

Value distribution of differential polynomials

Abstract

In this paper we deal with the value distribution of those differential polynomials which involve two distinct transcendental meromorphic functions and obtain analogous results of I. Lahiri [7] for these kind of differential polynomials.

Keywords and Phrases : Differential polynomials, monomials, transcendental meromorphic functions, small functions, value distribution.

1 Introduction, Definitions and Notations

We first consider the following definition.

Definition 1.1. Let $f(z)$ be a transcendental meromorphic function in the complex plane and let $m(r, f)$, $N(r, f)$, $T(r, f)$ have the usual meaning of Nevanlinna Theory. Let $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$ except possibly a set of finite linear measure. Then for a positive integer j , by a monomial in $f(z)$ we mean an expression of the form

$$M_j[f] = [f(z)]^{n_{0j}} [f^{(1)}(z)]^{n_{1j}} \dots [f^{(k)}(z)]^{n_{kj}}$$

where $n_{0j}, n_{1j}, \dots, n_{kj}$ are non-negative integers. We define $d(M_j) = \sum_{i=0}^k n_{ij}$ as the degree of $M_j[f]$ and $\Gamma_{M_j} = \sum_{i=0}^k (i+1)n_{ij}$ as the weight of $M_j[f]$. Next, let $a_j(z)$ be meromorphic functions in the plane satisfying $T(r, a_j) = S(r, f)$ for $j = 1, 2, \dots, n$. We define a differential polynomial in $f(z)$ as

$$P[f] = \sum_{j=1}^n a_j(z) M_j[f].$$

Also, we define

$$\bar{d}(P) = \max\{d(M_j) : 1 \leq j \leq n\},$$

$$\underline{d}(P) = \min\{d(M_j) : 1 \leq j \leq n\}$$

and

$$\Gamma_P = \max\{\Gamma_{M_j} : 1 \leq j \leq n\}$$

as the degree, the lower degree and the weight of $P[f]$ respectively. Also, k , the highest order derivative of f in $P[f]$ is called the order of $P[f]$.

If in particular, $\bar{d}(P) = \underline{d}(P)$, then $P[f]$ is called homogeneous and otherwise it

is called non-homogeneous.

In [11], H.X. Yi proved the following theorem.

Theorem 1.1. Let f be a transcendental meromorphic function in the complex plane and $Q_1[f], Q_2[f]$ be differential polynomials in f satisfying $Q_1[f] \not\equiv 0, Q_2[f] \not\equiv 0$, and

$$P[f] = a_n f^n + a_{n-1} f^{n-1} + \dots + a_0 \quad (a_n \neq 0). \tag{1.1}$$

Now if

$$F = P[f]Q_1[f] + Q_2[f] \tag{1.2}$$

then

$$[n - \bar{d}(Q_2)]T(r, f) \leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{P[f]}) + [\Gamma_{Q_2} - \bar{d}(Q_2) + 1]\bar{N}(r, f) + S(r, f).$$

Here it can be noted that the influence of $Q_1[f]$ on the value distribution of F is ignored. Later, in [7], I. Lahiri showed that Theorem 1.1. can further be improved if the influence of $Q_1[f]$ is taken into consideration and he proved the following theorem.

Theorem 1.2. Let f be a transcendental meromorphic function in the open complex plane, and $Q_1[f](\neq 0), Q_2[f](\neq 0)$ be two differential polynomials in f such that k and $\underline{d}(Q_1)$ be the order and lower degree of $Q_1[f]$, respectively and $P[f]$ is given by (1.1) where $a_n(\neq 0), a_{n-1}, \dots, a_0$ are small functions of f . Now if F is given by (1.2), then

$$[n - \bar{d}(Q_2)]T(r, f) \leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{P[f]}) + [\Gamma_{Q_2} - \bar{d}(Q_2) + 1]\bar{N}(r, f) - d[N(r, \frac{1}{f}) - N_{k+1}(r, \frac{1}{f})] + S(r, f),$$

where $d = \underline{d}(Q_1)$ if $n \geq \bar{d}(Q_2)$ and $d = 0$ if $n < \bar{d}(Q_2)$ and $N_{k+1}(r, \frac{1}{f})$ denotes the counting function of poles of f where a pole of f with multiplicity μ is counted μ times if $\mu \leq (k + 1)$ and $(k + 1)$ times if $\mu > (k + 1)$.

The proof of Theorem 1.2 is based on the following lemmas.

Lemma 1.1. [5] Let $Q^*[f]$ and $Q[f]$ denote differential polynomials in f with arbitrary meromorphic coefficients $q_1^*, q_2^*, \dots, q_n^*$ and q_1, q_2, \dots, q_k respectively. Suppose that $P[f]$ is given by (1.1). If $P[f]Q^*[f] = Q[f]$ and $\gamma_Q \leq n$, then

$$m(r, Q^*[f]) \leq \sum_{i=1}^n m(r, q_i^*) + \sum_{i=1}^k m(r, q_i) + S(r, f).$$

Lemma 1.2. [7] Let $P[f] = \sum_{j=1}^n a_j(z)M_j[f]$ be a differential polynomial in f of order and lower degree k and $\underline{d}(P)$, respectively. If z_0 is a zero of f with multiplicity $\mu (> k)$ and z_0 is not a pole of any of the coefficients a_j ($j = 1, 2, \dots, n$), then z_0 is a zero of $P[f]$ with multiplicity at least $(\mu - k)\underline{d}(P)$.

Lemma 1.3. [3] For a differential polynomial $P[f] = \sum_{j=1}^n a_j(z)M_j[f]$ the following inequality holds:

$$N(r, P[f]) \leq \bar{d}(P)N(r, f) + [\Gamma_P - \bar{d}(p)]\bar{N}(r, f) + S(r, f).$$

Lemma 1.4. [9] Let $P[f]$ be given by (1.1) where $a_n(\neq 0), a_{n-1}, \dots, a_0$ are small functions of f . Then

$$m(r, P[f]) = nm(r, f) + S(r, f).$$

Lemma 1.5. [5] If $Q[f]$ is a differential polynomial in f with arbitrary meromorphic coefficients $q_j, 1 \leq j \leq n$, then

$$m(r, Q[f]) \leq \bar{d}(Q)m(r, f) + \sum_{j=1}^n m(r, q_j) + S(r, f).$$

Lemma 1.6. [10] Let $P[f]$ be given by (1.1) where $a_n(\neq 0), a_{n-1}, \dots, a_0$ are small functions of f and $N(r, f) = S(r, f)$. Then

$$T(r, P[f]) = nT(r, f) + S(r, f).$$

Now we introduce the following analogous definition for homogeneous differential polynomials formed with two transcendental meromorphic functions f and g and try to prove analogous results of Theorem 1.2 for such kind of differential polynomials.

Definition 1.2. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions. Then we define a monomial in $f(z)$ and $g(z)$ as an expression of the form

$$M_j[f, g] = [f(z)]^{l_{0j}} [f^{(1)}(z)]^{l_{1j}} \dots [f^{(k)}(z)]^{l_{kj}} [g(z)]^{m_{0j}} [g^{(1)}(z)]^{m_{1j}} \dots [g^{(h)}(z)]^{m_{hj}}$$

where $l_{0j}, l_{1j}, \dots, l_{kj}, m_{0j}, m_{1j}, \dots, m_{hj}$ are non-negative integers. We define

$$d(M_j) = \sum_{i=0}^k l_{ij} + \sum_{i=0}^h m_{ij}$$

as the degree of $M_j[f, g]$ and

$$\Gamma_{M_j} = \sum_{i=0}^k (i+1)l_{ij} + \sum_{i=0}^h (i+1)m_{ij}$$

as the weight of $M_j[f, g]$.

Again, let $a_j(z)$ be meromorphic functions in the plane satisfying $T(r, a_j) =$

$S(r, f)$ and $T(r, a_j) = S(r, g)$ for $j = 1, 2, \dots, n$. Now we define a differential polynomial in $f(z)$ and $g(z)$ as

$$P[f, g] = \sum_{j=1}^n a_j(z)M_j[f, g]. \tag{1.3}$$

Here also we define

$$\begin{aligned} \bar{d}(P) &= \max\{d(M_j) : 1 \leq j \leq n\}, \\ \underline{d}(P) &= \min\{d(M_j) : 1 \leq j \leq n\} \end{aligned}$$

and

$$\Gamma_P = \max\{\Gamma_{M_j} : 1 \leq j \leq n\}$$

as the degree, the lower degree and the weight of $P[f, g]$ respectively. Also, $\max\{k, h\} = \nu$, the highest order derivative of f and g involved in $P[f, g]$ is called the order of $P[f, g]$.

If in particular, $\bar{d}(P) = \underline{d}(P)$, then $P[f, g]$ is called homogeneous and otherwise it is called non-homogeneous.

In addition, for our convenience we define the followings:

$$\bar{d}_f(P) = \max\left\{\sum_{i=0}^k l_{ij} : 1 \leq j \leq n\right\};$$

$$\underline{d}_f(P) = \min\left\{\sum_{i=0}^k l_{ij} : 1 \leq j \leq n\right\};$$

$$\bar{d}_g(P) = \max\left\{\sum_{i=0}^h m_{ij} : 1 \leq j \leq n\right\}$$

and

$$\underline{d}_g(P) = \min\left\{\sum_{i=0}^h m_{ij} : 1 \leq j \leq n\right\}.$$

In particular, if there exists atleast one $j(= j_1, \text{ say})$ such that

$$\bar{d}_f(P) = \sum_{i=0}^k l_{ij_1} \quad \text{and} \quad \bar{d}_g(P) = \sum_{i=0}^h m_{ij_1} \tag{1.4}$$

then

$$\bar{d}(P) = \bar{d}_f(P) + \bar{d}_g(P).$$

Similarly if there exists atleast one $j(= j_2, \text{ say})$ such that

$$\underline{d}_f(P) = \sum_{i=0}^k l_{ij_2} \quad \text{and} \quad \underline{d}_g(P) = \sum_{i=0}^h m_{ij_2}, \tag{1.5}$$

then

$$\underline{d}(P) = \underline{d}_f(P) + \underline{d}_g(P).$$

To clarify this scenario we refer [1].

We do not explain the standard notations and definitions of the value distribution theory as those are available in [6].

2 Main Results

In this section we present our main results of the paper. And to do that we first need the following lemmas.

