Hyperbolic Extensions of Generalized Pandita Numbers

Abstract. In this paper, we introduce the generalized hyperbolic Pandita numbers over the bidimensional Clifford algebra of hyperbolic numbers. As special cases, we deal with hyperbolic Pandita and hyperbolic Lucas numbers. We present Binet's formulas, generating functions, and the summation formulas for these numbers. Moreover, we give matrices related to these sequences.

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1. Introduction

The hypercomplex numbers systems, [10], are extensions of real numbers. Some commutative examples of hypercomplex number systems are complex numbers,

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\},$$

hyperbolic (double, split-complex) numbers, [7],

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\},$$

and dual numbers, [18],

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\},$$

Some non-commutative examples of hypercomplex number systems are quaternions, [29],

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\},$$

octonions [11] and sedenions [21]. The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are real algebras obtained from the real numbers $\mathbb{R}$ by a doubling procedure called the

Cayley-Dickson Process. This doubling process can be extended beyond the sedenions to form what are known as the 2^n-ions (see for example [4], [16], [6]).

Quaternions were invented by Irish mathematician W. R. Hamilton (1805-1865) [29] as an extension to the complex numbers. Hyperbolic numbers with complex coefficients are introduced by J. Cockle in 1848, [12]. H. H. Cheng and S. Thompson [8] introduced dual numbers with complex coefficients and called complex dual numbers. Akar, Yüce and Şahin [17] introduced dual hyperbolic numbers.

A dual hyperbolic number is a hyper-complex number and is defined by

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3,$$

where a_0, a_1, a_2 and a_3 are real numbers.

The set of all dual hyperbolic numbers are denoted by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3 : a_0, a_1, a_2, a_3 \in \mathbb{R},\ j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The base elements $\{1, j, \varepsilon, \varepsilon j\}$ of dual hyperbolic numbers satisfy the following properties (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j,\ \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0,\ j^2 = j.j = 1, \\ \varepsilon.j &= j.\varepsilon,\ \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0,\ j(\varepsilon j) = (\varepsilon j)j = \varepsilon, \end{aligned}$$

where ε denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j denotes the hyperbolic unit ($j^2 = 1$), and εj denotes the dual hyperbolic unit ($(j\varepsilon)^2 = 0$).

The product of two dual hyperbolic numbers $q = a_0 + ja_1 + \varepsilon a_2 + j\varepsilon a_3$ and $p = b_0 + jb_1 + \varepsilon b_2 + j\varepsilon b_3$ is

$$qp = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0) + \varepsilon(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + j\varepsilon(a_0b_3 + a_1b_2 + a_2b_1 + b_0a_3)$$

and addition of dual hyperbolic numbers is defined as componentwise.

The dual hyperbolic numbers form a commutative ring, real vector space and an algebra. But $\mathbb{H}_{\mathbb{D}}$ is not field because every dual hyperbolic numbers doesn't have an inverse. For more information on the dual hyperbolic numbers, see [17].

Here we use the set of hyperbolic numbers. The set of hyperbolic numbers $\mathbb{H}$ can be described as

$$\mathbb{H} = \{z = x + hy \mid h \notin \mathbb{R},\ h^2 = 1, x, y \in \mathbb{R}\}.$$

The hyperbolic ring $\mathbb{H}$ is a bidimensional Clifford algebra, see [14] for details. Hyperbolic numbers has been called in the mathematical literature with different names: Lorentz numbers, double numbers, duplex numbers, split complex numbers and perplex numbers. Hyperbolic numbers are useful for measuring distances in the Lorentz space-time plane (see Sobczyk [7]). For more information on hyperbolic numbers, see also [9,13,19,20].

Addition, substraction and multiplication of any two hyperbolic numbers z_1 and z_2 are defined by

$$\begin{aligned} z_1 \pm z_2 &= (x_1 + hy_1) \pm (x_2 + hy_2) = (x_1 \pm x_2) + h(y_1 \pm y_2), \\ z_1 \times z_2 &= (x_1 + hy_1) \times (x_2 + hy_2) = x_1 x_2 + y_1 y_2 + h(x_1 y_2 + y_1 x_2). \end{aligned}$$

and the division of two hyperbolic numbers are given by

$$\frac{z_1}{z_2} = \frac{x_1 + hy_1}{x_2 + hy_2} = \frac{(x_1 + hy_1)(x_2 - hy_2)}{(x_2 + hy_2)(x_2 - hy_2)} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 - y_2^2} + h\frac{x_1 y_2 + y_1 x_2}{x_2^2 - y_2^2}.$$

It is easy to see that this algebra of hyperbolic numbers is commutative and contains zero divisors. The hyperbolic conjugation of $z = x + hy$ is defined by

$$\overline{z} = z^{\dagger} = x - hy.$$

Note that $\overline{\overline{z}} = z$. Note also that for any hyperbolic numbers z_1, z_2, z we have

$$\begin{aligned} \overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2}, \\ \overline{z_1 \times z_2} &= \overline{z_1} \times \overline{z_2}, \\ \|z\|^2 &= z \times \overline{z} = x^2 - y^2. \end{aligned}$$

Now let us recall the definition of generalized Pandita numbers.

A generalized Pandita sequence $\{W_n\}_{n \geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n \geq 0}$ is defined by the fourth-order recurrence relations

$$W_n = 2W_{n-1} - W_{n-2} + W_{n-3} - W_{n-4} \tag{1.1}$$

with the initial values W_0, W_1, W_2, W_3 not all being zero. The sequence $\{W_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-1)} - W_{-(n-2)} + W_{-(n-3)} - W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

The first few generalized Pandita numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Pandita numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$W_0 - W_1 + 2W_2 - W_3$
2	W_2	$W_1 + W_2 - W_3$
3	W_3	$W_0 + W_1 - W_2$
4	$W_1 - W_0 - W_2 + 2W_3$	$2W_0 - 2W_1 + 2W_2 - W_3$
5	$W_1 - 2W_0 - W_2 + 3W_3$	$3W_2 - 2W_3$
6	$W_1 - 3W_0 - 2W_2 + 5W_3$	$3W_1 - 2W_2$
7	$2W_1 - 5W_0 - 4W_2 + 8W_3$	$3W_0 - 2W_1$
8	$3W_1 - 8W_0 - 6W_2 + 12W_3$	$W_0 - 3W_1 + 6W_2 - 3W_3$
9	$4W_1 - 12W_0 - 9W_2 + 18W_3$	$5W_1 - 2W_0 - W_2 - W_3$
10	$6W_1 - 18W_0 - 14W_2 + 27W_3$	$3W_0 + W_1 - 5W_2 + 2W_3$
11	$9W_1 - 27W_0 - 21W_2 + 40W_3$	$4W_0 - 8W_1 + 8W_2 - 3W_3$
12	$13W_1 - 40W_0 - 31W_2 + 59W_3$	$4W_1 - 4W_0 + 5W_2 - 4W_3$
13	$19W_1 - 59W_0 - 46W_2 + 87W_3$	$9W_1 - 12W_2 + 4W_3$

If we set $W_0 = 0, W_1 = 1, W_2 = 2, W_3 = 3$ then $\{W_n\}$ is the well-known Pandita sequence and if we set $W_0 = 4, W_1 = 2, W_2 = 2, W_3 = 5$ then $\{W_n\}$ is the well-known Pandita-Lucas sequence. In other words, Pandita sequence $\{P_n\}_{n\geq 0}$ and Pandita-Lucas sequence $\{S_n\}_{n\geq 0}$ are defined by the second-order recurrence relations

$$P_n = 2P_{n-1} - P_{n-2} + P_{n-3} - P_{n-4}, \qquad P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 3, \qquad n \geq 4, \tag{1.2}$$

and

$$S_n = 2S_{n-1} - S_{n-2} + S_{n-3} - S_{n-4}, \qquad S_0 = 4, S_1 = 2, S_2 = 2, S_3 = 5, \qquad n \geq 4 \tag{1.3}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{S_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = P_{-(n-1)} - P_{-(n-2)} + 2P_{-(n-3)} - P_{-(n-4)}$$

and

$$S_{-n} = S_{-(n-1)} - S_{-(n-2)} + 2S_{-(n-3)} - S_{-(n-4)},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.2) and (1.3) hold for all integer n.

