Original Research Article

Structural Characterization of T-Normal Graphs

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Abstract

A graph G is said to be T-normal if for any two disjoint cliques Q_1 and Q_2 in G there exists covers $\mathcal{D}_1$ and $\mathcal{D}_2$ of Q_1 and Q_2 respectively in G such that $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) = \emptyset$.

In this paper T-normal graphs have been discussed with examples. Also we have discussed conditions under which join of two graphs, union of two graphs, cartesian, tensor, and strong product of two graphs is T-normal. Also we have given a characterization of T-normal graphs using graph complement.

Keywords: T-regular graph, Join, Cartesian product, Tensor product, Strong product.

2010 MSC: 05C76

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1. Introduction

All the graphs considered here are finite and simple. In this paper we denote the set of vertices of G by $V(G)$, the set of edges of G by $E(G)$, the maximum degree of G by $\Delta(G)$ and the minimum degree of G by $\delta(G)$.

The *degree* [1] of a vertex v in graph G, denoted by $\deg(v)$, is the number of edges incident with v. A *pendant vertex* [2]

in a graph G is a vertex of degree one. A vertex v is *isolated* [1] if $\deg(v) = 0$. By an *empty graph* [3] we mean a graph with no edges. A simple graph is said to be *complete* [4] if every pair of distinct vertices of G are adjacent in G. A complete graph on n vertices is denoted by K_n. Given two graphs, G and H, we say H is an *induced subgraph*[5] of G if $V(H) \subseteq V(G)$, and two vertices of H are adjacent if and only if they are adjacent in G. In this case if $V(H) = S$, we write $H = G[S]$ or $H = \langle S \rangle$. An edge cover of a graph is a set of edges such that every vertex of the graph is incident to at least one edge of the set.[3]. The *union* [6] of two graphs G_1 and G_2 denoted by $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. If v is a vertex of a graph G, then the graph obtained from G by deleting the vertex v is denoted by $G - v$. More generally, if S is any set of vertices in G, $G - S$ is the graph obtained from G by deleting all the vertices in S, and all edges incident with any one of the vertices of S[7].

The *Cartesian product*[8] $G \Box H$ of two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is the graph with vertex set $V(G) \times V(H)$ where the vertex (u_1, v_1) is adjacent to the vertex (u_2, v_2) whenever $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$. The An *independent edge set*[9] of a graph G is a subset of the edges of G such that no two edges in the subset share a vertex of G. A hereditary property in graph usually means a property of a graph which also holds for (is "inherited" by) its induced subgraphs.

For a product $G_1 \star G_2$, where $\star = \Box$ or $\times$ or $\boxtimes$ or $*$ the projection [10] $p_i : G_1 \star G_2 \longrightarrow G_1$ is defined by

$$p_i(u_1, u_2) = \begin{cases} u_1 & \text{if} \quad i = 1 \\ u_2 & \text{if} \quad i = 2 \end{cases}$$

If U is a subgraph of $G_1 \star G_2$, then $p_i(U)$ is a subgraph of G_i with vertex set $\{p_i(v) : v \in U\}$ and edge set $\{p_i(u)p_i(v)/uv \in E(U), p_i(u) \neq p_i(v)\}$.

Lemma 1.1. Let Q be a clique in $G_1 \Box G_2$ then, either the first coordinates of all vertices of Q are same and the subgraph of G induced by the second coordinates of all vertices of Q is a clique in G_2, or the second coordinates of all vertices of Q are same and the the subgraph of G induced by the first coordinates of all vertices of Q is a clique in G_1.

2. T-normal Graphs

Let us introduce the T-normal graphs as

Definition 2.1. A graph G is said to be T-normal if for any two disjoint cliques Q_1 and Q_2 in G there exists covers $\mathcal{D}_1$ and $\mathcal{D}_2$ of Q_1 and Q_2 respectively in G such that $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) = \emptyset$.

Remark 2.1. Let Q_1 and Q_2 be two disjoint cliques in G_1 and G_2 respectively with $|V(Q_1)| = m$ and $|V(Q_2)| = n$. If both m and n are even then both Q_1 and Q_2 can be covered by its own edges. It is a trivial case. Hence it is enough to consider the cases in which one of m and n are odd; and in which both m and n are odd.

An example of a T- normal graph and a non-T-normal graph is depicted in Figure 1.

Figure 1: (a) A T-normal graph G. (b) A non-T-normal graph H.

Proposition 2.1. *Kn- Complete graph on n- vertices is T- normal for every n.*

Proof. As K_n is a complete graph there are no disjoint cliques in K_n.So condition holds trivially. □

Theorem 2.1. Give graph G is T-normal if and only if its complement $\overline{G}$ satisfies the following property: Given any two disjoint stable sets $I_1, I_2 \subseteq V(\overline{G})$, there exist stable-set covers $\mathcal{E}_1, \mathcal{E}_2$ of I_1 and I_2 in $\overline{G}$ such that

$$V(\mathcal{E}_1) \cap V(\mathcal{E}_2) = \emptyset$$

Here, an *stable-set cover*[11] of I_i means a collection of stable sets in $\overline{G}$ whose union contains I_i.