Lemma 2.1. Let $Q^*[f, g]$ and $Q[f, g]$ denote differential polynomials in f and g with arbitrary meromorphic coefficients $q_1^*, q_2^*, \dots, q_n^*$ and q_1, q_2, \dots, q_k respectively. Suppose that

$$P[f, g] = f^l g^m + a_{l-1} f^{l-1} g^m + b_{m-1} f^l g^{m-1} + a_{l-2} f^{l-2} g^m + b_{m-2} f^l g^{m-2} + \dots + a_0 g^m + b_0 f^l. \quad (2.1)$$

If

$$P[f, g]Q^*[f, g] = Q[f, g], \quad (2.2)$$

where $\bar{d}_f(Q) \leq l$, $\bar{d}_g(Q) \leq m$ (i.e., $\bar{d}(Q) \leq l + m$) and on the circle $|z| = r$, $|f(re^{i\theta})| = |g(re^{i\theta})|$, then

$$m(r, Q^*[f, g]) \leq \sum_{i=1}^n m(r, q_i^*) + \sum_{i=1}^k m(r, q_i) + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g).$$

Proof : We first prove the case when

$$a_{l-1} = a_{l-2} = \dots = a_0 = 0 = b_{m-1} = b_{m-2} = \dots = b_0.$$

In this case we can rewrite (2.2) as

$$f^l g^m Q^*[f, g] = Q[f, g]. \quad (2.3)$$

Now we suppose

$$Q[f, g] = \sum_{i=1}^k q_i M_i[f, g] \quad (2.4)$$

where

$$M_i[f, g] = f^{\alpha_{i0}} (f^{(1)})^{\alpha_{i1}} \dots (f^{(j)})^{\alpha_{ij}} g^{\beta_{i0}} (g^{(1)})^{\beta_{i1}} \dots (g^{(h)})^{\beta_{ih}} \quad \text{for } i = 1, 2, \dots, k;$$

with $\alpha_{i0}, \alpha_{i1}, \dots, \alpha_{ij}, \beta_{i0}, \beta_{i1}, \dots, \beta_{ih}, j, h$ being non-negative integers, and

$$Q^*[f, g] = \sum_{i=1}^n q_i^* M_i^*[f, g] \quad (2.5)$$

where

$$M_i^*[f, g] = f^{s_{i0}} (f^{(1)})^{s_{i1}} \dots (f^{(u)})^{s_{iu}} g^{t_{i0}} (g^{(1)})^{t_{i1}} \dots (g^{(v)})^{t_{iv}} \quad \text{for } i = 1, 2, \dots, n$$

with $s_{i0}, s_{i1}, \dots, s_{iu}, t_{i0}, t_{i1}, \dots, t_{iv}, u, v$ being non-negative integers.

First we consider $|f(re^{i\theta})| = |g(re^{i\theta})| > 1$.

Then from (2.3) and (2.4) we have

$$\begin{aligned}
 |Q^*[f, g]| &= |f^{-l}g^{-m}Q[f, g]| \\
 &= |f^{-l}g^{-m} \sum_{i=1}^k q_i M_i[f, g]| \\
 &= \left| \sum_{i=1}^k q_i \left(\frac{f^{(1)}}{f}\right)^{\alpha_{i1}} \dots \left(\frac{f^{(j)}}{f}\right)^{\alpha_{ij}} \left(\frac{g^{(1)}}{g}\right)^{\beta_{i1}} \dots \left(\frac{g^{(h)}}{g}\right)^{\beta_{ih}} f^{(\sum_{w=0}^j \alpha_{iw}-l)} g^{(\sum_{w=0}^h \beta_{iw}-m)} \right| \\
 &\leq \sum_{i=1}^k |q_i| \left| \frac{f^{(1)}}{f} \right|^{\alpha_{i1}} \dots \left| \frac{f^{(j)}}{f} \right|^{\alpha_{ij}} \left| \frac{g^{(1)}}{g} \right|^{\beta_{i1}} \dots \left| \frac{g^{(h)}}{g} \right|^{\beta_{ih}} |f|^{(\sum_{w=0}^j \alpha_{iw}-l)} |g|^{(\sum_{w=0}^h \beta_{iw}-m)} \\
 &\leq \sum_{i=1}^k |q_i| \left| \frac{f^{(1)}}{f} \right|^{\alpha_{i1}} \dots \left| \frac{f^{(j)}}{f} \right|^{\alpha_{ij}} \left| \frac{g^{(1)}}{g} \right|^{\beta_{i1}} \dots \left| \frac{g^{(h)}}{g} \right|^{\beta_{ih}} |f|^{(\bar{d}_f(Q)-l)} |g|^{(\bar{d}_g(Q)-m)} \\
 &\leq \sum_{i=1}^k |q_i| \left| \frac{f^{(1)}}{f} \right|^{\alpha_{i1}} \dots \left| \frac{f^{(j)}}{f} \right|^{\alpha_{ij}} \left| \frac{g^{(1)}}{g} \right|^{\beta_{i1}} \dots \left| \frac{g^{(h)}}{g} \right|^{\beta_{ih}}, \\
 &\quad \text{since } |f(re^{i\theta})| = |g(re^{i\theta})| > 1 \text{ and } \bar{d}_f(Q) - l \leq 0, \quad \bar{d}_g(Q) - m \leq 0.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \frac{1}{2\pi} \int_{|f|=|g|>1} \log^+ |Q^*[f, g]| d\theta &\leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left(\sum_{i=1}^k |q_i| \left| \frac{f^{(1)}}{f} \right|^{\alpha_{i1}} \dots \left| \frac{f^{(j)}}{f} \right|^{\alpha_{ij}} \left| \frac{g^{(1)}}{g} \right|^{\beta_{i1}} \dots \left| \frac{g^{(h)}}{g} \right|^{\beta_{ih}} \right) d\theta \\
 &\leq \sum_{i=1}^k \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left(|q_i| \left| \frac{f^{(1)}}{f} \right|^{\alpha_{i1}} \dots \left| \frac{f^{(j)}}{f} \right|^{\alpha_{ij}} \left| \frac{g^{(1)}}{g} \right|^{\beta_{i1}} \dots \left| \frac{g^{(h)}}{g} \right|^{\beta_{ih}} \right) d\theta + O(1) \\
 &= \sum_{i=1}^k \frac{1}{2\pi} \int_0^{2\pi} \log^+ |q_i| d\theta + \sum_{i=1}^k \left(\frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{f^{(1)}}{f} \right|^{\alpha_{i1}} d\theta + \dots \right. \\
 &\quad \dots + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{f^{(j)}}{f} \right|^{\alpha_{ij}} d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{g^{(1)}}{g} \right|^{\beta_{i1}} d\theta + \dots \\
 &\quad \left. \dots + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{g^{(h)}}{g} \right|^{\beta_{ih}} d\theta \right) + O(1) \\
 &= \sum_{i=1}^k m(r, q_i) + \sum_{i=1}^k \left[\alpha_{i1} m\left(r, \frac{f^{(1)}}{f}\right) + \dots + \alpha_{ij} m\left(r, \frac{f^{(j)}}{f}\right) \right. \\
 &\quad \left. + \beta_{i1} m\left(r, \frac{g^{(1)}}{g}\right) + \dots + \beta_{ih} m\left(r, \frac{g^{(h)}}{g}\right) \right] + O(1) \\
 &= \sum_{i=1}^k m(r, q_i) + \sum_{i=1}^k \left[\alpha_{i1} S(r, f) + \dots + \alpha_{ij} S(r, f) \right. \\
 &\quad \left. + \beta_{i1} S(r, g) + \dots + \beta_{ih} S(r, g) \right] + O(1), \\
 &\quad \text{using Milloux's Theorem} \\
 &= \sum_{i=1}^k m(r, q_i) + S(r, f) + S(r, g). \tag{2.6}
 \end{aligned}$$

Next we consider $|f(re^{i\theta})| = |g(re^{i\theta})| \leq 1$. From (2.5) we have

$$\begin{aligned} |Q^*[f, g]| &= \left| \sum_{i=1}^n q_i^* M_i^*[f, g] \right| \\ &= \left| \sum_{i=1}^n q_i^* f^{s_{i0}} (f^{(1)})^{s_{i1}} \dots (f^{(u)})^{s_{iu}} g^{t_{i0}} (g^{(1)})^{t_{i1}} \dots (g^{(v)})^{t_{iv}} \right| \\ &\leq \sum_{i=1}^n |q_i^*| \left| \frac{f^{(1)}}{f} \right|^{s_{i1}} \dots \left| \frac{f^{(u)}}{f} \right|^{s_{iu}} \left| \frac{g^{(1)}}{g} \right|^{t_{i1}} \dots \left| \frac{g^{(v)}}{g} \right|^{t_{iv}}. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{2\pi} \int_{|f|=|g|\leq 1} \log^+ |Q^*[f, g]| d\theta &\leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left(\sum_{i=1}^n |q_i^*| \left| \frac{f^{(1)}}{f} \right|^{s_{i1}} \dots \left| \frac{f^{(u)}}{f} \right|^{s_{iu}} \left| \frac{g^{(1)}}{g} \right|^{t_{i1}} \dots \left| \frac{g^{(v)}}{g} \right|^{t_{iv}} \right) d\theta \\ &\leq \sum_{i=1}^n \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left(|q_i^*| \left| \frac{f^{(1)}}{f} \right|^{s_{i1}} \dots \left| \frac{f^{(u)}}{f} \right|^{s_{iu}} \left| \frac{g^{(1)}}{g} \right|^{t_{i1}} \dots \left| \frac{g^{(v)}}{g} \right|^{t_{iv}} \right) d\theta + O(1) \\ &= \sum_{i=1}^n \frac{1}{2\pi} \int_0^{2\pi} \log^+ |q_i^*| d\theta + \sum_{i=1}^n \left(\frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{f^{(1)}}{f} \right|^{s_{i1}} d\theta + \dots \right. \\ &\quad \dots + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{f^{(u)}}{f} \right|^{s_{iu}} d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{g^{(1)}}{g} \right|^{t_{i1}} d\theta + \dots \\ &\quad \left. \dots + \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{g^{(v)}}{g} \right|^{t_{iv}} d\theta \right) + O(1) \\ &= \sum_{i=1}^n m(r, q_i^*) + \sum_{i=1}^n (s_{i1} m(r, \frac{f^{(1)}}{f}) + \dots + s_{iu} m(r, \frac{f^{(u)}}{f}) \\ &\quad + t_{i1} m(r, \frac{g^{(1)}}{g}) + \dots + t_{iv} m(r, \frac{g^{(v)}}{g})) + O(1) \\ &= \sum_{i=1}^n m(r, q_i^*) + \sum_{i=1}^n (s_{i1} S(r, f) + \dots + s_{iu} S(r, f) \\ &\quad + t_{i1} S(r, g) + \dots + t_{iv} S(r, g)) + O(1), \\ &\hspace{15em} \text{using Milloux's Theorem} \\ &= \sum_{i=1}^n m(r, q_i^*) + S(r, f) + S(r, g). \tag{2.7} \end{aligned}$$