We can list some important properties of generalized Pandita numbers that are needed.

- Binet formula of generalized Pandita sequence can be calculated using its characteristic equation which is given as

$$x^4 - 2x^3 + x^2 - x + 1 = (x^3 - x^2 - 1)(x - 1) = 0$$

The roots of characteristic equation are

$$\begin{aligned}
\alpha &= \frac{1}{3}+\left(\frac{29}{54}+\sqrt{\frac{31}{108}}\right)^{1/3}+\left(\frac{29}{54}-\sqrt{\frac{31}{108}}\right)^{1/3}, \\
\beta &= \frac{1}{3}+\omega\left(\frac{29}{54}+\sqrt{\frac{31}{108}}\right)^{1/3}+\omega^{2}\left(\frac{29}{54}-\sqrt{\frac{31}{108}}\right)^{1/3}, \\
\gamma &= \frac{1}{3}+\omega^{2}\left(\frac{29}{54}+\sqrt{\frac{31}{108}}\right)^{1/3}+\omega\left(\frac{29}{54}-\sqrt{\frac{31}{108}}\right)^{1/3}, \\
\delta &= 1,
\end{aligned}$$

where

$$\omega=\frac{-1+i\sqrt{3}}{2}=\exp(2\pi i/3).$$

Using these roots and the recurrence relation, Binet formula can be given as

$$\begin{aligned}
W_n &= \frac{z_1\alpha^n}{3\alpha-2}+\frac{z_2\beta^n}{3\beta-2}+\frac{z_3\gamma^n}{3\gamma-2}+z_4 \\
&= A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4,
\end{aligned}$$

where z_1, z_2 and z_3 are given below

$$\begin{aligned}
z_1 &= (\alpha W_3-\alpha(2-\alpha)W_2+(-\alpha^2+\alpha+1)W_1-W_0), \\
z_2 &= (\beta W_3-\beta(2-\beta)W_2+(-\beta^2+\beta+1)W_1-W_0), \\
z_3 &= (\gamma W_3-\gamma(2-\gamma)W_2+(-\gamma^2+\gamma+1)W_1-W_0), \\
z_4 &= -W_3+W_2+W_0.
\end{aligned}$$

and

$$\begin{aligned}
A_1 &= \frac{z_1}{3\alpha-2}, \\
A_2 &= \frac{z_2}{3\beta-2}, \\
A_3 &= \frac{z_3}{3\gamma-2}, \\
A_4 &= z_4.
\end{aligned} \tag{1.4}$$

Binet formula of Pandita and Pandita-Lucas sequences are

$$P_n=\frac{\alpha^{n+3}}{3\alpha-2}+\frac{\beta^{n+3}}{3\beta-2}+\frac{\gamma^{n+3}}{3\gamma-2}-1,$$

and

$$S_n=\alpha^n+\beta^n+\gamma^n+1,$$

respectively.

- The generating function for generalized Pandita numbers is

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 2W_0)x + (W_2 - 2W_1 + W_0)x^2 + (W_3 - 2W_2 + W_1 - W_0)x^3}{1 - 2x + x^2 - x^3 + x^4}.$$

For more details about generalized Pandita numbers, see [23].

Next, we give the exponential generating function of $\sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence W_n.

LEMMA 1. *[15, Lemma 1.4]Suppose that* $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ *is the exponential generating function of the generalized Pandita sequence* $\{W_n\}$.

Then $\sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is given by

$$\begin{aligned} \sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \frac{(\alpha W_3 - \alpha(2-\alpha)W_2 + (-\alpha^2 + \alpha + 1)W_1 - W_0)}{3\alpha - 2} e^{\alpha x} \\ &+ \frac{(\beta W_3 - \beta(2-\beta)W_2 + (-\beta^2 + \beta + 1)W_1 - W_0)}{3\beta - 2} e^{\beta x} \\ &+ \frac{(\gamma W_3 - \gamma(2-\gamma)W_2 + (-\gamma^2 + \gamma + 1)W_1 - W_0)}{3\gamma - 2} e^{\gamma x} \\ &+ (-W_3 + W_2 + W_0)e^x. \end{aligned}$$

The previous Lemma 1 gives the following results as particular examples.

COROLLARY 2. *Exponential generating function of Pandita and Pandita-Lucas numbers*

a): $\sum_{n=0}^{\infty} P_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (\frac{\alpha^{n+3}}{3\alpha - 2} + \frac{\beta^{n+3}}{3\beta - 2} + \frac{\gamma^{n+3}}{3\gamma - 2} - 1)\frac{x^n}{n!} = \frac{\alpha^3 e^{\alpha x}}{3\alpha - 2} + \frac{\beta^3 e^{\beta x}}{3\beta - 2} + \frac{\gamma^3 e^{\gamma x}}{3\gamma - 2} - e^x.$

b): $\sum_{n=0}^{\infty} S_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (\alpha^n + \beta^n + \gamma^n + 1)\frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^{\gamma x} + e^x.$

Next, we give some information on published papers related to hyperbolic and Dual hyperbolic numbers in literature.

- Cockle [12] presented the hyperbolic numbers with complex coefficients.
- Akar at al [17] introduced the dual hyperbolic numbers.
- Cheng and Thompson[8] studied dual numbers with complex coefficients.

Next, we give some information related to dual hyperbolic sequences presented in literature.

- Soykan at al [25] introduced dual hyperbolic generalized Pell numbers given by

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}$$

where generalized Pell numbers are given by $V_n = 2V_{n-1} + V_{n-2}$, $V_0 = a, V_1 = b$ $(n \geq 2)$ with the initial values V_0, V_1 not all being zero.

- Cihan at al [2] studied dual hyperbolic Fibonacci and Lucas numbers given by, respectively,

$$DHF_n = F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3},$$

$$DHL_n = L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3},$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1, L_n = L_{n-1} + L_{n-2}, L_0 = 2, L_1 = 1$.

- Soykan at al [27] introduced dual hyperbolic generalized Jacopsthal numbers given by

$$\widehat{J}_n = J_n + jJ_{n+1} + \varepsilon J_{n+2} + j\varepsilon J_{n+3}$$

where $J_n = J_{n-1} + 2J_{n-2}, J_0 = a, J_1 = b$.

