A Study on Hyperbolic Generalized Adrien Numbers

Abstract. In this study, we introduce and rigorously define a new class of number sequences known as Hyperbolic Adrien numbers, with particular emphasis on two distinct cases: the Hyperbolic Adrien numbers and the Hyperbolic Adrien-Lucas numbers. These sequences are constructed through hyperbolic analogues of classical Adrien and Adrien-Lucas formulations, offering novel perspectives within the framework of hypercomplex analysis. Following their definition, we conduct a comprehensive investigation into their structural and algebraic properties. Specifically, we derive and analyze a range of identities, explore their matrix representations, establish recurrence relations, and formulate explicit expressions via Binet-type formulas. Furthermore, we develop their generating functions and exponential representations, and examine their behavior through Simson-type identities and summation formulas. These results not only enrich the theoretical landscape of hyperbolic number sequences but also provide foundational tools for potential applications in discrete mathematics and mathematical physics.

Keywords. Hyperbolic Adrien numbers, Hyperbolic Adrien-Lucas numbers.

1. Introduction

In this section, we present the necessary background on the definition and fundamental properties of Adrien numbers.

1.1. Adrien Numbers. Numerous researchers have investigated the generalized (r, s, t, u) sequence, which encompasses various notable numerical constructs. Among these is the sequence of generalized Adrien numbers, formally introduced by Soykan [23]. Prior to presenting our original contributions, we briefly review several fundamental properties of the generalized Adrien numbers, including their recurrence relations, Binet-type formula, and generating function.

A generalized Adrien sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations

$$W_n = 3W_{n-1} - W_{n-2} - W_{n-4},\ n \geq 4, \tag{1.1}$$

with the initial values W_0, W_1, W_2, W_3 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-2)} + 3W_{-(n-3)} - W_{-(n-4)},$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n. Soykan has investigated this specific numerical sequence in a recent study, for more details, see [23].

Characteristic equation of $\{W_n\}$ is

$$z^4 - 3z^3 + z^2 + 1 = (z^3 - 2z^2 - z - 1)(z - 1) = 0.$$

The roots of characteristic equation are

$$\begin{aligned}
\alpha &= \frac{2}{3} + \left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}, \\
\beta &= \frac{2}{3} + \omega\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega^2\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}, \\
\gamma &= \frac{2}{3} + \omega^2\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}, \\
\delta &= 1.
\end{aligned}$$

Where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned}
\alpha + \beta + \gamma + \delta &= 3, \\
\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= 1, \\
\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= 0, \\
\alpha\beta\gamma\delta &= 1.
\end{aligned}$$

We see that

$$\begin{aligned}
\alpha + \beta + \gamma &= 2, \\
\alpha\beta + \alpha\gamma + \beta\gamma &= -1, \\
\alpha\beta\gamma &= 1.
\end{aligned}$$

Using the roots and the recurrence relation of $\{W_n\}$ the Binet's formula for the generalized Adrien numbers can be expressed for all integers n as follows

$$
\begin{aligned}
W_n &= \frac{p_1\alpha^n}{4\alpha^2+3\alpha-1}+\frac{p_2\beta^n}{4\beta^2+3\beta-1}+\frac{p_3\gamma^n}{4\gamma^2+3\gamma-1}+\frac{p_4\delta^n}{3} \qquad (1.2)\\
&= S_1\alpha^n+S_2\beta^n+S_3\gamma^n+S_4\delta^n.
\end{aligned}
$$

Where p_1, p_2, p_3 and p_4 are given below

$$
\begin{aligned}
p_1 &= (\alpha W_3-\alpha(3-\alpha)W_2+(-\alpha^2+(3-1)\alpha+1)W_1-W_0),\\
p_2 &= (\beta W_3-\beta(3-\beta)W_2+(-\beta^2+(3-1)\beta+1)W_1-W_0),\\
p_3 &= (\gamma W_3-\gamma(3-\gamma)W_2+(-\gamma^2+(3-1)\gamma+1)W_1-W_0),\\
p_4 &= -(W_3-2W_2-W_1-W_0).
\end{aligned}
$$

And

$$
\begin{aligned}
S_1 &= \frac{p_1}{4\alpha^2+3\alpha-1}, \qquad (1.3)\\
S_2 &= \frac{p_2}{4\beta^2+3\beta-1},\\
S_3 &= \frac{p_3}{4\gamma^2+3\gamma-1},\\
S_4 &= -\frac{(W_3-2W_2-W_1-W_0)}{3}.
\end{aligned}
$$

Binet's formula of Adrien and Adrien-Lucas sequences are

$$A_n=\frac{(2\alpha^2+\alpha+1)\alpha^n}{4\alpha^2+3\alpha-1}+\frac{(2\beta^2+\beta+1)\beta^n}{4\beta^2+3\beta-1}+\frac{(2\gamma^2+\gamma+1)\gamma^n}{4\gamma^2+3\gamma-1}-\frac{1}{3},$$

and

$$B_n=\alpha^n+\beta^n+\gamma^n+1,$$

respectively.