Adding (2.6) and (2.7) we get

$$\frac{1}{2\pi} \int_{|f|=|g|>1} \log^+ |Q^*[f, g]| d\theta + \frac{1}{2\pi} \int_{|f|=|g|\leq 1} \log^+ |Q^*[f, g]| d\theta \leq \sum_{i=1}^n m(r, q_i^*) + \sum_{i=1}^k m(r, q_i) + S(r, f) + S(r, g)$$

i.e.,

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ |Q^*[f, g]| d\theta \leq \sum_{i=1}^n m(r, q_i^*) + \sum_{i=1}^k m(r, q_i) + S(r, f) + S(r, g)$$

i.e.,

$$m(r, Q^*[f, g]) \leq \sum_{i=1}^n m(r, q_i^*) + \sum_{i=1}^k m(r, q_i) + S(r, f) + S(r, g).$$

Now we consider the general case when $a_{l-1}, a_{l-2}, \dots, a_0; b_{m-1}, b_{m-2}, \dots, b_0$ are any arbitrary meromorphic functions with smaller growth than f and g respectively. Then from (2.2) we can write

$$f^l g^m [(f^l g^m + a_{l-1} f^{l-1} g^m + b_{m-1} f^l g^{m-1} + \dots + a_0 g^m + b_0 f^l) Q^*[f, g]] = f^l g^m Q[f, g]$$

or, $f^l g^m R^*[f, g] = R[f, g]$, say where $R^*[f, g]$ and $R[f, g]$ are differential polynomials in f and g with meromorphic coefficients $q_1^*(1 + a_{l-1} + b_{m-1} + \dots + a_0 + b_0)$, $q_2^*(1 + a_{l-1} + b_{m-1} + \dots + a_0 + b_0)$, \dots , $q_n^*(1 + a_{l-1} + b_{m-1} + \dots + a_0 + b_0)$ and q_1, q_2, \dots, q_k respectively.

Hence by first case

$$\begin{aligned} m(r, R^*[f, g]) &\leq \sum_{i=1}^n m(r, q_i^*(1 + a_{l-1} + b_{m-1} + \dots + a_0 + b_0)) + \sum_{i=1}^k m(r, q_i) + S(r, f) + S(r, g) \\ &\leq \sum_{i=1}^n m(r, q_i^*) + \sum_{i=1}^k m(r, q_i) + S(r, f) + S(r, g). \end{aligned}$$

So,

$$\begin{aligned} m(r, Q^*[f, g]) &\leq m(r, R^*[f, g]) + m(r, \frac{1}{P[f, g]}) \\ &\leq \sum_{i=1}^n m(r, q_i^*) + \sum_{i=1}^k m(r, q_i) + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g). \end{aligned}$$

Hence the proof is complete.

Lemma 2.2 Let $P[f, g] = \sum_{j=1}^n a_j(z) M_j[f, g]$ be a differential polynomial in f and g of order and lower degree ν and $\underline{d}(P)$, respectively and let $P[f, g]$ satisfies the condition (1.5). If z_0 is a zero of f and g with multiplicities $\mu_1(> \nu)$ and $\mu_2(> \nu)$ respectively and z_0 is not a pole of any of the coefficients $a_j(j = 1, 2, \dots, n)$, then z_0 is a zero of $P[f, g]$ with multiplicity at least $(\mu - \nu)\underline{d}(P)$ where $\mu = \min\{\mu_1, \mu_2\}$.

Proof : Since z_0 is a zero of f and g with multiplicities $\mu_1(> \nu)$ and $\mu_2(> \nu)$ respectively, from Definition 1.2 we can clearly say that z_0 will be a zero of

$M_j[f, g]$ with multiplicity

$$\begin{aligned}
 &= [\mu_1 l_{0j} + (\mu_1 - 1)l_{1j} + \dots + (\mu_1 - k)l_{kj}] + [\mu_2 m_{0j} + (\mu_2 - 1)m_{1j} + \dots + (\mu_2 - h)m_{hj}] \\
 &= \mu_1 \sum_{i=0}^k l_{ij} - \sum_{i=1}^k i l_{ij} + \mu_2 \sum_{i=0}^h m_{ij} - \sum_{i=1}^h i m_{ij} \\
 &= (\mu_1 + 1) \sum_{i=0}^k l_{ij} - \sum_{i=0}^k (i + 1) l_{ij} + (\mu_2 + 1) \sum_{i=0}^h m_{ij} - \sum_{i=0}^h (i + 1) m_{ij} \\
 &= (\mu_1 + 1) \bar{d}_f(M_j) + (\mu_2 + 1) \bar{d}_g(M_j) - \Gamma_{M_j} \\
 &= \mu_1 \bar{d}_f(M_j) + \mu_2 \bar{d}_g(M_j) + \bar{d}(M_j) - \Gamma_{M_j} \\
 &= (\mu_1 - \nu) \bar{d}_f(M_j) + (\mu_2 - \nu) \bar{d}_g(M_j) + (\nu + 1) \bar{d}(M_j) - \Gamma_{M_j} \\
 &\geq (\mu_1 - \nu) \bar{d}_f(M_j) + (\mu_2 - \nu) \bar{d}_g(M_j) \\
 &\geq (\mu_1 - \nu) \underline{d}_f(P) + (\mu_2 - \nu) \underline{d}_g(P) \\
 &\geq (\mu - \nu) [\underline{d}_f(P) + \underline{d}_g(P)] \\
 &= (\mu - \nu) \underline{d}(P).
 \end{aligned}$$

Since z_0 is not a pole of any of the coefficients $a_j (j = 1, 2, \dots, n)$, therefore z_0 is a zero of $P[f, g]$ with multiplicity at least $(\mu - \nu) \underline{d}(P)$ where $\mu = \min\{\mu_1, \mu_2\}$.

Lemma 2.3. For a differential polynomial $P[f, g] = \sum_{j=1}^n a_j(z) M_j[f, g]$ the following inequality holds:

$$N(r, P[f, g]) \leq n[\bar{d}(P)\{N(r, f) + N(r, g)\} + \{\Gamma_P - \underline{d}(P)\}\{\bar{N}(r, f) + \bar{N}(r, g)\}] + S(r, f) + S(r, g).$$

Proof : Since $N(r, f^{(k)}) \leq N(r, f) + k\bar{N}(r, f)$, we have

$$\begin{aligned}
 N(r, P[f, g]) &= N(r, \sum_{j=1}^n a_j(z) M_j[f, g]) \\
 &\leq \sum_{j=1}^n N(r, a_j(z) M_j[f, g]) \\
 &\leq \sum_{j=1}^n [N(r, a_j(z)) + N(r, M_j[f, g])] \\
 &= \sum_{j=1}^n N(r, M_j[f, g]) + S(r, f) + S(r, g) \\
 &= \sum_{j=1}^n N(r, [f]^{l_{0j}} [f^{(1)}]^{l_{1j}} \dots [f^{(k)}]^{l_{kj}} [g]^{m_{0j}} [g^{(1)}]^{m_{1j}} \dots [g^{(h)}]^{m_{hj}}) + S(r, f) + S(r, g) \\
 &\leq \sum_{j=1}^n [N(r, [f]^{l_{0j}}) + N(r, [f^{(1)}]^{l_{1j}}) + \dots + N(r, [f^{(k)}]^{l_{kj}}) + N(r, [g]^{m_{0j}}) \\
 &\quad + N(r, [g^{(1)}]^{m_{1j}}) + \dots + N(r, [g^{(h)}]^{m_{hj}})] + S(r, f) + S(r, g) \\
 &\leq \sum_{j=1}^n [l_{0j} N(r, f) + l_{1j} N(r, f^{(1)}) + \dots + l_{kj} N(r, f^{(k)}) + m_{0j} N(r, g) \\
 &\quad + m_{1j} N(r, g^{(1)}) + \dots + m_{hj} N(r, g^{(h)})] + S(r, f) + S(r, g)
 \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=1}^n [l_{0j}N(r, f) + l_{1j}\{N(r, f) + \bar{N}(r, f)\} + \dots + l_{kj}\{N(r, f) + k\bar{N}(r, f)\} + m_{0j}N(r, g) \\
&\quad + m_{1j}\{N(r, g) + \bar{N}(r, g)\} + \dots + m_{hj}\{N(r, g) + h\bar{N}(r, g)\}] + S(r, f) + S(r, g) \\
&\leq \sum_{j=1}^n [(l_{0j} + l_{1j} + \dots + l_{kj})N(r, f) + (l_{1j} + 2l_{2j} + \dots + kl_{kj})\bar{N}(r, f) + (m_{0j} + m_{1j} \\
&\quad + \dots + m_{hj})N(r, g) + (m_{1j} + 2m_{2j} + \dots + hm_{hj})\bar{N}(r, g)] + S(r, f) + S(r, g) \\
&= \sum_{j=1}^n [\bar{d}_f(M_j)N(r, f) + \{(l_{0j} + 2l_{1j} + \dots + (k+1)l_{kj}) - (l_{0j} + l_{1j} + \dots + l_{kj})\}\bar{N}(r, f) \\
&\quad + \bar{d}_g(M_j)N(r, g) + \{(m_{0j} + 2m_{1j} + \dots + (h+1)m_{hj}) - (m_{0j} + m_{1j} \\
&\quad + \dots + m_{hj})\}\bar{N}(r, g)] + S(r, f) + S(r, g) \\
&\leq \sum_{j=1}^n [\bar{d}_f(M_j)N(r, f) + \bar{d}_g(M_j)N(r, g) + \{(l_{0j} + 2l_{1j} + \dots + (k+1)l_{kj} \\
&\quad + m_{0j} + 2m_{1j} + \dots + (h+1)m_{hj}) - (l_{0j} + l_{1j} + \dots + l_{kj} + m_{0j} + m_{1j} \\
&\quad + \dots + m_{hj})\}\{\bar{N}(r, f) + \bar{N}(r, g)\}] + S(r, f) + S(r, g) \\
&\leq \sum_{j=1}^n [\bar{d}(M_j)\{N(r, f) + N(r, g)\} + \{\Gamma_{M_j} - \bar{d}(M_j)\}\{\bar{N}(r, f) + \bar{N}(r, g)\}] + S(r, f) + S(r, g) \\
&\leq n[\bar{d}(P)\{N(r, f) + N(r, g)\} + \{\Gamma_P - \underline{d}(P)\}\{\bar{N}(r, f) + \bar{N}(r, g)\}] + S(r, f) + S(r, g).
\end{aligned}$$