- Bród at al [1] studied dual hyperbolic generalized Balancing numbers are

$$DHB_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}$$

where $B_n = 6B_{n-1} - B_{n-2}, B_0 = 0, B_1 = 1$.

- Yılmaz and Soykan [28] introduced dual hyperbolic generalized Guglielmo numbers are

$$\widehat{T}_0 = T_0 + jT_1 + \varepsilon T_2 + j\varepsilon T_3$$

where $T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, T_0 = 0, T_1 = 1, T_2 = 3$.

- Dikmen [3] introduced dual hyperbolic generalised Leonardo numbers given by

$$\widehat{l}_0 = l_0 + jl_1 + \varepsilon l_2 + j\varepsilon l_3$$

where $l_n = 2l_{n-1} - l_{n-3}, l_0 = 1, l_1 = 1, l_2 = 3$.

In this paper, we define the dual hyperbolic generalized Pandita numbers in the next section and give some properties of them.

2. Hyperbolic Generalized Pandita Numbers and their Generating Functions and Binet's Formulas

In this section, we define hyperbolic generalized Pandita numbers and present generating functions and Binet formulas for them. We now define hyperbolic generalized Pandita numbers over $\mathbb{H}_{\mathbb{D}}$. The nth hyperbolic generalized Pandita number is

$$HW_n = W_n + jW_{n+1}. \tag{2.1}$$

The sequence $\{HW_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$HW_{-n} = W_{-n} + jW_{-n+1},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.2) holds for all integer n.

Note that

$$\begin{aligned} HW_0 &= W_0 + jW_1 \\ HW_1 &= W_1 + jW_2 = W_1 + jW_2 \\ HW_2 &= W_2 + jW_3 = W_2 + jW_3 \\ HW_3 &= W_3 + jW_4 = W_3 + j(W_1 - W_0 - W_2 + 2W_3) \end{aligned}$$

It can be easily shown that

$$HW_n = 2HW_{n-1} - HW_{n-2} + HW_{n-3} - HW_{n-4} \tag{2.2}$$

and

$$HW_{-n} = HW_{-(n-1)} - HW_{-(n-2)} + 2HW_{-(n-3)} - HW_{-(n-4)}$$

The first few hyperbolic generalized Pandita numbers with positive subscript and negative subscript are given in the following Table 2.

Table 2. A few hyperbolic generalized Pandita numbers

n	HW_n	HW_{-n}
0	HW_0	HW_0
1	HW_1	$HW_0 - HW_1 + 2HW_2 - HW_3$
2	HW_2	$HW_1 + HW_2 - HW_3$
3	HW_3	$HW_0 + HW_1 - HW_2$
4	$HW_1 - HW_0 - HW_2 + 2HW_3$	$2HW_0 - 2HW_1 + 2HW_2 - HW_3$
5	$HW_1 - 2HW_0 - HW_2 + 3HW_3$	$3HW_2 - 2HW_3$
6	$HW_1 - 3HW_0 - 2HW_2 + 5HW_3$	$3HW_1 - 2HW_2$
7	$2HW_1 - 5HW_0 - 4HW_2 + 8HW_3$	$3HW_0 - 2HW_1$
8	$3HW_1 - 8HW_0 - 6HW_2 + 12HW_3$	$HW_0 - 3HW_1 + 6HW_2 - 3HW_3$
9	$4HW_1 - 12HW_0 - 9HW_2 + 18HW_3$	$5HW_1 - 2HW_0 - HW_2 - HW_3$
10	$6HW_1 - 18HW_0 - 14HW_2 + 27HW_3$	$3HW_0 + HW_1 - 5HW_2 + 2HW_3$
11	$9HW_1 - 27HW_0 - 21HW_2 + 40HW_3$	$4HW_0 - 8HW_1 + 8HW_2 - 3HW_3$
12	$13HW_1 - 40HW_0 - 31HW_2 + 59HW_3$	$4HW_1 - 4HW_0 + 5HW_2 - 4HW_3$
13	$19HW_1 - 59HW_0 - 46HW_2 + 87HW_3$	$9HW_1 - 12HW_2 + 4HW_3$

As special cases, the nth hyperbolic Pandita numbers and the nth hyperbolic Pandita Lucas numbers are given as

$$HP_n = P_n + jP_{n+1} \tag{2.3}$$

and

$$HS_n = S_n + jS_{n+1} \tag{2.4}$$

respectively. The sequences $\{HP_n\}_{n\geq 0}$ and $\{HS_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$HP_{-n} = P_{-(n-1)} - P_{-(n-2)} + 2P_{-(n-3)} - P_{-(n-4)}$$

and

$$HS_{-n} = S_{-(n-1)} - S_{-(n-2)} + 2S_{-(n-3)} - S_{-(n-4)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.3) and (2.4) holds for all integer n

For hyperbolic Pandita numbers (taking $W_n = P_n,\ \ P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 3,$) we get

$$\begin{aligned} HP_0 &= j, \\ HP_1 &= 2j+1, \\ HP_2 &= 3j+2, \end{aligned}$$

and for hyperbolic Pandita-Lucas numbers (taking $W_n = S_n,\ \ S_0 = 4, S_1 = 2, S_2 = 2, S_3 = 5,$) we get

$$\begin{aligned} HS_0 &= 2j+4, \\ HS_1 &= 2j+2. \\ HS_2 &= 5j+2 \end{aligned}$$

A few hyperbolic Pandita numbers and hyperbolic Pandita Lucas numbers with positive subscript and negative subscript are given in the following Table 3 and Table 4.

Table 3. hyperbolic Pandita numbers

n	HP_n	HP_{-n}
0	j	j
1	$2j+1$	0
2	$3j+2$	0
3	$5j+3$	-1
4	$8j+5$	$-j-1$
5	$12j+8$	$-j$

Table 4. hyperbolicPandita- Lucas numbers

n	HS_n	HS_{-n}
0	$2j+4$	$2j+4$
1	$2j+2$	$-4j+1$
2	$5j+2$	$j-1$
3	$6j+5$	$-j+4$
4	$7j+6$	$4j+3$
5	$11j++7$	$-3j-4$

Now, we will state Binet's formula for the hyperbolic generalized Pandita numbers and in the rest of the paper, we fix the following notations:

$$\widehat{\alpha} = 1+j\alpha, \tag{2.5}$$

$$\widehat{\beta} = 1+j\beta, \tag{2.6}$$

$$\widehat{\gamma} = 1+j\gamma \tag{2.7}$$

$$\widehat{\delta} = \widehat{1} = 1+j, \tag{2.8}$$

Note that we have the following identities:

$$\begin{aligned}
\widehat{\alpha}^2 &= 1+\alpha^2+2\alpha j, \\
\widehat{\beta}^2 &= 1+\beta^2+2j\beta, \\
\widehat{\alpha}\widehat{\beta} &= 1+\alpha\beta+(\alpha+\beta)\,j, \\
\widehat{\gamma}^2 &= 1+\gamma^2+2j\gamma, \\
\widehat{\delta}^2 &= \widehat{1}^2 = 2+2j, \\
\widehat{\gamma}\widehat{\delta} &= 1+\gamma+j+j\gamma
\end{aligned}$$