Now, we define two sequences related to third order Pell and third order Pell-Lucas numbers. Adrien and Adrien-Lucas numbers are defined as

$$A_n=2A_{n-1}+A_{n-2}+A_{n-3}+1, \quad \text{with } A_0=0, A_1=1, A_2=3, \quad n\geq 3,$$

and

$$B_n=2B_{n-1}+B_{n-2}+B_{n-3}-3, \quad \text{with } B_0=4, B_1=3, B_2=7, \quad n\geq 3,$$

respectively.

The first few values of Adrien and Adrien-Lucas numbers are

$$0, 1, 3, 8, 21, 54, 138, 352, 897, 2285, 5820, 14823, 37752, 96148, ...$$

and

$$4, 3, 7, 18, 43, 108, 274, 696, 1771, 4509, 11482, 29241, 74470, 189660, ...$$

respectively.

There are close relations between Adrien, Adrien-Lucas and third order Pell, third order Pell-Lucas numbers. For example, they satisfy the following interrelations:

$$\begin{aligned} P_{n+1} &= A_{n+1} - A_n, \\ 87P_n &= 11B_{n+2} - 12B_{n+1} - 14B_n + 15, \\ Q_n &= -3A_{n+3} + 8A_{n+2} + 3A_{n+1} - 8A_n, \\ 3Q_n &= B_{n+3} - 2B_{n+2} - B_{n+1} + 2B_n, \end{aligned}$$

For more details about generalized Adrien numbers, see [23].

If we set $W_0 = 0, W_1 = 1, W_2 = 3, W_3 = 8$ then $\{W_n\}$ is the well-known Adrien sequence and if we set $W_0 = 4, W_1 = 3, W_2 = 7, W_3 = 18$ then $\{W_n\}$ is the well-known Lucas sequence. In other words, Adrien sequence $\{A_n\}_{n\geq 0}$ and Adrien-Lucas sequence $\{B_n\}_{n\geq 0}$ are defined by the fourth-order recurrence relations as;

$$A_n = 3A_{n-1} - A_{n-2} - A_{n-4}\ ,\ A_0 = 0, A_1 = 1, A_2 = 3, A_3 = 8,\ n \geq 4, \tag{1.4}$$

$$B_n = 3B_{n-1} - B_{n-2} - B_{n-4}\ ,\ B_0 = 4, B_1 = 3, B_2 = 7, B_3 = 18,\ n \geq 4. \tag{1.5}$$

The sequences $\{A_n\}_{n\geq 0}$, $\{B_n\}_{n\geq 0}$, can be extended to negative subscripts by defining,

$$\begin{aligned} A_{-n} &= -A_{-(n-2)} + 3A_{-(n-3)} - A_{-(n-4)}, \\ B_{-n} &= -B_{-(n-2)} + 3B_{-(n-3)} - B_{-(n-4)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. As a result, recurrences (1.4), (1.5) hold for all integer n. Binet's formulas as follows.

Table 1 presents the initial generalized Adrien numbers corresponding to both positive and negative subscripts.

Table 1. A few generalized Adrien numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$3W_2 - W_1 - W_3$
2	W_2	$3W_1 - W_0 - W_2$
3	W_3	$3W_0 - 3W_2 + W_3$
4	$3W_3 - W_2 - W_0$	$10W_2 - 6W_1 - 3W_3$
5	$8W_3 - W_1 - 3W_2 - 3W_0$	$10W_1 - 6W_0 - 3W_2$
6	$21W_3 - 3W_1 - 9W_2 - 8W_0$	$10W_0 + 3W_1 - 18W_2 + 6W_3$
7	$54W_3 - 8W_1 - 24W_2 - 21W_0$	$3W_0 - 28W_1 + 36W_2 - 10W_3$
8	$138W_3 - 21W_1 - 62W_2 - 54W_0$	$33W_1 - 28W_0 - W_2 - 3W_3$

We next present the generating function that characterizes the generalized Adrien numbers.

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - 3W_0)x + (W_2 - 3W_1 + W_0)x^2 + (W_3 - 3W_2 + W_1)x^3}{1 - 3x + x^2 + x^4}.$$

Next, we give the exponential generating function of $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence W_n.

LEMMA 1. *[9]. Suppose that $f_{W_n}(x) = \sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is the exponential generating function of the generalized Adrien sequence $\{W_n\}$.*

Then $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is given by:

$$\begin{aligned}
\sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \frac{(\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2 + (3-1)\alpha + 1)W_1 - W_0)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} e^{\alpha x} \\
&+ \frac{(\beta W_3 - \beta(3-\beta)W_2 + (-\beta^2 + (3-1)\beta + 1)W_1 - W_0)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} e^{\beta x} \\
&+ \frac{(\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2 + (3-1)\gamma + 1)W_1 - W_0)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} e^{\gamma x} \\
&+ (\frac{W_3 - 2W_2 - W_1 - W_0}{-3}) e^{x}.
\end{aligned}$$

The previous Lemma 1 gives the following results as particular examples.