Lemma 2.4. Let $P[f, g]$ be given by (2.1) where $a_{l-1}, a_{l-2}, \dots, a_0$ are small functions of f and $b_{m-1}, b_{m-2}, \dots, b_0$ are small functions of g . Then

$$m(r, P[f, g]) \leq 2l \times m(r, f) + 2m \times m(r, g) + S(r, f) + S(r, g).$$

Proof : We have

$$\begin{aligned}
m(r, P[f, g]) &= m(r, f^l g^m + a_{l-1} f^{l-1} g^m + b_{m-1} f^l g^{m-1} + a_{l-2} f^{l-2} g^m + b_{m-2} f^l g^{m-2} + \dots + a_0 g^m + b_0 f^l) \\
&\leq m(r, f^l g^m + a_{l-1} f^{l-1} g^m + a_{l-2} f^{l-2} g^m + \dots + a_0 g^m) \\
&\quad + m(r, b_{m-1} f^l g^{m-1} + b_{m-2} f^l g^{m-2} + \dots + b_0 f^l) + O(1) \\
&= m(r, g^m \{f^l + a_{l-1} f^{l-1} + a_{l-2} f^{l-2} + \dots + a_0\}) \\
&\quad + m(r, f^l \{b_{m-1} g^{m-1} + b_{m-2} g^{m-2} + \dots + b_0\}) + O(1) \\
&\leq m(r, g^m) + m(r, f^l + a_{l-1} f^{l-1} + a_{l-2} f^{l-2} + \dots + a_0) \\
&\quad + m(r, f^l) + m(r, b_{m-1} g^{m-1} + b_{m-2} g^{m-2} + \dots + b_0) + O(1) \\
&\leq m \times m(r, g) + m(r, f^l + a_{l-1} f^{l-1} + a_{l-2} f^{l-2} + \dots + a_1 f) + m(r, a_0) \\
&\quad + l \times m(r, f) + m(r, b_{m-1} g^{m-1} + b_{m-2} g^{m-2} + \dots + b_1 g) + m(r, b_0) + O(1) \\
&\leq m \times m(r, g) + m(r, f \{f^{l-1} + a_{l-1} f^{l-2} + \dots + a_2 f + a_1\}) + l \times m(r, f) \\
&\quad + m(r, g \{b_{m-1} g^{m-2} + b_{m-2} g^{m-3} + \dots + b_2 g + b_1\}) + S(r, f) + S(r, g) \\
&\leq (l+1) \times m(r, f) + (m+1) \times m(r, g) + m(r, f^{l-1} + a_{l-1} f^{l-2} + \dots \\
&\quad + a_2 f + a_1) + m(r, b_{m-1} g^{m-2} + b_{m-2} g^{m-3} + \dots + b_2 g + b_1) + S(r, f) + S(r, g)
\end{aligned}$$

$$\begin{aligned} &\leq (l + 1) \times m(r, f) + (m + 1) \times m(r, g) + m(r, f\{f^{l-2} + a_{l-1}f^{l-3} + \dots \\ &\quad + a_3f + a_2\}) + m(r, g\{b_{m-1}g^{m-3} + b_{m-2}g^{m-4} + \dots + b_3g + b_2\}) + S(r, f) + S(r, g) \\ &\leq (l + 2) \times m(r, f) + (m + 2) \times m(r, g) + m(r, f^{l-2} + a_{l-1}f^{l-3} + \dots \\ &\quad + a_3f + a_2) + m(r, b_{m-1}g^{m-3} + b_{m-2}g^{m-4} + \dots + b_3g + b_2) + S(r, f) + S(r, g). \end{aligned}$$

Proceeding similarly, we ultimately get

$$\begin{aligned} m(r, P[f, g]) &\leq 2l \times m(r, f) + (2m - 1) \times m(r, g) + S(r, f) + S(r, g) \\ &\leq 2l \times m(r, f) + 2m \times m(r, g) + S(r, f) + S(r, g). \end{aligned}$$

Lemma 2.5. If $Q[f, g]$ is a differential polynomial in f and g with arbitrary meromorphic coefficients q_i , $1 \leq i \leq n$, then

$$m(r, Q[f, g]) \leq n\bar{d}(Q)[m(r, f) + m(r, g)] + \sum_{i=1}^n m(r, q_i) + S(r, f) + S(r, g).$$

Proof : Let

$$Q[f, g] = \sum_{j=1}^n q_j M_j[f, g] \tag{2.8}$$

where each

$$M_j[f, g] = [f(z)]^{l_{0j}} [f^{(1)}(z)]^{l_{1j}} \dots [f^{(k)}(z)]^{l_{kj}} [g(z)]^{m_{0j}} [g^{(1)}(z)]^{m_{1j}} \dots [g^{(h)}(z)]^{m_{hj}}$$

is a monomial in f and g of degree

$$d(M_j) = \sum_{i=0}^k l_{ij} + \sum_{i=0}^h m_{ij}$$

Here the degree of the differential polynomial $Q[f, g]$ is given by

$$\bar{d}(Q) = \max\{d(M_j) : 1 \leq j \leq n\}.$$

From (2.8) and using Milloux's Theorem we can have

$$\begin{aligned} m(r, Q[f, g]) &= m(r, \sum_{j=1}^n q_j M_j[f, g]) \\ &\leq \sum_{j=1}^n m(r, q_j M_j[f, g]) + O(1) \\ &\leq \sum_{j=1}^n m(r, q_j) + \sum_{j=1}^n m(r, M_j[f, g]) + O(1) \\ &= \sum_{i=j}^n m(r, q_j) + \sum_{j=1}^n m(r, [f]^{l_{0j}} [f^{(1)}]^{l_{1j}} \dots [f^{(k)}]^{l_{kj}} [g]^{m_{0j}} [g^{(1)}]^{m_{1j}} \dots [g^{(h)}]^{m_{hj}}) + O(1) \\ &\leq \sum_{i=j}^n m(r, q_j) + \sum_{j=1}^n [m(r, [f]^{\sum_{i=0}^k l_{ij}}) + m(r, [\frac{f^{(1)}}{f}]^{l_{1j}}) + \dots + m(r, [\frac{f^{(k)}}{f}]^{l_{kj}}) \\ &\quad + m(r, [g]^{\sum_{i=0}^h m_{ij}}) + m(r, [\frac{g^{(1)}}{g}]^{m_{1j}}) + \dots + m(r, [\frac{g^{(h)}}{g}]^{m_{hj}})] + O(1) \end{aligned}$$

$$\begin{aligned}
 &\leq \sum_{i=j}^n m(r, q_j) + \sum_{j=1}^n [\sum_{i=0}^k l_{ij} \times m(r, f) + l_{1j}m(r, \frac{f^{(1)}}{f}) + \dots + l_{kj}m(r, \frac{f^{(k)}}{f}) \\
 &\quad + \sum_{i=0}^h m_{ij} \times m(r, g) + m_{1j}m(r, \frac{g^{(1)}}{g}) + \dots + m_{hj}m(r, \frac{g^{(h)}}{g})] + O(1) \\
 &\leq \sum_{i=j}^n m(r, q_j) + \sum_{j=1}^n [\sum_{i=0}^k l_{ij} \times m(r, f) + \sum_{i=0}^h m_{ij} \times m(r, g)] + S(r, f) + S(r, g) \\
 &\leq \sum_{i=j}^n m(r, q_j) + \sum_{j=1}^n [\{\sum_{i=0}^k l_{ij} + \sum_{i=0}^h m_{ij}\} \times \{m(r, f) + m(r, g)\}] + S(r, f) + S(r, g) \\
 &= \sum_{i=j}^n m(r, q_j) + \sum_{j=1}^n [d(M_j) \times \{m(r, f) + m(r, g)\}] + S(r, f) + S(r, g) \\
 &\leq \sum_{i=j}^n m(r, q_j) + n\bar{d}(Q) \times \{m(r, f) + m(r, g)\} + S(r, f) + S(r, g).
 \end{aligned}$$

Hence the proof is complete.

Lemma 2.6. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with $N(r, f) = S(r, f)$ and $N(r, g) = S(r, g)$. Also let $P[f, g]$ be given by (2.1) where each a_0, a_1, \dots, a_{l-1} being small functions of f and b_0, b_1, \dots, b_{m-1} being small functions of g . Now if $|f(re^{i\theta})| = |g(re^{i\theta})| \geq 1$ on the circle $|z| = r$, then

$$\begin{aligned}
 \frac{1}{2}[lT(r, f) + mT(r, g)] &\leq T(r, P[f, g]) + S(r, f) + S(r, g) \\
 &\leq 2[lT(r, f) + mT(r, g)]. \tag{2.9}
 \end{aligned}$$

Proof : From (2.1) we have

$$P[f, g] = f^l g^m \left\{ 1 + \frac{a_{l-1}}{f} + \frac{b_{m-1}}{g} + \frac{a_{l-2}}{f^2} + \frac{b_{m-2}}{g^2} + \dots + \frac{a_0}{f^l} + \frac{b_0}{g^m} \right\}. \tag{2.10}$$

Now on the circle $|z| = r$, let

$$A(re^{i\theta}) = \max\{|a_{l-1}(re^{i\theta})|, |a_{l-2}(re^{i\theta})|^{\frac{1}{2}}, \dots, |a_0(re^{i\theta})|^{\frac{1}{l}}\}$$

and

$$B(re^{i\theta}) = \max\{|b_{m-1}(re^{i\theta})|, |b_{m-2}(re^{i\theta})|^{\frac{1}{2}}, \dots, |b_0(re^{i\theta})|^{\frac{1}{m}}\}.$$