Theorem 3. *(Binet's Formula) For any integer n, the nth hyperbolic generalized Pandita number is*

$$HW_n = A_1\alpha^n\widehat{\alpha} + A_2\beta^n\widehat{\beta} + A_3\gamma^n\widehat{\gamma} + \widehat{1}A_4. \tag{2.9}$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}, \widehat{\delta}$ are given as (2.5)-(2.8)

Proof. Using Binet's formula of the generalized Pandita numbers given below

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4.$$

where A_1, A_2, A_3, A_4 are given in (1.4) we get

$$\begin{aligned} HW_n &= W_n + jW_{n+1} \\ &= A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4 + j(A_1\alpha^{n+1} + A_2\beta^{n+1} + A_3\gamma^{n+1} + A_4) \\ &= A_1\alpha^n(1+j\alpha) + A_2\beta^n(1+j\beta) + A_3\gamma^n(1+j\gamma) + A_4(1+j) \\ &= A_1\alpha^n\widehat{\alpha} + A_2\beta^n\widehat{\beta} + A_3\gamma^n\widehat{\gamma} + \widehat{1}A_4. \end{aligned}$$

This proves (2.9). □

As special cases, for any integer n, the Binet's Formula of nth hyperbolic Pandita number is

$$HP_n = \frac{\alpha^{n+3}\widehat{\alpha}}{3\alpha-2} + \frac{\beta^{n+3}\widehat{\beta}}{3\beta-2} + \frac{\gamma^{n+3}\widehat{\gamma}}{3\gamma-2} - \widehat{1} \tag{2.10}$$

and the Binet's Formula of nth hyperbolic Pandita-Lucas number is

$$HS_n = \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n + \widehat{\gamma}\gamma^n + \widehat{1}, \tag{2.11}$$

Next, we present generating function.

THEOREM 4. *Let $f_{HW_n}(x) = \sum_{n=0}^{\infty} HW_n x^n$ denote the generating function of hyperbolic generalized Pandita numbers is given as follows:*

$f_{HW_n}(x) = \sum_{n=0}^{\infty} HW_n x^n = \frac{1}{1-2x+x^2-x^3+x^4}(HW_0 + (HW_1 - 2HW_0)x + (HW_2 - 2HW_1 + HW_0)x^2 + (HW_3 - 2HW_2 + HW_1 - HW_0)x^3)$.

Proof. Using the definition of hyperbolic Pandita numbers, and substracting $xf(x)$, $x^2f(x)$ and $x^3f(x)$ from $f(x)$ we obtain $(1-2x+x^2-x^3+x^4)f_{HW_n}(x)$

$$\begin{aligned} & (1-2x+x^2-x^3+x^4)f_{HW_n}(x) \\ =\ & \sum_{n=0}^{\infty} HW_n x^n - 2x\sum_{n=0}^{\infty} HW_n x^n + x^2\sum_{n=0}^{\infty} HW_n x^n - x^3\sum_{n=0}^{\infty} HW_n x^n + x^4\sum_{n=0}^{\infty} HW_n x^n, \\ =\ & \sum_{n=0}^{\infty} HW_n x^n - 2\sum_{n=0}^{\infty} HW_n x^{n+1} + \sum_{n=0}^{\infty} HW_n x^{n+2} - \sum_{n=0}^{\infty} HW_n x^{n+3} + \sum_{n=0}^{\infty} HW_n x^{n+4}, \\ =\ & \sum_{n=0}^{\infty} HW_n x^n - 2\sum_{n=1}^{\infty} HW_{(n-1)} x^n + \sum_{n=2}^{\infty} HW_{(n-2)} x^n - \sum_{n=3}^{\infty} HW_{(n-3)} x^n + \sum_{n=4}^{\infty} HW_{(n-4)} x^n, \\ =\ & (HW_0 + HW_1x + HW_2x^2 + HW_3x^3) - 2(HW_0x + HW_1x^2 + HW_2x^3) + (HW_0x^2 + HW_1x^3) - HW_0x^3 \\ & + \sum_{n=4}^{\infty}(HW_n - 2HW_{n-1} - HW_{n-2} - HW_{n-3} + HW_{n-4})x^n, \\ =\ & HW_0 + (HW_1 - 2HW_0)x + (HW_2 - 2HW_1 + HW_0)x^2 + (HW_3 - 2HW_2 + HW_1 - HW_0)x^3. \end{aligned}$$

And rearranging above equation, we get (4). □

The following results are immediate consequences of the preceding Theorem.

Corollary 5. *For all integers n, we have following identities:*

a): $\sum_{n=0}^{\infty} HP_n x^n = \dfrac{j+x}{1-2x+x^2-x^3+x^4}$.

b): $\sum_{n=0}^{\infty} HS_n x^n = \dfrac{(2j+4)+(-2j-6)x+(3j+2)x^2+(-4j+7)x^3}{1-2x+x^2-x^3+x^4}$.

Theorem (4) gives the following results as special cases,

$(1-2x+x^2-x^3+x^4)f_{HP_n}(x) = HP_0 + (HP_1-2HP_0)x + (HP_2-2HP_1+HP_0)x^2 + (HP_3-2HP_2+HP_1-HP_0)x^3 = j+x,$

$(1-2x+x^2-x^3+x^4)f_{HS_n}(x) = HS_0 + (HS_1-2HS_0)x + (HS_2-2HS_1+HS_0)x^2 + (HS_3-2HS_2+HS_1-HS_0)x^3 = (2j+4)+(-2j-6)x+(3j+2)x^2+(-4j+7)x^3.$

Next, we give the exponential hyperbolic generating function of $\sum\limits_{n=0}^{\infty} HW_n \frac{x^n}{n!}$ of the sequence HW_n.

Lemma 6. *Suppose that $f_{HW_n}(x) = \sum\limits_{n=0}^{\infty} HW_n \frac{x^n}{n!}$ is the exponential hyperbolic generating function of the generalized Pandita sequence $\{HW_n\}$.*

Then $\sum\limits_{n=0}^{\infty} HW_n \frac{x^n}{n!}$ is given by

$$\sum_{n=0}^{\infty} HW_n \frac{x^n}{n!} == A_1 e^{\alpha x}\widehat{\alpha} + A_2 e^{\beta x}\widehat{\beta} + A_3 e^{\gamma x}\widehat{\gamma} + A_4 e^{x}\widehat{1}.$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}, \widehat{\delta}$ are given as (2.5)-(2.8)

Proof. Using Binet's formula

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4.$$

where A_1, A_2, A_3, A_4 are given in (1.4) we get

$$\begin{aligned}
\sum_{n=0}^{\infty} HW_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty} W_n \frac{x^n}{n!} + j\sum_{n=0}^{\infty} W_{n+1}\frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty}(A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4)\frac{x^n}{n!} + j\sum_{n=0}^{\infty}(A_1\alpha^{n+1} + A_2\beta^{n+1} + A_3\gamma^{n+1} + A_4)\frac{x^n}{n!} \\
&= (A_1e^{\alpha x} + A_2e^{\beta x} + A_3e^{\gamma x} + A_4e^{x}) + j(A_1\alpha e^{\alpha x} + A_2\beta e^{\beta x} + A_3\gamma e^{\gamma x} + A_4e^{x}) \\
&= A_1e^{\alpha x}(1+j\alpha) + A_2e^{\beta x}(1+j\beta) + A_3e^{\gamma x}(1+j\gamma) + A_4e^{x}(1+j) \\
&= A_1e^{\alpha x}\widehat{\alpha} + A_2e^{\beta x}\widehat{\beta} + A_3e^{\gamma x}\widehat{\gamma} + A_4e^{x}\widehat{1}
\end{aligned}$$

This proves (6). □

The previous Lemma 6 gives the following results as particular examples.

