Original Research Article

A Note on Hyers-Ulam Stability of Functional Equations in 2-Banach spaces

Abstract

In this paper, we will make some corrections in the paper of S.C. Chung W. Park (3).

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1 Introduction

In this note, we would like to bring out some correction in the Theorems and proofs of Theorems in the paper of S. Chung and W. Park (3).

In the 1960s, S. Gähler (1) introduced the concept of linear 2-normed spaces.

Definition 1.1. Let X be a linear space over $\mathbb{R}$ with $\dim X > 1$ and let $\|\cdot,\cdot\| : X \times X \longrightarrow \mathbb{R}$ be a function satisfying the following properties:

1. $\|x, y\| = 0$ if and only if x and y are linearly dependent,
2. $\|x, y\| = \|y, x\|$,
3. $\|ax, y\| = |a|\|x, y\|$,
4. $\|x, y + z\| \leq \|x, y\| + \|x, z\|$

for each $x, y, z \in X$ and $a \in \mathbb{R}$. Then the function $\|\cdot,\cdot\|$ is called a 2-norm on X and $(X, \|\cdot,\cdot\|)$ is a called 2-normed space.

So it is clear that if $\|\cdot,\cdot\|$ is a 2-norm on X then $\|\cdot,\cdot\| : X \times X \longrightarrow \mathbb{R}$ is a function having some properties. Hence for finding $\|x, y\|$, it is necessary to take $x, y \in X$. Throughout the paper (3) authors have taken X, a normed linear space and Y, a 2-Banach space.

2 Main Results

In paper (3), Theorem (2.1) is as under. Let $\eta, \theta \in [0, \infty), p, q \in (0, 1)$ and let $f : X \longrightarrow Y$ be a mapping satisfying

$$\|f(x+y) - f(x) - f(y), z\| \le \eta\|x\|^p + \theta\|y\|^q \tag{2.1}$$

for all $x, y, z \in X$ and all $z \in Y$. Then there is a unique additive mapping $A : X \longrightarrow Y$ such that

$$\|f(x) - A(x), y\| \le \frac{\eta\|x\|^p}{2 - 2^p} + \frac{\theta\|x\|^q}{2 - 2^q}$$

for all $x \in X$ and all $y \in Y$.

Remark 2.1. No such function $f : X \longrightarrow Y$ satisfying (2.1) exists. If such function $f : X \longrightarrow Y$ exist then for each $n \in \mathbb{N}$, $z \in Y \Rightarrow nz \in Y$. By (2.1) we have

$$\|f(x+y) - f(x) - f(y), nz\| \le \eta\|x\|^p + \theta\|y\|^q$$

each $x, y \in X$ and each $z \in Y$. Therefore

$$n \le \frac{\eta\|x\|^p + \theta\|y\|^q}{\|f(x+y) - f(x) - f(y), z\|}$$

for each $n \in \mathbb{N}$. Where $\|f(x+y) - f(x) - f(y), z\| \neq 0$. So $\mathbb{N}$ would be bounded, which is absurd. Therefore no such function exists.

We correct the Theorem 2.1. of paper (3) as follow.

Let $\eta, \theta \in [0, \infty), p, q, r \in (0, \infty)$ and $p, q \in (0, 1)$ and $(X, \|\cdot\|)$ be a normed space and $(X, \|\cdot, \cdot\|)$ be a 2-Banach space and $f : (X, \|\cdot\|) \longrightarrow (X, \|\cdot, \cdot\|)$ be a mapping satisfying

$$\|f(x+y) - f(x) - f(y), z\| \le (\eta\|x\|^p + \theta\|y\|^q)\|z\|^r$$

for all $x, y, z \in X$. Then there is a unique additive mapping $A : X \longrightarrow X$ such that

$$\|f(x) - A(x), y\| \le \Big(\frac{\eta\|x\|^p}{2 - 2^p} + \frac{\theta\|x\|^q}{2 - 2^q}\Big)\|y\|^r$$

for all $x, y, z \in X$.

Correction in Theorem (2.2) of paper (3) is as under. Let $\eta, \theta \in [0, \infty), p, q, r \in (0, \infty)$ and $p, q \in (1, \infty)$ and $(X, \|\cdot\|)$ be a normed space and $(X, \|\cdot, \cdot\|)$ be a 2-Banach space and $f : (X, \|\cdot\|) \longrightarrow (X, \|\cdot, \cdot\|)$ be a mapping satisfying

$$\|f(x+y) - f(x) - f(y), z\| \le (\eta\|x\|^p + \theta\|y\|^q)\|z\|^r \tag{2.2}$$

for all $x, y, z \in X$. Then there is a unique additive mapping $A : X \longrightarrow X$ such that

$$\|f(x) - A(x), y\| \le \Big(\frac{\eta\|x\|^p}{2^p - 2} + \frac{\theta\|x\|^q}{2^q - 2}\Big)\|y\|^r$$

for all $x, y, z \in X$.

