{f(ax)}{a^2} - f(x), z\right\| \\
&\le \frac{1}{a^2}\left\|\frac{f(a^2 x)}{a^2} - f(ax), z\right\| + \left\|\frac{f(ax)}{a^2} - f(x), z\right\| \\
&\le \frac{1}{a^2}\Big[\frac{1}{a^2} G(ax, 0, z)\Big] + \frac{1}{a^2} G(x, 0, z) \\
&= \frac{1}{a^2}\Big[\frac{1}{a^2} G(ax, 0, z) + G(x, 0, z)\Big]
\end{aligned}$$

for each $x \in X, z \in Y$. By using induction on $n \in \mathbb{N}$, we get

$$\left\|\frac{f(a^n x)}{a^{2n}} - f(x), z\right\| \le \frac{1}{a^2} \sum_{j=0}^{n-1} \frac{1}{a^{2j}} G(a^j x, 0, z) \tag{3.7}$$

for each $x \in X, z \in Y$. For $m, n \in \mathbb{N}$

$$\begin{aligned}
\left\| \frac{f\big(a^{m+n}x\big)}{a^{2(m+n)}} - \frac{f(a^m x)}{a^{2m}}, z \right\| &= \frac{1}{a^{2m}} \left\| \frac{f\big(a^{m+n}x\big)}{a^{2n}} - f(a^m x), z \right\| \\
&\leq \frac{1}{a^{2m}} \left[ \frac{1}{a^2} \sum_{j=0}^{n-1} \frac{1}{a^{2j}} G(a^{m+j}x, 0, z) \right] \\
&= \frac{1}{a^2} \sum_{j=0}^{n-1} \frac{1}{a^{2(m+j)}} G(a^{m+j}x, 0, z) \\
&\leq \frac{1}{a^2} \sum_{j=0}^{\infty} \frac{1}{a^{2(m+j)}} G(a^{m+j}x, 0, z) \\
&= \frac{1}{a^2} \sum_{j=m}^{\infty} \frac{1}{a^{2j}} G(a^j x, 0, z) \\
&\longrightarrow 0 \ as\ m \to \infty
\end{aligned}$$

for each $x \in X, z \in Y$. Therefore $\left\{ \frac{f(a^m x)}{a^{2m}} \right\}$ is a 2-Cauchy sequence in $Y$, for each $x \in X$. Since $Y$ is a 2-Banach space, $\left\{ \frac{f(a^m x)}{a^{2m}} \right\}$ 2-converges, for each $x \in X$. Define $Q : X \longrightarrow Y$ as

$$Q(x) := \lim_{m \to \infty} \frac{f(a^m x)}{a^{2m}}$$

for each $x \in X$. Now, by (3.7), we get

$$\lim_{m \to \infty} \left\| \frac{f(a^m x)}{a^{2m}} - f(x), z \right\| \leq \frac{1}{a^2} \sum_{j=0}^{\infty} \frac{1}{a^{2j}} G(a^j x, 0, z)$$

for each $x \in X, z \in Y$. Therefore

$$\|f(x) - Q(x), z\| \leq \frac{1}{a^2} H(x, 0, z) \tag{3.8}$$

for each $x \in X, z \in Y$. Next we prove that $Q$ satisfies (1.2), for $x, y \in X$.

$$\begin{aligned}
\|D_Q(x, y), z\| &= \lim_{n \to \infty} \frac{1}{a^{2n}} \|D_f(a^n x, a^n y), z\| \\
&\leq \lim_{n \to \infty} \frac{1}{a^{2n}} G(a^n x, a^n y, z) \\
&= 0
\end{aligned}$$

for each $z \in Y$. Therefore $D_Q(x, y) = 0$, for each $x, y \in X$. Next we show that $Q$ is unique. Let $Q' : X \longrightarrow Y$ be another quadratic function satisfying (3.3). Since $Q$ and $Q'$ are quadratic functions, $Q(a^n x) = a^{2n} Q(x)$, $Q'(a^n x) = a^{2n} Q'(x)$, for each $x \in X$.

$$\begin{aligned}
\|Q(x) - Q'(x), z\| &= \frac{1}{a^{2n}} \|Q(a^n x) - Q'(a^n x), z\| \\
&\leq \frac{1}{a^{2n}} \big[ \|Q(a^n x) - f(a^n x), z\| + \|f(a^n x) - Q'(a^n x), z\| \big] \\
&\leq \frac{2}{a^{2n}} \frac{1}{a^2} H(a^n x, 0, z) \\
&\longrightarrow 0 \ as\ n \to \infty
\end{aligned}$$

for each $z \in Y$. Therefore $Q(x) = Q'(x)$, for each $x \in X$. $\square$

**Corollary 3.2.** *Let* $(X, \|\cdot\|)$ *be a real normed linear space and* $\|\cdot,\cdot\|$ *be a 2-norm on* $X$ *such that* $(X, \|\cdot,\cdot\|)$ *is a 2-Banach space,* $0 \leq p, q < 2$. $(X, \|\cdot\|) \longrightarrow (X, \|\cdot,\cdot\|)$ *be a function satisfying*

$$\|D_f(x, y), z\| \leq \|x, z\|^p + \|y, z\|^q \tag{3.9}$$

*for each* $x, y \in X, z \in Y$. *Then there exists a unique quadratic function* $Q : X \longrightarrow X$ *such that*

$$\|f(x) - Q(x), z\| \leq \frac{\|x, z\|^p}{a^2 - a^p} \tag{3.10}$$

*for each* $x, z \in X$.