Corollary 7. *Exponential hyperbolic generating function of Pandita and Pandita-Lucas numbers are*

a): $\sum\limits_{n=0}^{\infty} HP_n \frac{x^n}{n!} = \dfrac{\alpha^3 e^{\alpha x}\widehat{\alpha}}{3\alpha-2} + \dfrac{\beta^3 e^{\beta x}\widehat{\beta}}{3\beta-2} + \dfrac{\gamma^3 e^{\gamma x}\widehat{\gamma}}{3\gamma-2} - e^{x}\widehat{1}.$

b): $\sum\limits_{n=0}^{\infty} HS_n \frac{x^n}{n!} = e^{\alpha x}\widehat{\alpha} + e^{\beta x}\widehat{\beta} + e^{\gamma x}\widehat{\gamma} + e^{x}\widehat{1}.$

3. Obtaining Binet Formula From Generating Function

We next find Binet's formula generalized hyperbolic Pandita number $\{HW_n\}$ by the use of generating function for HW_n.

THEOREM 8. *Binet's formula of generalized hyperbolic Pandita numbers:*

$$HW_n = \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}. \tag{3.1}$$

where

$$\begin{aligned}
q_1 &= HW_0\alpha^3 + (HW_1 - 2HW_0)\,\alpha^2 + (HW_0 - 2HW_1 + HW_2)\,\alpha - HW_0 + HW_1 - 2HW_2 + HW_3, \\
q_2 &= HW_0\beta^3 + (HW_1 - 2HW_0)\,\beta^2 + (HW_0 - 2HW_1 + HW_2)\,\beta - HW_0 + HW_1 - 2HW_2 + HW_3, \\
q_3 &= HW_0\gamma^3 + (HW_1 - 2HW_0)\,\gamma^2 + (HW_0 - 2HW_1 + HW_2)\,\gamma - HW_0 + HW_1 - 2HW_2 + HW_3, \\
q_4 &= HW_0\delta^3 + (HW_1 - 2HW_0)\,\delta^2 + (HW_0 - 2HW_1 + HW_2)\,\delta - HW_0 + HW_1 - 2HW_2 + HW_3.
\end{aligned}$$

Proof. Let

$$h(x) = x^4 - x^3 + x^2 - 2x + 1.$$

Then for some α, β, γ and δ we write

$$h(x) = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x).$$

i.e.,

$$x^4 - x^3 + x^2 - 2x + 1 = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x). \tag{3.2}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ and $\frac{1}{\delta}$ are the roots of $h(x)$. This gives α, β, γ and δ as the roots of

$$h(\frac{1}{x}) = \frac{1}{x^2} - \frac{2}{x} - \frac{1}{x^3} + \frac{1}{x^4} + 1 = 0.$$

This implies $x^4 - x^3 + x^2 - 2x + 1 = 0$. Now, by it follows that

$$\sum_{n=0}^{\infty} HW_n x^n = \frac{(HW_1 - HW_0 - 2HW_2 + HW_3)\,x^3 + (HW_0 - 2HW_1 + HW_2)\,x^2 + (HW_1 - 2HW_0)\,x + HW_0}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}.$$

Then we write

$$\frac{(HW_1 - HW_0 - 2HW_2 + HW_3)\,x^3 + (HW_0 - 2HW_1 + HW_2)\,x^2 + (HW_1 - 2HW_0)\,x + HW_0}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)} \tag{3.3}$$

$$= \frac{B_1}{(1-\alpha x)} + \frac{B_2}{(1-\beta x)} + \frac{B_3}{(1-\gamma x)} + \frac{B_4}{(1-\delta x)}. \tag{3.4}$$

So

$$\begin{aligned}
& \left(HW_1 - HW_0 - 2HW_2 + HW_3\right) x^3 + \left(HW_0 - 2HW_1 + HW_2\right) x^2 + \left(HW_1 - 2HW_0\right) x + HW_0 \\
= & \; B_1(1-\beta x)(1-\gamma x)(1-\delta x) + B_2(1-\alpha x)(1-\gamma x)(1-\delta x) \\
& + B_3(1-\alpha x)(1-\beta x)(1-\delta x) + B_3(1-\alpha x)(1-\beta x)(1-\gamma x).
\end{aligned}$$

If we consider $x = \frac{1}{\alpha}$, we get $HW_0 + \frac{1}{\alpha^2}\left(HW_0 - 2HW_1 + HW_2\right) - \frac{1}{\alpha^3}\left(HW_0 - HW_1 + 2HW_2 - HW_3\right) + \frac{1}{\alpha}\left(HW_1 - 2HW_0\right) = -B_1\left(\frac{1}{\alpha}\beta - 1\right)\left(\frac{1}{\alpha}\gamma - 1\right)\left(\frac{1}{\alpha}\delta - 1\right).$

This gives

$$\begin{aligned}
B_1 &= \alpha^3(HW_0 + \frac{1}{\alpha^2}\left(HW_0 - 2HW_1 + HW_2\right) + \frac{1}{\alpha^3}\left(HW_1 - 5HW_0 - 4HW_2 + HW_3\right) + \frac{1}{\alpha}\left(HW_1 - 2HW_0\right)) \\
&= \frac{HW_0\alpha^3 + \left(HW_1 - 2HW_0\right)\alpha^2 + \left(HW_0 - 2HW_1 + HW_2\right)\alpha - HW_0 + HW_1 - 2HW_2 + HW_3}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
B_2 &= \frac{HW_0\beta^3 + \left(HW_1 - 2HW_0\right)\beta^2 + \left(HW_0 - 2HW_1 + HW_2\right)\beta - HW_0 + HW_1 - 2HW_2 + HW_3}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\
B_3 &= \frac{HW_0\gamma^3 + \left(HW_1 - 2HW_0\right)\gamma^2 + \left(HW_0 - 2HW_1 + HW_2\right)\gamma - HW_0 + HW_1 - 2HW_2 + HW_3}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\
B_4 &= \frac{HW_0\delta^3 + \left(HW_1 - 2HW_0\right)\delta^2 + \left(HW_0 - 2HW_1 + HW_2\right)\delta - HW_0 + HW_1 - 2HW_2 + HW_3}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}.
\end{aligned}$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty} HW_n x^n = B_1(1-\alpha x)^{-1} + B_2(1-\beta x)^{-1} + B_3(1-\gamma x)^{-1} + B_4(1-\delta x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} HW_n x^n = B_1\sum_{n=0}^{\infty}\alpha^n x^n + B_2\sum_{n=0}^{\infty}\beta^n x^n + B_3\sum_{n=0}^{\infty}\gamma^n x^n + B_4\sum_{n=0}^{\infty}\delta^n x^n = \sum_{n=0}^{\infty}(B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n)x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$HW_n = B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n.$$

The following identity establishes a relationship between the hyperbolic Pandita numbers and the Pandita–Lucas numbers.