COROLLARY 2. *Exponential generating function of Adrien and Adrien-Lucas numbers are given by:*

a): $\sum\limits_{n=0}^{\infty} A_n \dfrac{x^n}{n!} = \sum\limits_{n=0}^{\infty} ((\dfrac{(2\alpha^2+\alpha+1)\alpha^n}{4\alpha^2+3\alpha-1} + \dfrac{(2\beta^2+\beta+1)\beta^n}{4\beta^2+3\beta-1} + \dfrac{(2\gamma^2+\gamma+1)\gamma^n}{4\gamma^2+3\gamma-1} - \dfrac{1}{3})\dfrac{x^n}{n!}$
$= (\dfrac{(2\alpha^2+\alpha+1)}{4\alpha^2+3\alpha-1} e^{\alpha x} + \dfrac{(2\beta^2+\beta+1)}{4\beta^2+3\beta-1} e^{\beta x} + \dfrac{(2\gamma^2+\gamma+1)\gamma^n}{4\gamma^2+3\gamma-1} e^{\gamma x} - \dfrac{1}{3} e^{x}).$

b): $\sum\limits_{n=0}^{\infty} B_n \dfrac{x^n}{n!} = \sum\limits_{n=0}^{\infty} (\alpha^n + \beta^n + \gamma^n + 1)\dfrac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^{\gamma x} + e^{x}.$

For more details about generalized Adrien numbers, see [23].

1.2. The Importance and Applications of Number Theory. Number theory, often regarded as one of the most foundational branches of pure mathematics, is primarily concerned with the properties and relationships of integers. While traditionally viewed as abstract and theoretical, its influence extends deeply into both modern technology and scientific research.

1.2.1. *Real-Life Applications.* Despite its theoretical nature, number theory plays a crucial role in many aspects of daily life:

- **Cryptography and Cybersecurity:** Prime numbers and modular arithmetic form the backbone of encryption algorithms such as RSA, ensuring secure communication in banking, e-commerce, and data protection.
- **Computer Science and Algorithms:** Hashing functions, error detection codes, and pseudorandom number generators rely heavily on number-theoretic principles.
- **Telecommunications:** Signal processing and compression techniques often use number-theoretic transforms to optimize data transmission.
- **Barcode and QR Code Systems:** Check digit algorithms used in product identification systems are based on modular arithmetic and divisibility rules.

1.2.2. *Contributions to Research Fields.* Number theory also supports a wide range of academic disciplines:

- **Mathematical Physics:** Diophantine equations and modular forms appear in quantum mechanics and string theory.
- **Algebra and Geometry:** Concepts such as quadratic residues and continued fractions are foundational in algebraic number theory and geometric constructions.
- **Biological Modeling:** Discrete structures derived from number-theoretic sequences (e.g., Fibonacci, Pell) are used in modeling growth patterns, DNA sequencing, and biofluid dynamics.
- **Artificial Intelligence and Machine Learning:** Optimization algorithms and data encoding schemes often incorporate number-theoretic logic for efficiency and robustness.

1.2.3. *Educational and Philosophical Value.* Beyond applications, number theory cultivates logical reasoning, abstraction, and problem-solving skills. Its study encourages mathematical creativity and has historically inspired major breakthroughs in mathematics.

Next section, we define the hyperbolic generalized Adrien numbers and some properties, generating function and Binet's formula, of these numbers.

In this section, we introduce several number systems relevant to our study, with particular emphasis on the hypercomplex framework, which includes complex numbers, hyperbolic numbers, and dual numbers. Among these, hyperbolic numbers are of particular significance and will play a central role in our analysis. Notably, hyperbolic functions and numbers have found applications across various branches of engineering, including electrical engineering (e.g., transmission line modeling), control theory (e.g., system dynamics), and signal processing (e.g., filter design). Furthermore, they are utilized in diverse areas of engineering physics such as special relativity, wave propagation, fluid dynamics, optics, and heat conduction. While hyperbolic numbers possess intriguing mathematical properties, their practical utility is context-dependent and hinges on whether they offer computational or conceptual advantages over alternative number systems for the problem under consideration.

2. Background on Hypercomplex Number Systems

We begin by exploring hypercomplex number systems, which generalize the real number line and provide a broader algebraic framework for mathematical analysis. For a more detailed treatment, the reader is referred to [20]. Notably, several commutative special cases of hypercomplex system such as complex numbers, hyperbolic numbers, and dual number are widely utilized across various domains of mathematics and physics due to their unique algebraic structures and diverse applications. In the following sections, we present these number systems in a systematic manner, highlighting their foundational properties and relevance to our study.

- Complex numbers represent the simplest and most fundamental extension within the broader class of hypercomplex number systems. A complex number is defined as $z = a + ib$, where a and b are real numbers, and i denotes the imaginary unit satisfying the relation $i^2 = -1$. The components a and b are referred to as the real and imaginary parts of z, respectively, and are denoted by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$. Accordingly, the formal definition of complex numbers is given by:

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\}.$$

- Hyperbolic (double, split-complex) numbers, for more detail see [32], Split complex numbers, commonly recognized as hyperbolic numbers, defined as $h = a + jb$ where a and b real numbers and j hyperbolic unit that satisfy $j^2 = 1$. In addition that a and b named, respectively, $\mathbb{R}$ and $\mathbb{H}$. Thus, the definition of hyperbolic numbers given by,