Again if we suppose

$$E_1 = \{\theta \in [0, 2\pi] : |f(re^{i\theta})| > 4A(re^{i\theta})\}$$

and

$$E_2 = \{\theta \in [0, 2\pi] : |g(re^{i\theta})| > 4B(re^{i\theta})\},$$

then on $E_1 \cap E_2$, we have from (2.10)

$$\begin{aligned}
 |P[f, g]| &= |f|^l |g|^m \left| 1 + \frac{a_{l-1}}{f} + \frac{b_{m-1}}{g} + \frac{a_{l-2}}{f^2} + \frac{b_{m-2}}{g^2} + \dots + \frac{a_0}{f^l} + \frac{b_0}{g^m} \right| \\
 &\geq |f|^l |g|^m \left[1 - \left| \frac{a_{l-1}}{f} \right| - \left| \frac{a_{l-2}}{f^2} \right| - \dots - \left| \frac{a_0}{f^l} \right| - \left| \frac{b_{m-1}}{g} \right| - \left| \frac{b_{m-2}}{g^2} \right| - \dots - \left| \frac{b_0}{g^m} \right| \right] \\
 &\geq |f|^l |g|^m \left[1 - \left| \frac{A}{f} \right| - \left| \frac{A}{f} \right|^2 - \dots - \left| \frac{A}{f} \right|^l - \left| \frac{B}{g} \right| - \left| \frac{B}{g} \right|^2 - \dots - \left| \frac{B}{g} \right|^m \right] \\
 &\geq |f|^l |g|^m \left[1 - \frac{1}{4} - \left(\frac{1}{4} \right)^2 - \dots - \left(\frac{1}{4} \right)^l - \frac{1}{4} - \left(\frac{1}{4} \right)^2 - \dots - \left(\frac{1}{4} \right)^m \right], \\
 &= |f|^l |g|^m \left[1 - \left\{ \frac{1}{4} + \left(\frac{1}{4} \right)^2 + \dots + \left(\frac{1}{4} \right)^l \right\} - \left\{ \frac{1}{4} + \left(\frac{1}{4} \right)^2 + \dots + \left(\frac{1}{4} \right)^m \right\} \right] \\
 &\geq |f|^l |g|^m \left[1 - \left\{ \frac{\frac{1}{4}}{1 - \frac{1}{4}} \right\} - \left\{ \frac{\frac{1}{4}}{1 - \frac{1}{4}} \right\} \right] \\
 &= |f|^l |g|^m \left[1 - \frac{1}{3} - \frac{1}{3} \right] \\
 &= \frac{1}{3} |f|^l |g|^m.
 \end{aligned}$$

Hence on $E_1 \cap E_2$ we have

$$\begin{aligned}
 3|P[f, g]| &\geq |f|^l |g|^m \\
 \text{i.e., } \log^+ 3|P[f, g]| &\geq \log^+ |f|^l |g|^m.
 \end{aligned}$$

So,

$$l \log^+ |f| \leq \log 3 + \log^+ |P[f, g]| \quad (2.11)$$

and

$$m \log^+ |g| \leq \log 3 + \log^+ |P[f, g]|. \quad (2.12)$$

Therefore using (2.11) and by our hypothesis that on the circle $|z| = r$, $|f(re^{i\theta})| = |g(re^{i\theta})| \geq 1$, on E_1^c , $|f(re^{i\theta})| \leq 4A(re^{i\theta})$ and on $E_1 \cap E_2^c$, $|g(re^{i\theta})| \leq 4B(re^{i\theta})$, we have

$$\begin{aligned}
 l \times m(r, f) &= l \times \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta \\
 &= \frac{1}{2\pi} \int_{E_1 \cap E_2} l \log^+ |f(re^{i\theta})| d\theta + \frac{l}{2\pi} \int_{E_1 \cap E_2^c} \log^+ |f(re^{i\theta})| d\theta + \frac{l}{2\pi} \int_{E_1^c} \log^+ |f(re^{i\theta})| d\theta \\
 &\leq \frac{1}{2\pi} \int_{E_1 \cap E_2} (\log 3 + \log^+ |P[f, g]|) d\theta + \frac{l}{2\pi} \int_{E_1 \cap E_2^c} \log^+ |g(re^{i\theta})| d\theta + \frac{l}{2\pi} \int_{E_1^c} \log^+ 4A d\theta \\
 &\leq \frac{1}{2\pi} \int_{E_1 \cap E_2} \log 3 d\theta + \frac{1}{2\pi} \int_{E_1 \cap E_2} \log^+ |P[f, g]| d\theta + \frac{l}{2\pi} \int_{E_1 \cap E_2^c} \log^+ 4B d\theta + \frac{l}{2\pi} \int_{E_1^c} \log^+ 4A d\theta \\
 &\leq \frac{1}{2\pi} \int_0^{2\pi} \log 3 d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log^+ |P[f, g]| d\theta + \frac{l}{2\pi} \int_0^{2\pi} \log^+ 4B d\theta + \frac{l}{2\pi} \int_0^{2\pi} \log^+ 4A d\theta \\
 &= \log 3 + \frac{1}{2\pi} \int_0^{2\pi} \log^+ |P[f, g]| d\theta + \frac{l}{\pi} \int_0^{2\pi} \log^+ 4d\theta + \frac{l}{2\pi} \int_0^{2\pi} \log^+ B d\theta + \frac{l}{2\pi} \int_0^{2\pi} \log^+ A d\theta \\
 &= \log 3 + m(r, P[f, g]) + 2l \log 4 + l \times m(r, B) + l \times m(r, A) \\
 &= m(r, P[f, g]) + S(r, f) + S(r, g). \quad (2.13)
 \end{aligned}$$

Adding $l \times N(r, f)$ on both sides and recalling that $N(r, f) = S(r, f)$ we get

$$\begin{aligned} l \times T(r, f) &\leq m(r, P[f, g]) + S(r, f) + S(r, g) \\ &\leq T(r, P[f, g]) + S(r, f) + S(r, g). \end{aligned} \quad (2.14)$$

Similarly, using (2.12) and by our hypothesis that on the circle $|z| = r$, $|f(re^{i\theta})| = |g(re^{i\theta})| \geq 1$, on E_2^c , $|g(re^{i\theta})| \leq 4B(re^{i\theta})$ and on $E_1^c \cap E_2$, $|f(re^{i\theta})| \leq 4A(re^{i\theta})$, we have

$$\begin{aligned} m \times m(r, g) &= m \times \frac{1}{2\pi} \int_0^{2\pi} \log^+ |g(re^{i\theta})| d\theta \\ &= \frac{1}{2\pi} \int_{E_1 \cap E_2} m \log^+ |g(re^{i\theta})| d\theta + \frac{m}{2\pi} \int_{E_1^c \cap E_2} \log^+ |g(re^{i\theta})| d\theta + \frac{m}{2\pi} \int_{E_2^c} \log^+ |g(re^{i\theta})| d\theta \\ &\leq \frac{1}{2\pi} \int_{E_1 \cap E_2} (\log 3 + \log^+ |P[f, g]|) d\theta + \frac{m}{2\pi} \int_{E_1^c \cap E_2} \log^+ |f(re^{i\theta})| d\theta + \frac{m}{2\pi} \int_{E_2^c} \log^+ 4B d\theta \\ &\leq \frac{1}{2\pi} \int_{E_1 \cap E_2} \log 3 d\theta + \frac{1}{2\pi} \int_{E_1 \cap E_2} \log^+ |P[f, g]| d\theta + \frac{m}{2\pi} \int_{E_1^c \cap E_2} \log^+ 4A d\theta + \frac{m}{2\pi} \int_{E_2^c} \log^+ 4B d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \log 3 d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log^+ |P[f, g]| d\theta + \frac{m}{2\pi} \int_0^{2\pi} \log^+ 4A d\theta + \frac{m}{2\pi} \int_0^{2\pi} \log^+ 4B d\theta \\ &= \log 3 + \frac{1}{2\pi} \int_0^{2\pi} \log^+ |P[f, g]| d\theta + \frac{m}{\pi} \int_0^{2\pi} \log^+ 4d\theta + \frac{m}{2\pi} \int_0^{2\pi} \log^+ A d\theta + \frac{m}{2\pi} \int_0^{2\pi} \log^+ B d\theta \\ &= \log 3 + m(r, P[f, g]) + 2m \log 4 + m \times m(r, A) + m \times m(r, B) \\ &= m(r, P[f, g]) + S(r, f) + S(r, g). \end{aligned} \quad (2.15)$$

Adding $m \times N(r, g)$ on both sides and recalling that $N(r, g) = S(r, g)$ we get

$$\begin{aligned} m \times T(r, g) &\leq m(r, P[f, g]) + S(r, f) + S(r, g) \\ &\leq T(r, P[f, g]) + S(r, f) + S(r, g). \end{aligned} \quad (2.16)$$

Now adding (2.14) and (2.16), we get

$$\begin{aligned} l \times T(r, f) + m \times T(r, g) &\leq 2T(r, P[f, g]) + S(r, f) + S(r, g) \\ \text{i.e., } \frac{1}{2}[l \times T(r, f) + m \times T(r, g)] &\leq T(r, P[f, g]) + S(r, f) + S(r, g). \end{aligned} \quad (2.17)$$

Next from Lemma 2.4 we have

$$m(r, P[f, g]) \leq 2l \times m(r, f) + 2m \times m(r, g) + S(r, f) + S(r, g). \quad (2.18)$$

Finally using (2.18) and by our hypotheses that $N(r, f) = S(r, f)$ and $N(r, g) = S(r, g)$, we get

$$\begin{aligned} T(r, P[f, g]) &= m(r, P[f, g]) + N(r, P[f, g]) \\ &\leq 2l \times m(r, f) + 2m \times m(r, g) + S(r, f) + S(r, g) \\ &\leq 2[l \times T(r, f) + m \times T(r, g)] + S(r, f) + S(r, g). \end{aligned} \quad (2.19)$$

Combining (2.17) and (2.19) we obtain the desired result.

Lemma 2.7. Suppose that $M_j[f, g]$ is a monomial in f and g . If f and g has

a pole at $z = z_0$ of order p and q respectively, then z_0 is a pole of $M_j[f, g]$ of order atmost $(p + q - 2)d(M_j) + \Gamma_{M_j}$.

Proof : Let

$$M_j[f, g] = (f(z))^{l_{0j}}(f^{(1)}(z))^{l_{1j}} \dots (f^{(k)}(z))^{l_{kj}}(g(z))^{m_{0j}}(g^{(1)}(z))^{m_{1j}} \dots (g^{(h)}(z))^{m_{hj}}$$

be a monomial in f and g of degree

$$d(M_j) = \sum_{i=0}^k l_{ij} + \sum_{i=0}^h m_{ij}$$

and of weight

$$\Gamma_{M_j} = \sum_{i=0}^k (i + 1)l_{ij} + \sum_{i=0}^h (i + 1)m_{ij}.$$

where $l_{0j}, l_{1j}, \dots, l_{kj}, m_{0j}, m_{1j}, \dots, m_{hj}$ are non-negative integers.