Corollary 9. *For all integers m, n the following identities holds:*

$$HW_{m+n} = P_{m-2}HW_{n+3} + (P_{m-4} - P_{m-3} - P_{m-5})HW_{n+2} + (P_{m-3} - P_{m-4})HW_{n+1} - HW_n P_{m-3}.$$

Proof. First we assume that $m, n \geq 0$.The Theorem (9) can be proved by mathematical induction on m. If $m = 0$ we get

$$HW_n = P_{-2}HW_{n+3} + (P_{-4} - P_{-3} - P_{-5})HW_{n+2} + (P_{-3} - P_{-4})HW_{n+1} - HW_n P_{-3}.$$

which is true since $P_{-2}=0, P=-1, P_{-4}=-1, P_{-5}=0$. Assume that the equality holds for $m\leq k$. For $m=k+1$, we get

$$
\begin{aligned}
HW_{k+1+n} &= 2HW_{n+k}-HW_{n+k-1}+HW_{n+k-2}-HW_{n+k-3},\\
& 2(P_{m-2}HW_{n+3}+(P_{m-4}-P_{m-3}-P_{m-5})HW_{n+2}+(P_{m-3}-P_{m-4})HW_{n+1}-HW_nP_{m-3})\\
& -(P_{m-3}HW_{n+3}+(P_{m-5}-P_{m-4}-P_{m-6})HW_{n+2}+(P_{m-4}-P_{m-5})HW_{n+1}-HW_nP_{m-4})\\
& +(P_{m-4}HW_{n+3}+(P_{m-6}-P_{m-5}-P_{m-7})HW_{n+2}+(P_{m-5}-P_{m-6})HW_{n+1}-HW_nP_{m-5})\\
& -(P_{m-5}HW_{n+3}+(P_{m-7}-P_{m-6}-P_{m-8})HW_{n+2}+(P_{m-6}-P_{m-7})HW_{n+1}-HW_nP_{m-6}).
\end{aligned}
$$

Consequently, by mathematical induction on m, this proves Theorem 9.

The other cases of m,n can be proved smilarly for all integers m,n. □

Taking $HW_n=HP_n$ or $HW_n=HS_n$ in above Theorem, respectively, we get:

Corollary 10.

$$
\begin{aligned}
HP_{m+n} &= P_{m-2}HP_{n+3}+(P_{m-4}-P_{m-3}-P_{m-5})HP_{n+2}+(P_{m-3}-P_{m-4})HP_{n+1}-HP_nP_{m-3},\\
HS_{m+n} &= P_{m-2}HS_{n+3}+(P_{m-4}-P_{m-3}-P_{m-5})HS_{n+2}+(P_{m-3}-P_{m-4})HS_{n+1}-HS_nP_{m-3}.
\end{aligned}
$$

4. SIMSON'S FORMULA

In this section, we present Simpson's formula for the hyperbolic generalized Pandita numbers . This is a special case of [22, Theorem 4.1].

Theorem 11. *(Simpson's formula for hyperbolic generalized Pandita numbers) For all integers n we have,*

$$
\begin{vmatrix}
HW_{n+3} & HW_{n+2} & HW_{n+1} & HW_n\\
HW_{n+2} & HW_{n+1} & HW_n & HW_{n-1}\\
HW_{n+1} & HW_n & HW_{n-1} & HW_{n-2}\\
HW_n & HW_{n-1} & HW_{n-2} & HW_{n-3}
\end{vmatrix}
=
\begin{vmatrix}
HW_3 & HW_2 & HW_1 & HW_0\\
HW_2 & HW_1 & HW_0 & HW_{-1}\\
HW_1 & HW_0 & HW_{-1} & HW_{-2}\\
HW_0 & HW_{-1} & HW_{-2} & HW_{-3}
\end{vmatrix}
$$

$= (HW_0+HW_2-HW_3)(-HW_3^3+3HW_2^3-HW_1^3+HW_0^3+(5HW_2-2HW_1)HW_3^2+(4HW_0-5HW_1-8HW_3)HW_2^2+(4HW_0+4HW_2-5HW_3)HW_1^2$

$+(HW_2-3HW_1-HW_3)HW_0^2+9HW_1HW_2HW_3-3HW_0HW_2HW_3+5HW_0HW_1HW_3-7HW_0HW_1HW_2)$

.

Proof. Using Theorem 3 it can be proved by using induction use [22, Theorem 4.1]

From the Theorem 11 we get the following Corollary.

Corollary 12. *For all integers n, the Simson's formulas of hyperbolic Pandita numbers and hyperbolic Pandita Lucas numbers are given as,*

a): $\begin{vmatrix} HP_{n+3} & HP_{n+2} & HP_{n+1} & HP_n \\ HP_{n+2} & HP_{n+1} & HP_n & HP_{n-1} \\ HP_{n+1} & HP_n & HP_{n-1} & HP_{n-2} \\ HP_n & HP_{n-1} & HP_{n-2} & HP_{n-3} \end{vmatrix} = 3j+3,$

b): $\begin{vmatrix} HS_{n+3} & HS_{n+2} & HS_{n+1} & HS_n \\ HS_{n+2} & HS_{n+1} & HS_n & HS_{n-1} \\ HS_{n+1} & HS_n & HS_{n-1} & HS_{n-2} \\ HS_n & HS_{n-1} & HS_{n-2} & HS_{n-3} \end{vmatrix} = -93j-93,$

respectively.

5. Linear Sums

In this section, we give the summation formulas of the hyperbolic generalized Pandita numbers with positive and negatif subscripts.

Now, we present the summation formulas of the generalized Pandita numbers.

Theorem 13. *For the generalized Pandita numbers, we have the following formulas:*

(a): $\sum_{k=0}^{n} W_k = -(n+3)W_{n+3} + (n+4)W_{n+2} + (n+4)W_n + 3W_3 - 4W_2 - 3W_0.$

(b): $\sum_{k=0}^{n} W_{2k} = \frac{1}{3}(-3(n+2)W_{2n+2} + (3n+8)W_{2n+1} + 2W_{2n} + (3n+7)W_{2n-1} + 7W_3 - 8W_2 - W_1 - 6W_0).$

(c): $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{3}(-(3n+4)W_{2n+2} + (3n+8)W_{2n+1} + W_{2n} + 3(n+2)W_{2n-1} + 6W_3 - 8W_2 + W_1 - 7W_0).$

Proof. For the proof, see Soykan [24, Theorem 3.12]. □

Theorem 14. *For the hyperbolic Pandita numbers, we have the following formulas:*

(a): $\sum_{k=0}^{n} HW_k = -(n+3)HW_{n+3} + (n+4)HW_{n+2} + (n+4)HW_n + 3HW_3 - 4HW_2 - 3HW_0.$

(b): $\sum_{k=0}^{n} HW_{2k} = \frac{1}{3}(-3(n+2)HW_{2n+2} + (3n+8)HW_{2n+1} + 2HW_{2n} + (3n+7)HW_{2n-1} + 7HW_3 - 8HW_2 - HW_1 - 6HW_0).$

(c): $\sum_{k=0}^{n} HW_{2k+1} = \frac{1}{3}(-(3n+4)HW_{2n+2} + (3n+8)HW_{2n+1} + HW_{2n} + 3(n+2)HW_{2n-1} + 6HW_3 - 8HW_2 + HW_1 - 7HW_0).$

Proof. Use Theorem 13 and the definition of HW_n. □

As a special case of the theorem 14, we present the following Corollary.