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\},$$

- Dual numbers, as discussed in [13], are defined in the form $d = a + \varepsilon b$, where a and b are real numbers, and ε is the dual unit satisfying $\varepsilon^2 = 0$. The components a and b are referred to as the real part and the dual part of d, respectively, and are denoted by $\mathbb{R}$ and and $\mathbb{D}$. Accordingly, the formal definition of dual numbers is given as follows:

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

- A dual hyperbolic number constitutes a distinct subclass of hypercomplex numbers, integrating the structural characteristics of both dual and hyperbolic number systems. A dual hyperbolic number is defined by,

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$ and the set of all dual hyperbolic numbers are defined by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3 : a_0, a_1, a_2, a_3 \in \mathbb{R},\ j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The $\{1, j, \varepsilon, \varepsilon j\}$ is linear independent and $\mathbb{H}_{\mathbb{D}} = sp\{1, j, \varepsilon, \varepsilon j\}$ so that $\{1, j, \varepsilon, \varepsilon j\}$ is a basis of $\mathbb{H}_{\mathbb{D}}$. For more detail, see [1].

The next properties are true for the base elements $\{1, j, \varepsilon, \varepsilon j\}$ (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j,\ \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0,\ j^2 = j.j = 1 \\ \varepsilon.j &= j.\varepsilon,\ \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0,\ j(\varepsilon j) = (\varepsilon j)j = \varepsilon \end{aligned}$$

where ε satisfy the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j satisfy the hyperbolic unit ($j^2 = 1$), and εj satisfy the dual hyperbolic unit ($(j\varepsilon)^2 = 0$).

Furthermore, we introduce additional hypercomplex number system namely, quaternions, octonions, and sedenion each of which will be examined in detail in the subsequent sections.

- Quaternions, which represent a non-commutative subclass of hypercomplex number systems, constitute a four-dimensional extension of complex numbers.
- They are expressed as $a_0+ia_1+ja_2+ka_3$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$, and i, j, and k are the quaternion units that satisfy specific multiplication rules. For more detail,see [16]. Quaternion numbers are defined by

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\},$$

- The octonions, denoted by $\mathbb{O}$, form an algebraic set in which each element is expressed as a linear combination of the eight unit octonions $\{e_i : i = 0, 1.2, ..., 7\}$. Octonions are defined by,

$$\mathbb{O} = \{\sum_{i=0}^{7} a_i e_i : a_i \in \mathbb{R}, e_0 e_i = e_i e_0 = e_i,\ e_i e_j = -\delta_{ij} e_0 + \varepsilon_{ijk} e_k\ \}$$

where $e_e = 1$, δ_{ij} is Kroneker delta (equal to 1 if and only if $i = j$), ε_{ijk} is anti-symetric tensor. For more detaıl see [18, 29]

- Sedenions is a set, every element of the set linear combinations of unit sedenions $\{e_i : i = 0, 1.2, ..., 15\}$, denoted by $\mathbb{S}$. It can be seen from here that ever sedenion can be written as

$$\sum_{i=0}^{15} a_i e_i,$$

where a_i is real number. For more detail, see[22, 29].

Next we give some proporties on two hyperbolic numbers, $h_1 = a + jb$ and $h_2 = c + jd$, as

$$\begin{aligned}
h_1 + h_2 &= (a+b) + j(c+d), \\
h_1.h_2 &= (ac+bd) + j(ad+bc), \\
\overline{h_1} &= a - jb \\
\frac{h_1}{h_2} &= \frac{(ac-bd)+j(cb-ad)}{c^2-d^2}, \\
h_1 &= h_2 \text{ if only if } a = c \text{ and } b = d, \\
\langle h_1, h_2 \rangle &= (ac+bd) + j(bc+ad), \\
\|h_1\| &= \sqrt{|a^2-b^2|}, \text{ called norm of } h_1, \\
\text{if } \left|a^2-b^2\right| &> 0, \ h_1 \text{ is named spacelike vector}, \\
\text{if } \left|a^2-b^2\right| &< 0, \ h_1 \text{ is named timelike vector}, \\
\text{if } \left|a^2-b^2\right| &= 0, \ h_1 \text{ is named null(light-like) vector}.
\end{aligned}$$

Note that$\{\mathbb{R}^2, H, \langle , \rangle\}$ is called Lorentz plane and denoted as $\mathbb{R}^2_1$. There is an isomorphism relationship between the Lorentz plane and hyperbolic numbers. For more detail, see [29].Hence, the algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are all real algebras derived from the field of real numbers $\mathbb{R}$ through successive applications of a doubling procedure known as the Cayley–Dickson process.This doubling process can be extended beyond the sedenions to generate higher-dimensional algebras collectively referred to as the 2^n-ions, where n denotes the number of doublings applied starting from the real numbers (see for example [4, 16, 17, 21, 14].

Several authors have conducted research on dual numbers, hyperbolic numbers, dual hyperbolic numbers, and other specialized numerical systems, exploring their algebraic structures, geometric interpretations, and applications across various fields.We now present a selection of information from published papers in the literature.

- Cockle [8] explored hyperbolic numbers with complex coefficients, contributing to the early development of hypercomplex algebra.
- Eren and Soykan [12] studied the generalized Generalized Woodall Numbers.
- Cheng and Thompson [6] introduced dual numbers with complex coefficients, expanding the algebraic versatility of dual number systems for applications in polynomial equations and transformation theory.
- Akar at al [1] introduced the concept of dual hyperbolic numbers, combining characteristics of dual and hyperbolic systems into a unified algebraic structure.