Proof. Necessary Part. Suppose that G is a T -normal graph. Then, for any two disjoint cliques Q_1 and Q_2 in G there exists covers $\mathcal{D}_1$ and $\mathcal{D}_2$ of Q_1 and Q_2 respectively in G such that $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) = \emptyset$. Let I_1, I_2 be two disjoint stable sets in $\overline{G}$. Then I_1 and I_2 are disjoint cliques in G. As G is T-normal, there exist clique covers $\mathcal{C}_1$ and $\mathcal{C}_2$ of I_1 and I_2 in G, respectively, satisfying $V(\mathcal{C}_1) \cap V(\mathcal{C}_2) = \emptyset$. Each clique in the cover $\mathcal{C}_i$ becomes a stable set in $\overline{G}$, so $\mathcal{E}_i = \{\overline{C} : C \in \mathcal{C}_i\}$ forms an stable-set cover of I_i in $\overline{G}$, preserving vertex disjointness. Hence, $\overline{G}$ holds the required property.

Sufficent Part. Suppose $\overline{G}$ satisfies the stated property. Now we have to show that $G$is T-normal. Let Q_1 and Q_2 be two disjoint cliques in G. Then Q_1 and Q_2 are disjoint stable sets in $\overline{G}$. By assumption there exist stable-set covers $\mathcal{E}_1, \mathcal{E}_2$ of Q_1, Q_2 in $\overline{G}$ with $V(\mathcal{E}_1) \cap V(\mathcal{E}_2) = \emptyset$. Each stable set in $\overline{G}$ corresponds to a clique in G, so $\mathcal{C}_i = \{\overline{D} : D \in \mathcal{E}_i\}$ are clique covers of Q_i in G, which are vertex disjoint. Therefore, G is a T -normal graph. □

Proposition 2.2. *If G_1 and G_2 are two disjoint T-normal graphs then $G = G_1 \cup G_2$ is T-normal.*

Proof. Given that $G_1$and G_2 are disjoint T- normal graphs. Therefore,$V(G_1) \cap V(G)_2 = \emptyset$..Let $G = G_1 \cup G_2$.Then G is the disjoint union of two components G_1 and G_2.Let Q_1 and Q_2 be two cliquesin G.

Case 1. The cliques Q_1 and Q_2 lie in different components.

Suppose that $Q_1 \subseteq G_1$ and $Q_2 \subseteq G_2$. Since G_1 and G_2 are T-normal, there exist clique covers $\mathcal{D}_1$ of Q_1 in G_1 and $\mathcal{D}_2$ of Q_2 in G_2 with $V(\mathcal{D}_1)$ is contained in $V(G_1)$ and $V(\mathcal{D}_2)$ is contained in $V(G_2)$, respectively. Hence $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) = \emptyset$.

Case 2. Both cliques Q_1 and Q_2 lie in the same component of G, say G_1.

As $G_1$is T-normal, there exist disjoint clique covers for Q_1 and Q_2 within G_1 itself. Hence disjointness condition holds in G.

Therefore,in all cases given any two disjoint cliques Q_1, Q_2 in G, there exist disjoint clique covers $\mathcal{D}_1$ and $\mathcal{D}_2$ in G. Therefore G is T-normal. □

Remark 2.2. T normality in graphs is not a heriditory property.That is if G is T-normal then every induced subraph of G need not be normal.

3. *T*-normal in Cartesian Product of Graphs

Theorem 3.1. Let G_1 and G_2 be two graphs with $\delta(G_1)$ and $\delta(G_2) \geq 2$, then $G_1 \square G_2$ is T-normal.

Proof. Let Q_1 and Q_2 be two disjoint cliques in G_1 and G_2 respectively. Let $|V(Q_1)| = m$ and $|V(Q_2)| = n$. By Lemma 1.1 either the first coordinate of vertices of Q_1 are same and the span of second coordinate of vertices of Q_1 forms a clique in G_2, or the second coordinate of vertices of Q_1 are same and the span of second coordinate of vertices of Q_1 forms a clique in G_1. Similarly, the case of Q_2 also. First of all suppose that second coordinate of each vertex of Q_1 is x and that of Q_2 is y. Assume that $n \geq m$. Let $V(Q_1) = \{(u_1, x), (u_2, x), (u_3, x), \ldots, (u_m, x)\}$ and $V(Q_2) = \{(v_1, y), (v_2, y), (v_3, y), \ldots, (v_n, y)\}$. As $\delta(G_1) \geq 2$, $m \geq 2$ and $n \geq 2$. Therefore, we can suppose $v_1 \neq u_m$. To get covers $\mathcal{D}_1$ and $\mathcal{D}_2$ of Q_1 and Q_2 respectively in G such that $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) = \emptyset$, consider the following two cases.