Corollary 15. *For $n \geq 0$, hyperbolic Pandita numbers have the following properties:*

(a): $\sum_{k=0}^{n} HW_k = -(n+3)HW_{n+3} + (n+4)HW_{n+2} + (n+4)HW_n + 1 - 5j\varepsilon - 2\varepsilon.$

(b): $\sum_{k=0}^{n} HW_{2k} = \frac{1}{3}(-3(n+2)HW_{2n+2} + (3n+8)HW_{2n+1} + 2HW_{2n} + (3n+7)HW_{2n-1} + 3j + \varepsilon - 3j\varepsilon + 4).$

(c): $\sum_{k=0}^{n} HW_{2k+1} = \frac{1}{3}(-(3n+4)HW_{2n+2}+(3n+8)HW_{2n+1}+HW_{2n}+3(n+2)HW_{2n-1}+j-3\varepsilon-8j\varepsilon+3)$.

COROLLARY 16. *For $n \geq 0$, hyperbolic Pandita Lucas numbers have the following properties.*

(a): $\sum_{k=0}^{n} HS_k = -(n+3)HS_{n+3} + (n+4)HS_{n+2} + (n+4)HS_n - 8j - 9\varepsilon - 10j\varepsilon - 5$.

(b): $\sum_{k=0}^{n} HS_{2k} = \frac{1}{3}(-3(n+2)HS_{2n+2} + (3n+8)HS_{2n+1} + 2HS_{2n} + (3n+7)HS_{2n-1} + -12j - 16\varepsilon - 15j\varepsilon - 7)$.

(c): $\sum_{k=0}^{n} HS_{2k+1} = \frac{1}{3}(-(3n+4)HS_{2n+2} + (3n+8)HS_{2n+1} + HS_{2n} + 3(n+2)HS_{2n-1} + -16j - 15\varepsilon - 19j\varepsilon - 12)$.

Next, we give the ordinary generating functions of some special cases of hyperbolic generalized Pandita numbers.

THEOREM 17. *The ordinary generating functions of the sequences HW_{2n}, HW_{2n+1} are given as follows:*

(a): $\sum_{n=0}^{\infty} HW_{2n}x^n = \dfrac{HW_2\left(x^3+3x^2-x\right) + HW_0\left(2x^2+2x-1\right) - HW_1\left(x^2-x^3\right) - HW_3\left(x^3+2x^2\right)}{-x^4-x^3+x^2+2x-1}$.

(b): $\sum_{n=0}^{\infty} HW_{2n+1}x^n = \dfrac{HW_0\left(x^3+2x^2\right) - HW_3\left(x^3+x^2+x\right) - HW_1\left(x^3-2x+1\right) + HW_2\left(2x^3+x^2\right)}{-x^4-x^3+x^2+2x-1}$.

Proof. Similary, the proof can be constructed as in [4, Theorem 4].

From the last Theorem, we have the following Corollary which gives sum formula of hyperbolic Pandita numbers (Take $HW_n = HP_n$ whit $HP_0 = j$,$HP_1 = 2j+1, HP_2 = 3j+2, HP_3 = 5j+3$)

COROLLARY 18. *For $n \geq 0$ hyperbolic Pandita numbers have the following properties.*

(a): $\sum_{n=0}^{\infty} HW_{2n}x^n = \dfrac{j+x}{1-2x+x^2-x^3+x^4}$,

(b): $\sum_{n=0}^{\infty} HW_{2n+1}x^n = \dfrac{(2j+4)+(-2j-6)x+(3j+2)x^2+(-4j+7)x^3}{1-2x+x^2-x^3+x^4}$.

6. Matrices related with Hyperbolic Generalized Pandita Numbers

In this sectiıon, using hyperbolic Pandita numbers, we give some matrices related to hyperbolic Pandita numbers.

We define the square matrix A of order 4 as

$$A = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

uch that $detA = 1$. Note that

$$A^n = \begin{pmatrix} P_{n+1} & -P_n + P_{n-1} - P_{n-2} & P_n - P_{n-1} & -P_n \\ P_n & -P_{n-1} + P_{n-2} - P_{n-3} & P_{n-1} - P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-2} + P_{n-3} - P_{n-4} & P_{n-2} - P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-3} + P_{n-4} - P_{n-5} & P_{n-3} - P_{n-4} & -P_{n-3} \end{pmatrix}$$

for the proof see [26].

Then we give the following lemma.

LEMMA 19. *For $n \geq 0$ the following identitiy is true:*

$$\begin{pmatrix} HW_{n+3} \\ HW_{n+2} \\ HW_{n+1} \\ HW_{n} \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{n} \begin{pmatrix} HW_{3} \\ HW_{2} \\ HW_{1} \\ HW_{0} \end{pmatrix}.$$

Proof. The identitiy (19) can be proved by mathematical induction on n. If $n = 0$ we obtain

$$\begin{pmatrix} HW_{3} \\ HW_{2} \\ HW_{1} \\ HW_{0} \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{0} \begin{pmatrix} HW_{3} \\ HW_{2} \\ HW_{1} \\ HW_{0} \end{pmatrix}$$

which is true. We assume that the identity given holds for $n = k$. Thus the following identitiy is true

$$\begin{pmatrix} HW_{k+3} \\ HW_{k+2} \\ HW_{k+1} \\ HW_{k} \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} HW_{3} \\ HW_{2} \\ HW_{1} \\ HW_{0} \end{pmatrix}.$$

For $n = k + 1$, we get

$$\begin{aligned} \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} HW_{3} \\ HW_{2} \\ HW_{1} \\ HW_{0} \end{pmatrix} &= \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} HW_{3} \\ HW_{2} \\ HW_{1} \\ HW_{0} \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} HW_{k+3} \\ HW_{k+2} \\ HW_{k+1} \\ HW_{k} \end{pmatrix} \\ &= \begin{pmatrix} HW_{k+4} \\ HW_{k+3} \\ HW_{k+2} \\ HW_{k+1} \end{pmatrix}. \end{aligned}$$

Consequently, by mathematical induction on n, the proof completed. □

We define

$$N_{HW} = \begin{pmatrix} HW_3 & HW_2 & HW_1 & HW_0 \\ HW_2 & HW_1 & HW_0 & HW_{-1} \\ HW_1 & HW_0 & HW_{-1} & HW_{-2} \\ HW_0 & HW_{-1} & HW_{-2} & HW_{-3} \end{pmatrix}, \tag{6.1}$$

$$E_{HW} = \begin{pmatrix} HW_{n+3} & HW_{n+2} & HW_{n+1} & HW_n \\ HW_{n+2} & HW_{n+1} & HW_n & HW_{n-1} \\ HW_{n+1} & HW_n & HW_{n-1} & HW_{n-2} \\ HW_n & HW_{n-1} & HW_{n-2} & HW_{n-3} \end{pmatrix}. \tag{6.2}$$