Next, we present a selection of information from the literature concerning hyperbolic numbers, including their algebraic properties, historical development, and applications.

- Aydın [2] ntroduced the concept of hyperbolic Fibonacci numbers, defined by the following expression:

$$\widetilde{F}_n = F_n + hF_{n+1},$$

where Fibonacci numbers are given by $F_{n+2} = F_{n+1} + F_n$, with the initial conditation $F_0 = 0$, $F_1 = 1$.

- Soykan and Taşdemir [24] studied hyperbolic generalized Jacobsthal numbers given by

$$\widetilde{V}_n = V_n + hV_{n+1},$$

where generalized Jacobsthal numbers are $V_{n+2} = V_{n+1} + 2V_n$ with the initial conditation $V_0 = a$, $V_1 = b$.

- Taş [31] introduced hyperbolic Jacobsthal-Lucas sequence written by

$$HJ_n = J_n + hJ_{n+1},$$

where Jacobsthal-Lucas numbers given by $J_{n+2} = J_{n+1} + 2J_n$ with the inintial conditation $J_0 = 2$, $J_1 = 1$.

- Dikmen and Altınsoy, [11] introduced On Third Order Hyperbolic Jacobsthal Numbers are

$$\begin{aligned} \widehat{J}_n^{(3)} &= J_n^{(3)} + hJ_{n+1}^{(3)}, \\ \widehat{j}_n^{(3)} &= j_n^{(3)} + hj_{n+1}^{(3)}, \end{aligned}$$

  where Jacobsthal numbers, respectively, given by $J_n^{(3)} = J_{n-1}^{(3)} + J_{n-2}^{(3)} + 2J_{n-3}^{(3)}$, $J_0^{(3)} = 0$, $J_1^{(3)} = 1$, $J_2^{(3)} = 1$, $j_n^{(3)} = j_{n-1}^{(3)} + j_{n-2}^{(3)} + 2j_{n-3}^{(3)}$, $j_0^{(3)} = 2$, $j_1^{(3)} = 1$, $j_2^{(3)} = 5$.

Following this, we provide details on dual hyperbolic sequences as they are presented in literature.

- Soykan at al [25] introduced dual hyperbolic generalized Pell numbers are

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3},$$

where generalized Pell numbers, with the initial values V_0, V_1 not all being zero, are given by $V_n = 2V_{n-1} + V_{n-2}$, $V_0 = a$, $V_1 = b$ $(n \geq 2)$.

- Cihan at al [7] introduced dual hyperbolic Fibonacci and Lucas numbers are,

$$\begin{aligned} DHF_n &= F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3}, \\ DHL_n &= L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3}, \end{aligned}$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$, $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$.

- Soykan at al [24] introduced dual hyperbolic generalized Jacopsthal numbers are

$$\widehat{J}_n = J_n + jJ_{n+1} + \varepsilon J_{n+2} + j\varepsilon J_{n+3},$$

where $J_n = J_{n-1} + 2J_{n-2},\ J_0 = a,\ J_1 = b.$

- Bród et al [5] investigated dual hyperbolic generalized balancing numbers, examining their algebraic formulation, recurrence relations, and potential applications within number theory and symbolic computation

$$DHB_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3},$$

where $B_n = 6B_{n-1} - B_{n-2},\ B_0 = 0,\ B_1 = 1.$

- Yılmaz and Soykan [33] introduced dual hyperbolic generalized Guglielmo numbers are

$$\widehat{T}_0 = T_0 + jT_1 + \varepsilon T_2 + j\varepsilon T_3,$$

where $T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, T_0 = 0, T_1 = 1, T_2 = 3.$

- Kalça and Soykan [19] introduced dual hyperbolic generalized Pandita numbers are

$$\widehat{P}_0 = P_0 + jP_1 + \varepsilon P_2 + j\varepsilon P_3,$$

where $P_n = 2P_{n-1} - P_{n-2} + P_{n-3} - P_{n-4}, P_0 = 0P_1 = 1, P_2 = 2, P_3 = 3.$

- Demirci and Soykan [10] introduced dual hyperbolic generalized Adrien numbers are

$$\widehat{A}_0 = A_0 + jA_1 + \varepsilon A_2 + j\varepsilon A_3,$$

  where $A_n = 3A_{n-1} - A_{n-2} - A_{n-4}$, $A_0 = 0, A_1 = 1, A_2 = 3, A_3 = 8.$
- In [15], the outhers introduce the dual generalized Fibonacci matrices.

Next, the hyperbolic Fibonacci sequence and Pell numbers will be introduced, followed by an explanation of their relationship with Adrien numbers. Subsequently, the practical applications and significance of Adrien numbers in daily life will be discussed.

2.1. Hyperbolic Fibonacci and Pell Numbers. Hyperbolic extensions of classical number sequences offer rich algebraic and geometric interpretations, particularly within the context of hypercomplex systems and special functions.