Correction in Theorem (3.1) of paper (3) is as under.

Let $\theta \in [0, \infty), p, q, r \in (0, \infty)$ and $p, q \in (0, 1)$ and $(X, \|\cdot\|)$ be a normed space and $(X, \|\cdot, \cdot\|)$ be a 2-Banach space and $f : (X, \|\cdot\|) \longrightarrow (X, \|\cdot, \cdot\|)$ be a mapping satisfying $f(0) = 0$ and

$$\Big\|2f\Big(\frac{x+y}{2}\Big) - f(x) - f(y), z\Big\| \le (\eta\|x\|^p + \theta\|y\|^q)\|z\|^r$$

for all $x, y, z \in X$. Then there is a unique Jensen mapping $J : X \longrightarrow X$ such that

$$\|f(x) - J(x), y\| \leq \Big(\frac{2\eta}{3 - 3^p}\|x\|^p + \theta\frac{1 + 3^q}{3 - 3^q}\|x\|^q\Big)\|y\|^r$$

for all $x, y, z \in X$.

Correction in Theorem (3.2) of paper (3) is as under.

Let $\theta \in [0, \infty), p, q, r \in (0, \infty)$ and $p, q \in (1, \infty)$ and $(X, \| \cdot \|)$ be a normed space and $(X, \|\cdot, \cdot\|)$ be a 2-Banach space and $f : (X, \| \cdot \|) \longrightarrow (X, \|\cdot, \cdot\|)$ be a mapping satisfying $f(0) = 0$ and

$$\Big\|2f\Big(\frac{x + y}{2}\Big) - f(x) - f(y), z\Big\| \leq (\eta\|x\|^p + \theta\|y\|^q)\|z\|^r$$

for all $x, y, z \in X$. Then there is a unique Jensen mapping $J : X \longrightarrow X$ such that

$$\|f(x) - J(x), y\| \leq \Big(\frac{2\eta}{3^p - 3}\|x\|^p + \theta\frac{1 + 3^q}{3^q - 3}\|x\|^q\Big)\|y\|^r$$

for all $x, y, z \in X$.

In paper (3), Theorem (4.1) is as under.

Let $\eta, \theta \in [0, \infty), p, q, r \in (0, \infty)$ and $p, q \in (0, 2)$ and let $f : X \longrightarrow Y$ be a mapping satisfying

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y), z\| \leq \eta\|x\|^p + \theta\|y\|^q \tag{2.3}$$

for all $x, y, z \in X$. Then there is a unique quadratic mapping $Q : X \longrightarrow Y$ such that

$$\|f(x) - Q(x), y\| \leq \frac{\eta\|x\|^p}{4 - 2^p} + \frac{\theta\|x\|^q}{4 - 2^q} + \frac{1}{3}\|f(0), y\|$$

for all $x \in X$ and all $y \in Y$.

Correction in Theorem (4.1) of paper (3) is as under.

Let $\eta, \theta \in [0, \infty), p, q, r \in (0, \infty)$ and $p, q \in (0, 2)$ and $(X, \| \cdot \|)$ be a normed space and $(X, \|\cdot, \cdot\|)$ be a 2-Banach space and $f : (X, \| \cdot \|) \longrightarrow (X, \|\cdot, \cdot\|)$ be a mapping satisfying

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y), z\| \leq (\eta\|x\|^p + \theta\|y\|^q)\|z\|^r \tag{2.4}$$

for all $x, y, z \in X$. Then there is a unique quadratic mapping $Q : X \longrightarrow X$ such that

$$\|f(x) - Q(x), y\| \leq \Big(\frac{\eta\|x\|^p}{4 - 2^p} + \frac{\theta\|x\|^q}{4 - 2^q}\Big)\|z\|^r$$

for all $x, y \in X$.

Remark 2.2. By taking $x = y = 0$ in (2.4), we get $\|2f(0), z\| = 0$, for each $z \in X$. Therefore $f(0) = 0$.

Correction in Theorem (4.1) of paper (3) is as under.

Let $\eta, \theta \in [0, \infty), p, q, r \in (0, \infty)$ and $p, q \in (2, \infty)$ and $(X, \| \cdot \|)$ be a normed space and $(X, \|\cdot, \cdot\|)$ be a 2-Banach space and $f : (X, \| \cdot \|) \longrightarrow (X, \|\cdot, \cdot\|)$ be a mapping satisfying

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y), z\| \leq (\eta\|x\|^p + \theta\|y\|^q)\|z\|^r$$

for all $x, y, z \in X$. Then there is a unique quadratic mapping $Q : X \longrightarrow X$ such that

$$\|f(x) - Q(x), y\| \leq \Big(\frac{\eta\|x\|^p}{2^p - 4} + \frac{\theta\|x\|^q}{2^q - 4}\Big)\|z\|^r$$

for all $x, y \in X$.

3 CONCLUSIONS

We hope that this research work will be useful in the theory of functional equations.

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References

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