Now, we have the following theorem with N_{HW} and E_{HW}

THEOREM 20. *Using N_{HW} and E_{HW}, we get*

$$A^n N_{HW} = E_{HW}.$$

Proof. Note that we get

$$\begin{aligned} A^n N_{HW} &= \begin{pmatrix} P_{n+1} & -P_n + P_{n-1} - P_{n-2} & P_n - P_{n-1} & -P_n \\ P_n & -P_{n-1} + P_{n-2} - P_{n-3} & P_{n-1} - P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-2} + P_{n-3} - P_{n-4} & P_{n-2} - P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-3} + P_{n-4} - P_{n-5} & P_{n-3} - P_{n-4} & -P_{n-3} \end{pmatrix} \begin{pmatrix} HW_3 & HW_2 & HW_1 & HW_0 \\ HW_2 & HW_1 & HW_0 & HW_{-1} \\ HW_1 & HW_0 & HW_{-1} & HW_{-2} \\ HW_0 & HW_{-1} & HW_{-2} & HW_{-3} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \end{aligned}$$

where

$$
\begin{aligned}
a_{11} &= HW_1\left(P_n - P_{n-1}\right) - HW_2\left(P_n - P_{n-1} + P_{n-2}\right) - HW_0 P_n + W_3 P_{n+1} = HW_{n+3}, \\
a_{12} &= HW_0\left(P_n - P_{n-1}\right) - HW_1\left(P_n - P_{n-1} + P_{n-2}\right) - P_n HW_{-1} + HW_2 P_{n+1} = HW_{n+2}, \\
a_{13} &= HW_{-1}\left(P_n - P_{n-1}\right) - HW_0\left(P_n - P_{n-1} + P_{n-2}\right) - P_n HW_{-2} + HW_1 P_{n+1} = HW_{n+1}, \\
a_{14} &= HW_{-2}\left(P_n - P_{n-1}\right) - HW_{-1}\left(P_n - P_{n-1} + P_{n-2}\right) - P_n HW_{-3} + HW_0 P_{n+1} = HW_n, \\
a_{21} &= HW_3 P_n - HW_2\left(P_{n-1} - P_{n-2} + P_{n-3}\right) + HW\left(P_{n-1} - P_{n-2}\right) - HW_0 P_{n-1} = HW_{n+2}, \\
a_{22} &= HW_2 P_n - HW_{-1} P_{n-1} - HW_1\left(P_{n-1} - P_{n-2} + P_{n-3}\right) + HW\left(P_{n-1} - P_{n-2}\right) = HW_{n+1}, \\
a_{23} &= HW_{-1}\left(P_{n-1} - P_{n-2}\right) - HW_{-2} P_{n-1} + HW_1 P_n - HW_0\left(P_{n-1} - P_{n-2} + P_{n-3}\right) = HW_n, \\
a_{24} &= HW_{-2}\left(P_{n-1} - P_{n-2}\right) - HW_{-3} P_{n-1} + HW_0 P_n - HW_{-1}\left(P_{n-1} - P_{n-2} + P_{n-3}\right) = HW_{n-1}, \\
a_{31} &= HW_1\left(P_{n-2} - P_{n-3}\right) - HW_2\left(P_{n-2} - P_{n-3} + P_{n-4}\right) - HW_0 P_{n-2} + HW_3 P_{n-1} = HW_{n+1}, \\
a_{32} &= HW_0\left(P_{n-2} - P_{n-3}\right) - HW_1\left(P_{n-2} - P_{n-3} + P_{n-4}\right) - HW_{-1} P_{n-2} + HW_2 P_{n-1} = HW_n, \\
a_{33} &= HW_{-1}\left(P_{n-2} - P_{n-3}\right) - HW_{-2} P_{n-2} - HW_0\left(P_{n-2} - P_{n-3} + P_{n-4}\right) + HW_1 P_{n-1} = HW_{n-1}, \\
a_{34} &= HW_{-2}\left(P_{n-2} - P_{n-3}\right) - HW_{-3} P_{n-2} - HW_{-1}\left(P_{n-2} - P_{n-3} + P_{n-4}\right) + HW_0 P_{n-1} = HW_{n-2}, \\
a_{41} &= HW_1\left(P_{n-3} - P_{n-4}\right) - HW_2\left(P_{n-3} - P_{n-4} + P_{n-5}\right) - HW_0 P_{n-3} + HW_3 P_{n-2} = HW_n, \\
a_{42} &= HW_0\left(P_{n-3} - P_{n-4}\right) - HW_1\left(P_{n-3} - P_{n-4} + P_{n-5}\right) - HW_{-1} P_{n-3} + HW_2 P_{n-2} = HW_{n-1}, \\
a_{43} &= HW_{-1}\left(P_{n-3} - P_{n-4}\right) - HW_{-2} P_{n-3} - HW_0\left(P_{n-3} - P_{n-4} + P_{n-5}\right) + HW_1 P_{n-2} = HW_{n-2}, \\
a_{44} &= HW_{-2}\left(P_{n-3} - P_{n-4}\right) - HW_{-3} P_{n-3} - HW_{-1}\left(P_{n-3} - P_{n-4} + P_{n-5}\right) + HW_0 P_{n-2} = HW_{n-3}.
\end{aligned}
$$

Using the theorem (9) the proof is done. □

By taking $HW_n = HP_n$ with HP_0, HP_1, HP_2, HP_3 in (6.1) and (6.2)

$HW_n = S_n$ with HS_0, HS_1, HS_2, HS_3 in (6.1) and (6.2) respectively, we get:

$$N_{HP} = \begin{pmatrix} 5j+3 & 3j+2 & 2j+1 & j \\ 3j+2 & 2j+1 & j & 0 \\ 2j+1 & j & 0 & 0 \\ j & 0 & 0 & -1 \end{pmatrix},$$

$$E_{HP} = \begin{pmatrix} HP_{n+3} & HP_{n+2} & HP_{n+1} & HP_n \\ HP_{n+2} & HP_{n+1} & HP_n & HP_{n-1} \\ HP_{n+1} & HP_n & HP_{n-1} & HP_{n-2} \\ HP_n & HP_{n-1} & HP_{n-2} & HP_{n-3} \end{pmatrix},$$

$$N_{HS} = \begin{pmatrix} 6j+5 & 5j+2 & 2j+2 & 2j+4 \\ 5j+2 & 2j+2 & 2j+4 & 4j+1 \\ 2j+2 & 2j+4 & -4j+1 & j-1 \\ 2j+4 & -4j+1 & j-1 & -j+4 \end{pmatrix},$$

$$E_{HS} = \begin{pmatrix} HS_{n+3} & HS_{n+2} & HS_{n+1} & HS_n \\ HS_{n+2} & HS_{n+1} & HS_n & HS_{n-1} \\ HS_{n+1} & S_n & HS_{n-1} & HS_{n-2} \\ HS_n & HS_{n-1} & HS_{n-2} & HS_{n-3} \end{pmatrix}.$$

From Theorem [20], we can write the following corollary.

Corollary 21. *The following identities are hold:*

- **a):** $A^n N_{HP} = E_{HP}$.
- **b):** $A^n N_{HS} = E_{HS}$.

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