Now since z_0 is a pole of f and g of order p and q respectively, z_0 will be a pole of $M_j[f, g]$ of order

$$\begin{aligned} &= \{pl_{0j} + (p + 1)l_{1j} + \dots + (p + k)l_{kj}\} + \{qm_{0j} + (q + 1)m_{1j} + \dots + (q + h)m_{hj}\} \\ &= p \sum_{i=0}^k l_{ij} + \sum_{i=1}^k il_{ij} + q \sum_{i=0}^h m_{ij} + \sum_{i=1}^h im_{ij} \\ &= (p - 1) \sum_{i=0}^k l_{ij} + \sum_{i=0}^k (i + 1)l_{ij} + (q - 1) \sum_{i=0}^h m_{ij} + \sum_{i=0}^h (i + 1)m_{ij} \\ &\leq (p - 1)d(M_j) + (q - 1)d(M_j) + \sum_{i=0}^k (i + 1)l_{ij} + \sum_{i=0}^h (i + 1)m_{ij} \\ &= (p + q - 2)d(M_j) + \Gamma_{M_j}. \end{aligned}$$

Hence the proof is complete.

Lemma 2.8. Suppose that $Q[f, g]$ is a differential polynomial in f and g . Let z_0 be a pole of f and g of order p and q respectively and not a zero or a pole of coefficients of $Q[f, g]$. Then z_0 is a pole of $Q[f, g]$ of order atmost $(p + q - 1)\gamma_Q + (\Gamma_Q - \gamma_Q)$.

Proof : Let

$$Q[f, g] = \sum_{j=1}^n a_j M_j[f, g]$$

where each

$$M_j[f, g] = (f(z))^{l_{0j}}(f^{(1)}(z))^{l_{1j}} \dots (f^{(k)}(z))^{l_{kj}}(g(z))^{m_{0j}}(g^{(1)}(z))^{m_{1j}} \dots (g^{(h)}(z))^{m_{hj}}$$

is a monomial in f and g of degree

$$d(M_j) = \sum_{i=0}^k l_{ij} + \sum_{i=0}^h m_{ij}$$

and of weight

$$\Gamma_{M_j} = \sum_{i=0}^k (i+1)l_{ij} + \sum_{i=0}^h (i+1)m_{ij}.$$

Here the degree and weight of the differential polynomial $Q[f, g]$ is given by

$$\bar{d}(Q) = \max\{d(M_j) : 1 \leq j \leq n\}$$

and

$$\Gamma_Q = \max\{\Gamma_{M_j} : 1 \leq j \leq n\}$$

respectively.

Now since z_0 is a pole of f and g of order p and q respectively and not a zero or a pole of the coefficients of $Q[f, g]$, by using the Lemma 2.7 we can say that z_0 is a pole of $Q[f, g]$ of order

$$\begin{aligned} &\leq \max_{1 \leq j \leq n} \{(p+q-2)d(M_j) + \Gamma_{M_j}\} \\ &\leq \max_{1 \leq j \leq n} \{(p+q-2)d(M_j)\} + \max_{1 \leq j \leq n} \{\Gamma_{M_j}\} \\ &= (p+q-2)\bar{d}(Q) + \Gamma_Q \\ &= (p+q-1)\bar{d}(Q) + [\Gamma_Q - \bar{d}(Q)]. \end{aligned}$$

Hence the proof is complete.

We now present the main result of the paper.

Theorem 2.1 Let f and g be transcendental meromorphic functions in the open complex plane and $Q_1[f, g](\neq 0)$, $Q_2[f, g](\neq 0)$ be two differential polynomials in f and g such that ν and $\underline{d}(Q_1)$ be the order and lower degree of $Q_1[f, g]$, respectively and $Q_2[f, g]$ consisting of n arbitrary meromorphic coefficients which are small functions of f and g . Also let $P[f, g]$ be given by (2.1) where each a_0, a_1, \dots, a_{l-1} are small functions of f and b_0, b_1, \dots, b_{m-1} are small functions of g . If

$$F = P[f, g]Q_1[f, g] + Q_2[f, g] \tag{2.20}$$

and on the circle $|z| = r$, $|f(re^{i\theta})| = |g(re^{i\theta})| \geq 1$, then

$$\begin{aligned} &[\frac{l}{2} - n\bar{d}(Q_2)]T(r, f) + [\frac{m}{2} - n\bar{d}(Q_2)]T(r, g) \\ &\leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{P[f, g]}) + [\Gamma_{Q_2} - 2\bar{d}(Q_2)][\bar{N}(r, f) + \bar{N}(r, g)] \\ &- d[\{N(r, \frac{1}{f}) - N_{\nu+1}(r, \frac{1}{f})\} + \{N(r, \frac{1}{g}) - N_{\nu+1}(r, \frac{1}{g})\}] + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g), \end{aligned}$$

where $d = \underline{d}(Q_1)$ if $l \geq 2n\bar{d}(Q_2)$ and $m \geq 2n\bar{d}(Q_2)$ whereas $d = 0$ if $l < 2n\bar{d}(Q_2)$ and $m < 2n\bar{d}(Q_2)$; $N_{\nu+1}(r, \frac{1}{f})$ denotes the counting function of poles of f where a pole of f with multiplicity μ is counted μ times if $\mu \leq \nu + 1$ and $(\nu + 1)$ times if $\mu > (\nu + 1)$.

Proof : Without loss of generality we may suppose $l > 2n\bar{d}(Q_2)$ and $m >$

$2n\bar{d}(Q_2)$.

From (2.20) we have

$$1 = \frac{1}{F}P[f, g]Q_1[f, g] + \frac{1}{F}Q_2[f, g]$$

i.e.,

$$F' = \frac{F'}{F}P[f, g]Q_1[f, g] + \frac{F'}{F}Q_2[f, g]. \quad (2.21)$$

Again differentiating (2.20) we have

$$F' = (P[f, g])'Q_1[f, g] + P[f, g](Q_1[f, g])' + (Q_2[f, g])'. \quad (2.22)$$

Now comparing (2.21) and (2.22) we have

$$\frac{F'}{F}P[f, g]Q_1[f, g] + \frac{F'}{F}Q_2[f, g] = (P[f, g])'Q_1[f, g] + P[f, g](Q_1[f, g])' + (Q_2[f, g])'$$

$$\text{i.e.,} \quad \frac{F'}{F}P[f, g]Q_1[f, g] - (P[f, g])'Q_1[f, g] - P[f, g](Q_1[f, g])' = (Q_2[f, g])' - \frac{F'}{F}Q_2[f, g]$$

$$\text{i.e.,} \quad P[f, g]\left\{\frac{F'}{F}Q_1[f, g] - \frac{(P[f, g])'}{P[f, g]}Q_1[f, g] - (Q_1[f, g])'\right\} = (Q_2[f, g])' - \frac{F'}{F}Q_2[f, g]$$

i.e.,

$$P[f, g]Q^*[f, g] = Q[f, g] \quad (2.23)$$

where

$$Q^*[f, g] = \frac{F'}{F}Q_1[f, g] - \frac{(P[f, g])'}{P[f, g]}Q_1[f, g] - (Q_1[f, g])' \quad (2.24)$$

and

$$Q[f, g] = (Q_2[f, g])' - \frac{F'}{F}Q_2[f, g]. \quad (2.25)$$

First we suppose that $Q^*[f, g] \neq 0$. Then from (2.23) we have $Q[f, g] \neq 0$.

Now a monomial in f and g is an expression of the form

$$M_j[f, g] = [f(z)]^{l_{0j}}[f^{(1)}(z)]^{l_{1j}} \dots [f^{(k)}(z)]^{l_{kj}} [g(z)]^{m_{0j}} [g^{(1)}(z)]^{m_{1j}} \dots [g^{(h)}(z)]^{m_{hj}}$$

with degree

$$d(M_j) = \sum_{i=0}^k l_{ij} + \sum_{i=0}^h m_{ij},$$

where $l_{0j}, l_{1j}, \dots, l_{kj}, m_{0j}, m_{1j}, \dots, m_{hj}$ are non-negative integers.

On differentiation, we get

$$\begin{aligned} (M_j[f, g])' &= l_{0j}(f)^{l_{0j}-1}(f^{(1)})^{l_{1j}+1} \dots (f^{(k)})^{l_{kj}}(g)^{m_{0j}}(g^{(1)})^{m_{1j}} \dots (g^{(h)})^{m_{hj}} \\ &\quad + l_{1j}(f)^{l_{0j}}(f^{(1)})^{l_{1j}-1} \dots (f^{(k)})^{l_{kj}}(g)^{m_{0j}}(g^{(1)})^{m_{1j}} \dots (g^{(h)})^{m_{hj}} + \dots \\ &\quad \dots + l_{kj}(f)^{l_{0j}}(f^{(1)})^{l_{1j}} \dots (f^{(k)})^{l_{kj}-1} f^{(k+1)}(g(z))^{m_{0j}}(g^{(1)})^{m_{1j}} \dots (g^{(h)})^{m_{hj}} \\ &\quad + m_{0j}(f)^{l_{0j}}(f^{(1)})^{l_{1j}} \dots (f^{(k)})^{l_{kj}}(g)^{m_{0j}-1}(g^{(1)})^{m_{1j}+1} \dots (g^{(h)})^{m_{hj}} \\ &\quad + m_{1j}(f)^{l_{0j}}(f^{(1)})^{l_{1j}} \dots (f^{(k)})^{l_{kj}}(g)^{m_{0j}}(g^{(1)})^{m_{1j}-1} \dots (g^{(h)})^{m_{hj}} + \dots \\ &\quad \dots + m_{hj}(f)^{l_{0j}}(f^{(1)})^{l_{1j}} \dots (f^{(k)})^{l_{kj}}(g)^{m_{0j}}(g^{(1)})^{m_{1j}} \dots (g^{(h)})^{m_{hj}-1} g^{(h+1)}. \end{aligned}$$

Hence

$$\begin{aligned}
 d(M'_j) &= \max\{[(l_{0j} - 1) + (l_{1j} + 1) + \dots + l_{kj} + m_{0j} + m_{1j} + \dots + m_{hj}], \\
 &\quad [l_{0j} + (l_{1j} - 1) + (l_{2j} + 1) + \dots + l_{kj} + m_{0j} + m_{1j} + \dots + m_{hj}], \dots \\
 &\quad \dots[l_{0j} + l_{1j} + \dots + (l_{kj} - 1) + 1 + m_{0j} + m_{1j} + \dots + m_{hj}], \\
 &\quad [l_{0j} + l_{1j} + \dots + l_{kj} + (m_{0j} - 1) + (m_{1j} + 1) + \dots + m_{hj}], \\
 &\quad [l_{0j} + l_{1j} + \dots + l_{kj} + m_{0j} + (m_{1j} - 1) + (m_{2j} + 1) \dots + m_{hj}], \dots \\
 &\quad \dots[l_{0j} + l_{1j} + \dots + l_{kj} + m_{0j} + m_{1j} + \dots + (m_{hj} - 1) + 1]\} \\
 &= \max\{d(M_j), d(M_j), \dots, d(M_j)\} \\
 &= d(M_j).
 \end{aligned}$$

Thus we can write $\bar{d}(Q_2) = \bar{d}(Q'_2)$. Hence from (2.25) we have $\bar{d}(Q) = \bar{d}(Q_2)$.