Hyperbolic Fibonacci Numbers. Hyperbolic Fibonacci numbers generalize the classical Fibonacci sequence using hyperbolic functions. One such formulation involves the hyperbolic sine function defined as:

$$\sinh_F(x) = \frac{\phi^x - \psi^x}{\phi - \psi}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$ are the golden ratio and its conjugate, respectively. This expression yields values closely related to the classical Fibonacci numbers for integer inputs:

$$\sinh_F(n) \approx F_n$$

Further generalizations include hyperbolic cosine and tangent functions that encode Fibonacci-related ratios. These constructions are useful in analytic number theory and combinatorial identities [30]

Hyperbolic Pell Numbers. Hyperbolic Pell numbers extend the classical Pell sequence into hyperbolic number systems, such as Clifford algebras or Lorentzian geometry. The classical Pell recurrence is given by:

$$P_n = 2P_{n-1} + P_{n-2}$$

In hyperbolic settings, these numbers are interpreted through algebraic structures that allow Binet-type formulas and generating functions to be adapted accordingly. Applications include Lorentz-Minkowski geometry and theoretical physics, where hyperbolic rotations and metric spaces are involved [3]

2.2. Applications and Relevance of Adrien Numbers. Adrien numbers, introduced as a generalization of classical recursive sequences, possess rich algebraic structures that make them suitable for both theoretical exploration and practical modeling. Although primarily studied within pure mathematics, their properties—such as recurrence relations, matrix representations, and closed-form expressions—enable interdisciplinary applications.

2.2.1. *Potential Applications in Daily Life.* While Adrien numbers are not yet widely known in applied engineering or consumer technologies, their structural similarities to Fibonacci, Pell, and Lucas numbers suggest promising future uses:

- **Digital Signal Processing:** Recursive sequences like Adrien numbers can be used to model waveforms, filter designs, and compression algorithms, especially in systems requiring layered or third-order recurrence behavior.
- **Cryptography and Coding Theory:** The algebraic and modular properties of Adrien numbers may contribute to the design of secure key generation schemes and error-correcting codes.
- **Pattern Recognition and Image Processing:** Adrien-based matrices and transformations can be adapted for feature extraction in visual data, particularly in systems with periodic or recursive structures.
- **Biological and Fluid Modeling:** As shown in recent studies, generalized number systems—including Adrien-Lucas variants—can model cilia-driven flow, microorganism propulsion, and mucus dynamics in low Reynolds number environments.

2.2.2. *Contributions to Research Fields.* Adrien numbers support advanced mathematical modeling in:

- **Hypercomplex Systems:** Their extension into Gaussian, hyperbolic, and Clifford algebras allows for simulations in non-Euclidean geometries and relativistic frameworks.
- **Special Functions and Combinatorics:** Adrien sequences yield new identities, generating functions, and summation formulas that enrich analytic number theory.
- **Numerical Methods:** Their recurrence structure is compatible with finite-difference and iterative schemes used in computational fluid dynamics and bioengineering simulations.

These emerging applications demonstrate that Adrien numbers are not merely abstract constructs but hold potential for integration into real-world systems where recursive logic, symmetry, and algebraic generalization are essential.

3. Hyperbolic Generalized Adrien Numbers and their Generating Functions and Binet's Formulas

In this section, we introduce the concept of hyperbolic generalized Adrien numbers, formulated within the framework of the hyperbolic algebra $\mathbb{H}$. Based on this definition, we proceed to derive their corresponding generating function and Binet type formula. We now examine the structure of these numbers in the algebra $\mathbb{H}$, where the nth hyperbolic generalized Adrien number is defined as follows:

$$HW_n = W_n + jW_{n+1} \tag{3.1}$$

with the initial values HW_0, HW_1, HW_2, HW_3. (3.1). The hyperbolic Adrien numbers, as defined above, can be extended to negative subscripts by introducing the following definition,

$$HW_{-n} = W_{-n} + jW_{-n+1} \tag{3.2}$$

so identity (3.1) holds for all integers n.

We now define several special cases of the hyperbolic generalized Adrien numbers, highlighting particular parameter choices that yield notable variations or simplifications. The nth hyperbolic Adrien numbers, the nth hyperbolic Adrien-Lucas numbers, respectively, are given as the nth hyperbolic Adrien numbers is given $HA_n = A_n + jA_{n+1}$, with the initial values

$$\begin{aligned} HA_0 &= A_0 + jA_1, \\ HA_1 &= A_1 + jA_2, \\ HA_2 &= A_2 + jA_3, \end{aligned}$$

the nth hyperbolic Adrien-Lucas numbers is given $HB_n = B_n + jB_{n+1}$ with the initial values

$$\begin{aligned} HB_0 &= B_0 + jB_1, \\ HB_1 &= B_1 + jB_2, \\ HB_2 &= B_2 + jB_3, \end{aligned}$$

Note that, hyperbolic Adrien numbers (by using $W_n = A_n$, $A_0 = 0$, $A_1 = 1$, $A_2 = 3$) we get

$$\begin{aligned} HA_0 &= j, \\ HA_1 &= 1 + 3j, \\ HA_2 &= 3 + 8j, \end{aligned}$$

for hyperbolic Adrien-Lucas numbers (bu using $W_n = B_n$, $B_0 = 4$, $B_1 = 3$, $B_2 = 7$) we obtain

$$\begin{aligned} HB_0 &= 4 + 3j, \\ HB_1 &= 3 + 7j, \\ HB_2 &= 7 + 18j. \end{aligned}$$

So, using (3.1), we can write the following identity for non negative integers n,

$$HW_n = 3HW_{n-1} - HW_{n-2} - HW_{n-4}, \tag{3.3}$$

and the sequence $\{HW_n\}_{n\geq 0}$ can be given as

$$HW_{-n} = -HW_{-(n-2)} + 3HW_{-(n-3)} - HW_{-(n-4)},$$

for $n = 1, 2, 3, ...$ by using (3.2). As a result, recurrence (3.3) holds for all integer n.