Case 1. m is even and n is odd

As m is even Q_1 can be covered by its own edges. As $\delta(G_2) \geq 2$, there exists a vertex w different from x such that y is adjacent to w. Then $\{(v_1, y)(v_2, y), (v_3, y)(v_4, y), \ldots, (v_{n-2}, y)(v_{n-1}, y),$ $(v_n, y)(v_n, w)\}$ is a cover of Q_2 which is not incident with any edges of Q_1.

Case 2. Both m and n are odd

As $\delta(G_2) \geq 2$, there exists a vertex w of G_2 different from y such that w is adjacent to x. Similarly, there exists a vertex z of G_2 different from x such that z is adjacent to y. Then $\mathcal{D}_1 = \{(u_1, x)(u_2, x), (u_3, x)(u_4, x), \ldots, (u_m, x)(u_m, w)\}$ and $\mathcal{D}_2 = \{(v_1, z)(v_1, y), (v_2, y)(v_3, y), (v_4, y)(v_5, y), \ldots, (v_{n-3}, y)(v_{n-2}, y), (v_{n-1}, y)(v_n, y)\}$ are covers of Q_1 and Q_2 respectively and $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) = \emptyset$.

Now, assume the second coordinate of each vertex of Q_1 is x and the first coordinate of each vertex of Q_2 is u. Also assume $n \geq m$. Let $V(Q_1) = \{(u_1, x), (u_2, x), (u_3, x), \ldots, (u_m, x)\}$ and $V(Q_2) = \{(u, v_1), (u, v_2), (u, v_3), \ldots, (u, v_n)\}$. The vertex u may or may not belongs to $\{u_1, u_2, \ldots, u_m\}$. If $u \in \{u_1, u_2, \ldots, u_m\}$, let $u = u_1$. Assume $y \notin \{v_1, v_2, \ldots, v_n\}$. To get covers $\mathcal{D}_1$ and $\mathcal{D}_2$ of Q_1 and Q_2 respectively in G such that $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) = \emptyset$, consider the following two cases.

Case 1. m is even and n is odd

Since m is even, Q_1 can be covered by its own edges. As $\delta(G_1) \geq 2$, there exists a vertex w different from u_1 such that u is adjacent to w. Then, $\{(w, v_1)(u, v_1), (u, v_2)(u, v_3), (u, v_4)(u, v_5), \ldots, (u, v_{n-1})(u, v_n)\}$ is a cover of Q_2 which is not incident with any edges of Q_1.

Case 2. Both m and n are odd

As $\delta(G_1) \geq 2$, there exists a vertex w of G_1 different from u_1 such that w is adjacent to u. As $\delta(G_2) \geq 2$, there exists a vertex z of G_2 different from v_1 such that z is adjacent to y. Then $\mathcal{D}_1 = \{(u_1, z)(u_1, y), (u_2, y), (u_3, y), (u_4, y)(u_5, y), \ldots, (u_{m-1}, y)(u_m, y)\}$ and $\mathcal{D}_2 = \{(w, v_1)(u, v_1), (u, v_2)(u, v_3), (u, v_4)(u, v_5), \ldots, (u, v_{n-1})(u, v_n)\}$ are covers of Q_1 and Q_2 respectively and $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) = \emptyset$.

Now, assume $y \in \{v_1, v_2, \ldots, v_n\}$. Let $y = v_1$. To get covers $\mathcal{D}_1$ and $\mathcal{D}_2$ of Q_1 and Q_2 respectively in G such that $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) = \emptyset$, consider the following two cases.

Case 1. m is even and n is odd

Since m is even, Q_1 can be covered by its own edges. As $\delta(G_1) \geq 2$, there exists a vertex w different from u_1 such that u is adjacent to w. Then $\{(u, v_1)(u, v_2), (u, v_3)(u, v_4), (u, v_5)(u, v_6), \ldots, (u, v_{n-2})(u, v_{n-1}), (u, v_n)(w, v_n)\}$ is a cover of Q_2 which is not incident with any edges of Q_1.

Case 2. Both m and n are odd

As $\delta(G_1) \geq 2$, there exists a vertex w of G_1 different from u_1 such that w is adjacent to u. As $\delta(G_2) \geq 2$, there exists a vertex z of G_2 different from v_1 such that z is adjacent to y. Then $\mathcal{D}_1 = \{(u_1, z)(u_1, y), (u_2, y), (u_3, y), (u_4, y)(u_5, y), \ldots, (u_{m-1}, y)(u_m, y)\}$ and $\mathcal{D}_2 = \{(u, v_1)(u, v_2), (u, v_3)(u, v_4),$

$(u, v_5)(u, v_6), \ldots, (u, v_{n-2})(u, v_{n-1}), (u, v_n)(w, v_n)\}$ are covers of Q_1 and Q_2 respectively and $V(\mathcal{D}_1) \cap V(\mathcal{D}_2) = \emptyset$.

Hence the theorem. □

4. Conclusion

In this paper T- normal graphs have been discussed with examples. Also we have discussed conditions under which join of two graphs, union of two graphs cartesian, tensor, and strong product of two graphs is T-normal. Also we have given a characterization of T-normal graphs using graph complement.

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