Now using Lemma 2.1 on (2.23) and assuming that the coefficients of $Q^*[f, g]$ and $Q[f, g]$ are small functions of f and g , we get

$$m(r, Q^*[f, g]) = m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g). \quad (2.26)$$

Again from (2.23) we have

$$\begin{aligned}
 m(r, P[f, g]) &= m(r, \frac{Q[f, g]}{Q^*[f, g]}) \\
 &\leq m(r, Q[f, g]) + m(r, \frac{1}{Q^*[f, g]}). \quad (2.27)
 \end{aligned}$$

Using (2.25), Lemma 2.5, Milloux's theorem and the fact that the coefficients of $Q_2[f, g]$ are small functions of f and g we have

$$\begin{aligned}
 m(r, Q[f, g]) &= m(r, (Q_2[f, g])' - \frac{F'}{F} Q_2[f, g]) \\
 &= m(r, Q_2[f, g]) \{ \frac{(Q_2[f, g])'}{Q_2[f, g]} - \frac{F'}{F} \} \\
 &\leq m(r, Q_2[f, g]) + m(r, \frac{(Q_2[f, g])'}{Q_2[f, g]}) + m(r, -\frac{F'}{F}) + O(1) \\
 &= m(r, Q_2[f, g]) + S(r, f) + S(r, g) \\
 &\leq n\bar{d}(Q_2)[m(r, f) + m(r, g)] + S(r, f) + S(r, g). \quad (2.28)
 \end{aligned}$$

Also, by the First Fundamental Theorem and by using (2.26) we have

$$\begin{aligned}
 T(r, \frac{1}{Q^*[f, g]}) &= T(r, Q^*[f, g]) + O(1) \\
 \text{i.e., } m(r, \frac{1}{Q^*[f, g]}) + N(r, \frac{1}{Q^*[f, g]}) &= m(r, Q^*[f, g]) + N(r, Q^*[f, g]) + O(1) \\
 \text{i.e., } m(r, \frac{1}{Q^*[f, g]}) + N(r, \frac{1}{Q^*[f, g]}) &= m(r, \frac{1}{P[f, g]}) + N(r, Q^*[f, g]) + S(r, f) + S(r, g) \\
 \text{i.e., } m(r, \frac{1}{Q^*[f, g]}) &= N(r, Q^*[f, g]) - N(r, \frac{1}{Q^*[f, g]}) + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g). \quad (2.29)
 \end{aligned}$$

Using (2.28) and (2.29) in (2.27) we can write

$$m(r, P[f, g]) \leq n\bar{d}(Q_2)[m(r, f) + m(r, g)] + N(r, Q^*[f, g]) - N(r, \frac{1}{Q^*[f, g]}) + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g). \quad (2.30)$$

Now using (2.13) and (2.15) in (2.30) we get

$$\frac{l}{2} \times m(r, f) + \frac{m}{2} \times m(r, g) \leq n\bar{d}(Q_2)[m(r, f) + m(r, g)] + N(r, Q^*[f, g]) - N(r, \frac{1}{Q^*[f, g]}) + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g).$$

$$i.e., [\frac{l}{2} - n\bar{d}(Q_2)]m(r, f) + [\frac{m}{2} - n\bar{d}(Q_2)]m(r, g) \leq N(r, Q^*[f, g]) - N(r, \frac{1}{Q^*[f, g]}) + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g). \quad (2.31)$$

Obviously, from (2.24) we can say that the poles of $Q^*[f, g]$ occurs possibly only from the zeros of F and $P[f, g]$, the poles of f and g and the zeros and poles of the coefficients. Also we can note that the zeros of F and $P[f, g]$ are at most simple poles of $Q^*[f, g]$.

Now suppose that z_0 is a pole of f and g of order p and q respectively and not a zero or a pole of the coefficients of $P[f, g]$, $Q_1[f, g]$ and $Q_2[f, g]$.

Then from (2.1) we can say that z_0 is a pole of $P[f, g]$ of order $pl + qm$.

Also by using Lemma 2.8 on (2.25) we can say that z_0 is a pole of $Q[f, g]$ of order atmost $(p + q - 2)\bar{d}(Q_2) + \Gamma_{Q_2}$.

Now from (2.23) we have

$$Q^*[f, g] = \frac{Q[f, g]}{P[f, g]}$$

and hence we can say that if z_0 is a pole of $Q^*[f, g]$, it will be a pole of $Q^*[f, g]$ of order atmost

$$(p + q - 2)\bar{d}(Q_2) + \Gamma_{Q_2} - (pl + qm) = [\Gamma_{Q_2} - 2\bar{d}(Q_2)] - p[l - \bar{d}(Q_2)] - q[m - \bar{d}(Q_2)].$$

Thus we obtain

$$N(r, Q^*[f, g]) \leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{P[f, g]}) + [\Gamma_{Q_2} - 2\bar{d}(Q_2)][\bar{N}(r, f) + \bar{N}(r, g)] - [l - \bar{d}(Q_2)]N(r, f) - [m - \bar{d}(Q_2)]N(r, g) + S(r, f) + S(r, g). \quad (2.32)$$

Again we can say that the order of the differential polynomial $(Q_1[f, g])'$ is $\nu + 1$. Now let z_0 be a zero of f and g with multiplicities $\mu_1 (> \nu + 1)$ and $\mu_2 (> \nu + 1)$. Then by Lemma 2.2, we can see that z_0 is a zero of $Q_1[f, g]$ with multiplicity at least $(\mu - \nu)\bar{d}(Q_1)$ where $\mu = \min\{\mu_1, \mu_2\}$. Also z_0 may be a pole

of $[\frac{F'}{F} - \frac{(P[f,g])'}{P[f,g]}]$ of order atmost 1. Hence z_0 is a zero of $[\frac{F'}{F} - \frac{(P[f,g])'}{P[f,g]}]Q_1[f, g]$ with multiplicity at least $(\mu - \nu)\underline{d}(Q_1) - 1$.

Again, since $\underline{d}(Q_1') = \underline{d}(Q_1)$, it follows from Lemma 2.2 that z_0 is a zero of $(Q_1[f, g])'$ with multiplicity at least $(\mu - \nu - 1)\underline{d}(Q_1)$.

Hence, assuming $\underline{d}(Q_1) \geq 1$, from (2.24) we can say that z_0 is a zero of $Q^*[f, g]$ with multiplicity at least $(\mu - \nu - 1)\underline{d}(Q_1)$.

Thus if $N(r, \frac{1}{Q^*[f, g]} | f, g = 0, > \nu + 1)$ denotes the counting function of those poles of $Q^*[f, g]$, counted with proper order, which are zeros of f and g with multiplicities greater than $\nu + 1$; $N(r, \frac{1}{f} | > \nu + 1)$ and $\bar{N}(r, \frac{1}{f} | > \nu + 1)$ denote the counting functions of those zeros of f whose multiplicities are greater than $\nu + 1$, counted according to its multiplicities and ignoring its multiplicities respectively, then

$$\begin{aligned} N(r, \frac{1}{Q^*[f, g]}) &\geq N(r, \frac{1}{Q^*[f, g]} | f, g = 0, > \nu + 1) \\ &\geq \underline{d}(Q_1)[N(r, \frac{1}{f} | > \nu + 1) + N(r, \frac{1}{g} | > \nu + 1)] \\ &\quad - \underline{d}(Q_1)(\nu + 1)[\bar{N}(r, \frac{1}{f} | > \nu + 1) + \bar{N}(r, \frac{1}{g} | > \nu + 1)] \\ &\quad + S(r, f) + S(r, g) \\ &= \underline{d}(Q_1)[N(r, \frac{1}{f}) - \{N(r, \frac{1}{f} | \leq \nu + 1) + (\nu + 1)\bar{N}(r, \frac{1}{f} | > \nu + 1)\}] \\ &\quad + \underline{d}(Q_1)[N(r, \frac{1}{g}) - \{N(r, \frac{1}{g} | \leq \nu + 1) + (\nu + 1)\bar{N}(r, \frac{1}{g} | > \nu + 1)\}] \\ &\quad + S(r, f) + S(r, g) \\ &= \underline{d}(Q_1)[N(r, \frac{1}{f}) - N_{\nu+1}(r, \frac{1}{f})] + \underline{d}(Q_1)[N(r, \frac{1}{g}) - N_{\nu+1}(r, \frac{1}{g})] \\ &\quad + S(r, f) + S(r, g) \end{aligned} \tag{2.33}$$

If $\underline{d}(Q_1) = 0$, (2.33) obviously holds.

Now using (2.32) and (2.33) in (2.31) we get

$$\begin{aligned} &[\frac{l}{2} - n\bar{d}(Q_2)]m(r, f) + [\frac{m}{2} - n\bar{d}(Q_2)]m(r, g) \\ &\leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{P[f, g]}) + [\Gamma_{Q_2} - 2\bar{d}(Q_2)][\bar{N}(r, f) + \bar{N}(r, g)] \\ &\quad - [l - \bar{d}(Q_2)]N(r, f) - [m - \bar{d}(Q_2)]N(r, g) - \underline{d}(Q_1)[N(r, \frac{1}{f}) - N_{\nu+1}(r, \frac{1}{f})] \\ &\quad - \underline{d}(Q_1)[N(r, \frac{1}{g}) - N_{\nu+1}(r, \frac{1}{g})] + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g) \end{aligned}$$

$$\begin{aligned} i.e., &[\frac{l}{2} - n\bar{d}(Q_2)]m(r, f) + [\frac{m}{2} - n\bar{d}(Q_2)]m(r, g) \\ &\leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{P[f, g]}) + [\Gamma_{Q_2} - 2\bar{d}(Q_2)][\bar{N}(r, f) + \bar{N}(r, g)] \\ &\quad - [\frac{l}{2} - n\bar{d}(Q_2)]N(r, f) - [\frac{m}{2} - n\bar{d}(Q_2)]N(r, g) - \underline{d}(Q_1)[N(r, \frac{1}{f}) - N_{\nu+1}(r, \frac{1}{f})] \\ &\quad - \underline{d}(Q_1)[N(r, \frac{1}{g}) - N_{\nu+1}(r, \frac{1}{g})] + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g) \end{aligned}$$

$$\begin{aligned}
 & \text{i.e., } [\frac{l}{2} - n\bar{d}(Q_2)]T(r, f) + [\frac{m}{2} - n\bar{d}(Q_2)]T(r, g) \\
 & \leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{P[f, g]}) + [\Gamma_{Q_2} - 2\bar{d}(Q_2)][\bar{N}(r, f) + \bar{N}(r, g)] \\
 & \quad - \underline{d}(Q_1)[N(r, \frac{1}{f}) - N_{\nu+1}(r, \frac{1}{f})] - \underline{d}(Q_1)[N(r, \frac{1}{g}) - N_{\nu+1}(r, \frac{1}{g})] \\
 & \quad + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g).
 \end{aligned}$$