Table 2 displays the initial values of the hyperbolic generalized Adrien numbers HW_n, incorporating both positive and negative indices to provide a comprehensive representation of the sequence's symmetric behavior.

Table 2. A few hyperbolic generalized Adrien numbers

n	HW_n	HW_{-n}
0	HW_0	HW_0
1	HW_1	$3HW_2 - HW_1 - HW_3$
2	HW_2	$3HW_1 - HW_0 - HW_2$
3	HW_3	$3HW_0 - 3HW_2 + HW_3$
4	$3HW_3 - HW_2 - HW_0$	$10HW_2 - 6HW_1 - 3HW_3$
5	$8HW_3 - HW_1 - 3HW_2 - 3HW_0$	$10HW_1 - 6HW_0 - 3HW_2$
6	$21HW_3 - 3HW_1 - 9HW_2 - 8HW_0$	$10HW_0 + 3HW_1 - 18HW_2 + 6HW_3$
7	$54HW_3 - 8HW_1 - 24HW_2 - 21HW_0$	$3HW_0 - 28HW_1 + 36HW_2 - 10HW_3$
8	$138HW_3 - 21HW_1 - 62HW_2 - 54HW_0$	$33HW_1 - 28HW_0 - HW_2 - 3HW_3$

Note that

$$\begin{aligned} HW_0 &= W_0 + jW_1, \\ HW_1 &= W_1 + jW_2, \\ HW_2 &= W_2 + jW_3. \end{aligned}$$

A selection of hyperbolic Adrien numbers and hyperbolic Adrien–Lucas numbers with both positive and negative subscripts are presented in Table 3 and Table 4, respectively.

Table 3.Some hyperbolic Adrien numbers

n	HA_n	HA_{-n}
0	j	j
1	$1+3j$	0
2	$3+8j$	0
3	$8+21j$	-1
4	$21+54j$	$-j$
5	$54+138j$	1
6	$138+352j$	$-3+j$
7	$352+897j$	$-3j$
8	$897+2285j$	6

Table 4. Some hyperbolic Adrien-Lucas numbers

n	HB_n	HB_{-n}
0	$4+3j$	$4+3j$
1	$3+7j$	$4j$
2	$7+18j$	-2
3	$18+43j$	$9-2j$
4	$43+108j$	$-2+9j$
5	$108+274j$	$-15-2j$
6	$274+696j$	$31-15j$
7	$696+1771j$	$31j$
8	$1771+4509j$	-74

Now, we will give some expressions that we will use in the rest of the paper and then we define Binet's formula for the hyperbolic generalized Adrien numbers. First, we define

$$\tilde{\alpha} = 1 + j\alpha, \tag{3.4}$$

$$\tilde{\beta} = 1 + j\beta, \tag{3.5}$$

$$\tilde{\gamma} = 1 + j\gamma, \tag{3.6}$$

$$\tilde{\lambda} = 1 + j, \tag{3.7}$$

note that using above equalities we can write the following identities:

$$\begin{aligned}
\tilde{\alpha}^2 &= 1+2j\alpha+\alpha^2,\\
\tilde{\beta}^2 &= 1+2j\beta+\beta^2,\\
\tilde{\gamma}^2 &= 1+2j\gamma+\gamma^2,\\
\tilde{\lambda}^2 &= 2+2j,\\
\tilde{\alpha}\tilde{\beta} &= 1+j(\alpha+\beta)+\alpha\beta,\\
\tilde{\alpha}\tilde{\gamma} &= 1+j(\alpha+\gamma)+\alpha\gamma,\\
\tilde{\gamma}\tilde{\beta} &= 1+j(\alpha+\beta)+\alpha\beta,\\
\tilde{\alpha}\tilde{\gamma} &= 1+j(1+\alpha)+\alpha.
\end{aligned}$$

THEOREM 3. *(Binet's Formula) For any integer n, the nth hyperbolic generalized Adrien number is*

$$\widetilde{W}_n = \tilde{\alpha}S_1\alpha^n + \tilde{\beta}S_2\beta^n + \tilde{\gamma}S_3\gamma^n + \tilde{\delta}S_4, \tag{3.8}$$

where $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, $\tilde{\delta}$ are given as (3.4), (3.5), (3.6), (3.7).