**Theorem 3.3.** *Let* $G : X \times X \times Y \longrightarrow [0, \infty)$ *be a function such that*

$$H(x, y, z) := \sum_{n=0}^{\infty} a^{2n} G\Big(\frac{x}{a^n}, \frac{y}{a^n}, z\Big) < \infty \tag{3.11}$$

*for each* $x, y \in X, z \in Y, a \in \mathbb{N}$. *Let* $f : X \longrightarrow Y$ *be a function satisfying*

$$\|D_f(x, y), z\| \leq G(x, y, z) \tag{3.12}$$

*for each* $x, y \in X, z \in Y$. *Then there exists a unique quadratic function* $Q : X \longrightarrow Y$ *such that*

$$\|f(x) - Q(x), z\| \leq H\Big(\frac{x}{a}, 0, z\Big) \tag{3.13}$$

*for each* $x \in X, z \in Y$.

*Proof.* Letting $y = 0$ in (3.12), we get

$$\|f(ax) - a^2 f(x), z\| \leq G(x, 0, z) \tag{3.14}$$

for each $x \in X, z \in Y$. Replacing $x$ by $\frac{x}{a}$ in (3.14), we get

$$\Big\|f(x) - a^2 f\Big(\frac{x}{a}\Big), z\Big\| \leq G\Big(\frac{x}{a}, 0, z\Big) \tag{3.15}$$

for each $x \in X, z \in Y$. Replacing $x$ by $\frac{x}{a}$ in (3.15), we get

$$\Big\|f\Big(\frac{x}{a}\Big) - a^2 f\Big(\frac{x}{a^2}\Big), z\Big\| \leq G\Big(\frac{x}{a^2}, 0, z\Big) \tag{3.16}$$

for each $x \in X, z \in Y$. By (3.15) and (3.16), we get

$$\begin{aligned}\Big\|f(x) - a^4 f\Big(\frac{x}{a^2}\Big), z\Big\| &\leq \Big\|f(x) - a^2 f\Big(\frac{x}{a}\Big), z\Big\| + \Big\|a^2 f\Big(\frac{x}{a}\Big) - a^4 f\Big(\frac{x}{a^2}\Big), z\Big\| \\ &\leq G\Big(\frac{x}{a}, 0, z\Big) + a^2 G\Big(\frac{x}{a^2}, 0, z\Big)\end{aligned}$$

for each $x \in X, z \in Y$. By using induction on $n \in \mathbb{N}$, we get

$$\Big\|f(x) - a^{2n} f\Big(\frac{x}{a^n}\Big), z\Big\| \leq \sum_{j=0}^{n-1} a^{2j} G\Big(\frac{x}{a^{j+1}}, 0, z\Big) \tag{3.17}$$

for each $x \in X, z \in Y$. For $m, n \in \mathbb{N}$

$$\begin{aligned}\Big\|a^{2(m+n)} f\Big(\frac{x}{a^{m+n}}\Big) - a^{2m} f\Big(\frac{x}{a^m}\Big), z\Big\| &= a^{2m}\Big\|a^{2n} f\Big(\frac{x}{a^{m+n}}\Big) - f\Big(\frac{x}{a^m}\Big), z\Big\| \\ &\leq a^{2m}\sum_{j=0}^{n-1} a^{2j} G\Big(\frac{x}{a^{m+j+1}}, 0, z\Big) \\ &= \sum_{j=0}^{n-1} a^{2(m+j)} G\Big(\frac{x}{a^{m+j+1}}, 0, z\Big) \\ &\leq \sum_{j=0}^{\infty} a^{2(m+j)} G\Big(\frac{x}{a^{m+j+1}}, 0, z\Big) \\ &\longrightarrow 0 \ as \ m \to \infty\end{aligned}$$

for each $x \in X, z \in Y$. Therefore $\Big\{a^{2m} f\Big(\frac{x}{a^m}\Big)\Big\}$ is a 2-Cauchy sequence in $Y$, for each $x \in Y$. Since $Y$ is a 2-Banach space, $\Big\{a^{2m} f\Big(\frac{x}{a^m}\Big)\Big\}$ 2-converges, for each $x \in X$. Define $Q : X \longrightarrow Y$ as

$$Q(x) := \lim_{n\to\infty} a^{2n} f\Big(\frac{x}{a^n}\Big)$$

for each $x \in X$. By (3.17), we get

$$\lim_{n\to\infty} \Big\|f(x) - a^{2n} f\Big(\frac{x}{a^n}\Big), z\Big\| \leq \sum_{j=0}^{\infty} a^{2j} G\Big(\frac{x}{a^{j+1}}, 0, z\Big)$$

for each $x \in X, z \in Y$. Therefore

$$\|f(x) - Q(x), z\| \leq H\Big(\frac{x}{a}, 0, z\Big)$$

for each $x \in X, z \in Y$. The further part of the proof is similar to that of the proof of Theorem 3.1. $\square$

**Corollary 3.4.** *Let $(X, \|\cdot\|)$ be a real normed linear space and $\|\cdot,\cdot\|$ be a 2-norm on $X$ such that $(X, \|\cdot,\cdot\|)$ is a 2-Banach space, $p, q > 2$. $f : (X, \|\cdot\|) \longrightarrow (X, \|\cdot,\cdot\|)$ be a function satisfying*

$$\|D_f(x, y), z\| \leq \|x, z\|^p + \|y, z\|^q \tag{3.18}$$

*for each $x, y \in X, z \in Y$. Then there exists a unique quadratic function $Q : X \longrightarrow X$ such that*

$$\|f(x) - Q(x), z\| \leq \frac{\|x, z\|^p}{a^p - a^2} \tag{3.19}$$

*for each $x, z \in X$.*

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