Next let us suppose that $Q^*[f, g] \equiv 0$, then from (2.23) we have

$$Q[f, g] \equiv 0,$$

which implies from (2.25),

$$\begin{aligned}
 & (Q_2[f, g])' - \frac{F'}{F}Q_2[f, g] = 0 \\
 & \text{i.e., } \frac{(Q_2[f, g])'}{Q_2[f, g]} = \frac{F'}{F}.
 \end{aligned}$$

Integrating we get

$$\begin{aligned}
 & F = C_1Q_2[f, g] \\
 & \text{i.e., } P[f, g]Q_1[f, g] + Q_2[f, g] = C_1Q_2[f, g], \quad \text{using (2.20)} \\
 & \text{i.e., } P[f, g]Q_1[f, g] = CQ_2[f, g], \quad (2.34) \\
 & \text{assuming } C_1 - 1 = C (\neq 0) \text{ is a finite complex number.}
 \end{aligned}$$

Now using the Lemma 2.1 and assuming that the coefficients of $Q_1[f, g]$ and $Q_2[f, g]$ are small functions of f and g , we get

$$m(r, Q_1[f, g]) = m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g). \quad (2.35)$$

Again from (2.34) we have

$$\begin{aligned}
 & m(r, P[f, g]) = m(r, C \frac{Q_2[f, g]}{Q_1[f, g]}) \\
 & \leq m(r, Q_2[f, g]) + m(r, \frac{1}{Q_1[f, g]}) + O(1). \quad (2.36)
 \end{aligned}$$

Using (2.13) and (2.15), we can rewrite (2.36) as following

$$\frac{l}{2} \times m(r, f) + \frac{m}{2} \times m(r, g) \leq m(r, Q_2[f, g]) + m(r, \frac{1}{Q_1[f, g]}) + S(r, f) + S(r, g). \quad (2.37)$$

Again if $Q_2[f, g]$ consists n arbitrary meromorphic coefficients then applying Lemma 2.5 and using the hypothesis that the coefficients of $Q_2[f, g]$ to be small functions of f and g , we get

$$m(r, Q_2[f, g]) \leq n\bar{d}(Q_2)[m(r, f) + m(r, g)] + S(r, f) + S(r, g). \quad (2.38)$$

Also, by the First Fundamental Theorem and using (2.35) we have

$$\begin{aligned}
 T(r, \frac{1}{Q_1[f, g]}) &= T(r, Q_1[f, g]) + O(1) \\
 \text{i.e., } m(r, \frac{1}{Q_1[f, g]}) + N(r, \frac{1}{Q_1[f, g]}) &= m(r, Q_1[f, g]) + N(r, Q_1[f, g]) + O(1) \\
 \text{i.e., } m(r, \frac{1}{Q_1[f, g]}) + N(r, \frac{1}{Q_1[f, g]}) &= N(r, Q_1[f, g]) + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g) \\
 \text{i.e., } m(r, \frac{1}{Q_1[f, g]}) &= N(r, Q_1[f, g]) - N(r, \frac{1}{Q_1[f, g]}) + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g).
 \end{aligned} \tag{2.39}$$

Now using (2.38) and (2.39) in (2.37) we get

$$\begin{aligned}
 \frac{l}{2} \times m(r, f) + \frac{m}{2} \times m(r, g) &\leq n\bar{d}(Q_2)[m(r, f) + m(r, g)] + N(r, Q_1[f, g]) - N(r, \frac{1}{Q_1[f, g]}) \\
 &\quad + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g) \\
 \text{i.e., } [\frac{l}{2} - n\bar{d}(Q_2)]m(r, f) + [\frac{m}{2} - n\bar{d}(Q_2)]m(r, g) \\
 &\leq N(r, Q_1[f, g]) - N(r, \frac{1}{Q_1[f, g]}) + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g)
 \end{aligned} \tag{2.40}$$

It is clear that a pole of $Q_1[f, g]$ is either a pole of f or g , or a pole of the coefficients of $Q_1[f, g]$.

Now suppose that z_0 is a pole of f and g of order p and q respectively and z_0 is not a zero or a pole of the coefficients of $P[f, g]$, $Q_1[f, g]$ and $Q_2[f, g]$. Then from (2.1) we can say that z_0 is a pole of $P[f, g]$ of order $pl + qm$.

Also from Lemma 2.8 we can say that z_0 is a pole of $Q_2[f, g]$ of order atmost $(p + q - 2)\bar{d}(Q_2) + \Gamma_{Q_2}$.

Now from (2.34) we have

$$Q_1[f, g] = C \frac{Q_2[f, g]}{P[f, g]}$$

and hence, we can say that z_0 is a pole of $Q_1[f, g]$ of order atmost

$$\begin{aligned}
 &(p + q - 2)\bar{d}(Q_2) + \Gamma_{Q_2} - (pl + qm) \\
 &= [\Gamma_{Q_2} - 2\bar{d}(Q_2)] - p[l - \bar{d}(Q_2)] - q[m - \bar{d}(Q_2)].
 \end{aligned}$$

Thus we obtain

$$\begin{aligned}
 N(r, Q_1[f, g]) &\leq [\Gamma_{Q_2} - 2\bar{d}(Q_2)][\bar{N}(r, f) + \bar{N}(r, g)] - [l - \bar{d}(Q_2)]N(r, f) \\
 &\quad - [m - \bar{d}(Q_2)]N(r, g) + S(r, f) + S(r, g).
 \end{aligned} \tag{2.41}$$

Again if z_0 be a zero of f and g with multiplicities $\mu_1 (> \nu + 1)$ and $\mu_2 (> \nu + 1)$ then by Lemma 2.2, we can see that z_0 is a zero of $Q_1[f, g]$ with multiplicity

greater than $(\mu - \nu - 1)\underline{d}(Q_1)$ where $\mu = \min\{\mu_1, \mu_2\}$.

Thus we have

$$\begin{aligned}
 N(r, \frac{1}{Q_1[f, g]}) &\geq N(r, \frac{1}{Q_1[f, g]} | f, g = 0, > \nu + 1) \\
 &\geq \underline{d}(Q_1)[N(r, \frac{1}{f} | > \nu + 1) + N(r, \frac{1}{g} | > \nu + 1)] \\
 &\quad - \underline{d}(Q_1)(\nu + 1)[\bar{N}(r, \frac{1}{f} | > \nu + 1) + \bar{N}(r, \frac{1}{g} | > \nu + 1)] \\
 &\quad + S(r, f) + S(r, g) \\
 &= \underline{d}(Q_1)[N(r, \frac{1}{f}) - \{N(r, \frac{1}{f} | \leq \nu + 1) + (\nu + 1)\bar{N}(r, \frac{1}{f} | > \nu + 1)\}] \\
 &\quad + \underline{d}(Q_1)[N(r, \frac{1}{g}) - \{N(r, \frac{1}{g} | \leq \nu + 1) + (\nu + 1)\bar{N}(r, \frac{1}{g} | > \nu + 1)\}] \\
 &\quad + S(r, f) + S(r, g) \\
 &= \underline{d}(Q_1)[N(r, \frac{1}{f}) - N_{\nu+1}(r, \frac{1}{f})] + \underline{d}(Q_1)[N(r, \frac{1}{g}) - N_{\nu+1}(r, \frac{1}{g})] \\
 &\quad + S(r, f) + S(r, g) \tag{2.42}
 \end{aligned}$$

Now using (2.41) and (2.42) in (2.40) we get

$$\begin{aligned}
 &[\frac{l}{2} - n\bar{d}(Q_2)]m(r, f) + [\frac{m}{2} - n\bar{d}(Q_2)]m(r, g) \\
 &\leq [\Gamma_{Q_2} - 2\bar{d}(Q_2)][\bar{N}(r, f) + \bar{N}(r, g)] \\
 &\quad - [l - \bar{d}(Q_2)]N(r, f) - [m - \bar{d}(Q_2)]N(r, g) - \underline{d}(Q_1)[N(r, \frac{1}{f}) - N_{\nu+1}(r, \frac{1}{f})] \\
 &\quad - \underline{d}(Q_1)[N(r, \frac{1}{g}) - N_{\nu+1}(r, \frac{1}{g})] + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g)
 \end{aligned}$$

$$\begin{aligned}
 \text{i.e., } &[\frac{l}{2} - n\bar{d}(Q_2)]m(r, f) + [\frac{m}{2} - n\bar{d}(Q_2)]m(r, g) \\
 &\leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{P[f, g]}) + [\Gamma_{Q_2} - 2\bar{d}(Q_2)][\bar{N}(r, f) + \bar{N}(r, g)] \\
 &\quad - [\frac{l}{2} - n\bar{d}(Q_2)]N(r, f) - [\frac{m}{2} - n\bar{d}(Q_2)]N(r, g) - \underline{d}(Q_1)[N(r, \frac{1}{f}) - N_{\nu+1}(r, \frac{1}{f})] \\
 &\quad - \underline{d}(Q_1)[N(r, \frac{1}{g}) - N_{\nu+1}(r, \frac{1}{g})] + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g)
 \end{aligned}$$

$$\begin{aligned}
 \text{i.e., } &[\frac{l}{2} - n\bar{d}(Q_2)]T(r, f) + [\frac{m}{2} - n\bar{d}(Q_2)]T(r, g) \\
 &\leq \bar{N}(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{P[f, g]}) + [\Gamma_{Q_2} - 2\bar{d}(Q_2)][\bar{N}(r, f) + \bar{N}(r, g)] \\
 &\quad - \underline{d}(Q_1)[N(r, \frac{1}{f}) - N_{\nu+1}(r, \frac{1}{f})] - \underline{d}(Q_1)[N(r, \frac{1}{g}) - N_{\nu+1}(r, \frac{1}{g})] \\
 &\quad + m(r, \frac{1}{P[f, g]}) + S(r, f) + S(r, g).
 \end{aligned}$$

Hence the proof is complete.

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