Proof. Using Binet's formula of the generalized Adrien numbers given below

$$W_n = S_1\alpha^n + S_2\beta^n + S_3\gamma^n + S_4,$$

where S_1, S_2, S_2, S_4 are given (1.3) we get

$$\begin{aligned}
HW_n &= W_n + jW_{n+1},\\
&= S_1\alpha^n + S_2\beta^n + S_3\gamma^n + S_4\\
&\quad +j(S_1\alpha^{n+1} + S_2\beta^{n+1} + S_3\gamma^{n+1} + S_4)\\
&= \tilde{\alpha}S_1\alpha^n + \tilde{\beta}S_2\beta^n + \tilde{\gamma}S_3\gamma^n + \tilde{\delta}S_4,
\end{aligned}$$

This proves (3.8). □

As special cases, for any integer n, the Binet's Formula of nth hyperbolic Adrien number is

$$\tilde{A}_n = \frac{(2\alpha^2+\alpha+1)\alpha^n\tilde{\alpha}}{4\alpha^2+3\alpha-1} + \frac{(2\beta^2+\beta+1)\beta^n\tilde{\beta}}{4\beta^2+3\beta-1} + \frac{(2\gamma^2+\gamma+1)\gamma^n\tilde{\gamma}}{4\gamma^2+3\gamma-1} - \frac{\tilde{1}}{3}, \tag{3.9}$$

and the Binet's Formula of nth hyperbolic Adrien-Lucas number is

$$\tilde{B}_n = \tilde{\alpha}\alpha^n + \tilde{\beta}\beta^n + \tilde{\gamma}\gamma^n + 1. \tag{3.10}$$

In the following section, we present the generating function associated with the hyperbolic generalized Adrien numbers.

THEOREM 4. *The generating function for the hyperbolic generalized Adrien numbers is*

$$\sum_{n=0}^{\infty} HW_n x^n = \frac{HW_0+(HW_1-3HW_0)x+(HW_2-3HW_1+HW_0)x^2+(HW_3-3HW_2+HW_1)x^3}{1-3x+x^2+x^4}. \tag{3.11}$$

Proof. Let

$$f_{HW_n}(x)=\sum_{n=0}^{\infty} HW_n x^n$$

be generating function of the hyperbolic generalized Adrien numbers. Then, using the definition of the hyperbolic generalized Adrien numbers, and substracting $xf_{HW_n}(x)$ and $x^2 f_{HW_n}(x)$ from $f_{HW_n}(x)$, we obtain $(1-3x+x^2+x^4)f_{HW_n}(x)$

$$\begin{aligned}
(1-3x+x^2+x^4)f_{HW_n}(x) &= \sum_{n=0}^{\infty} HWx^n - 3x\sum_{n=0}^{\infty} HWx^n + x^2\sum_{n=0}^{\infty} HWx^n + x^4\sum_{n=0}^{\infty} HWx^n, \\
&= \sum_{n=0}^{\infty} HWx^n - 3\sum_{n=0}^{\infty} HWx^{n+1} + \sum_{n=0}^{\infty} HWx^{n+2} + \sum_{n=0}^{\infty} HWx^{n+4}, \\
&= \sum_{n=0}^{\infty} HWx^n - 3\sum_{n=1}^{\infty} HW_{(n-1)}x^n + \sum_{n=2}^{\infty} HW_{(n-2)}x^n + \sum_{n=4}^{\infty} HW_{(n-4)}x^n, \\
&= (HW_0+HW_1x+HW_2x^2+HW_3x^3)-3(HW_0x+HW_1x^2+HW_2x^3) \\
&\quad +(HW_0x^2+HW_1x^3)+\sum_{n=4}^{\infty}(HW_n-3HW_{n-1}+HW_{n-2}+HW_{n-4})x^n, \\
&= HW_0+(HW_1-3HW_0)x+(HW_2-3HW_1+HW_0)x^2 \\
&\quad +(HW_3-3HW_2+HW_1)x^3.
\end{aligned}$$

Note that, using the recurrence relation $\tilde{A}=3\tilde{A}_{n-1}-\tilde{A}_{n-2}-\tilde{A}_{n-4}$ and rearranging above equation the (3.11) has been obtained. □

Now we can write the generating functions of the hyperbolic Adrien and Adrien-Lucas numbers as:

(a): $f_{\tilde{A}_n}(x)=\sum\limits_{n=0}^{\infty}\tilde{A}_n x^n=\frac{j+x}{1-3x+x^2+x^4}$,

(b): $f_{\tilde{B}_n}(x)=\sum\limits_{n=0}^{\infty}\tilde{B}_n x^n=\frac{-4jx^3+2x^2+(-2j-9)x+3j+4}{1-3x+x^2+x^4}$.

LEMMA 5. *Suppose that $f_{HW_n}(x)=\sum\limits_{n=0}^{\infty} HW_n\frac{x^n}{n!}$ is the exponential generating function of the hyperbolic generalized Adrien sequence $\{HW_n\}$.*

Then $\sum\limits_{n=0}^{\infty} HW_n\frac{x^n}{n!}$ is given by

$$\sum_{n=0}^{\infty} HW_n\frac{x^n}{n!}=S_1e^{\alpha x}\tilde{\alpha}+S_2e^{\beta x}\tilde{\beta}+S_3e^{\gamma x}\tilde{\gamma}+S_4e^{x}\tilde{1}.$$

where $\tilde{\alpha}$,$\tilde{\beta}$, $\tilde{\gamma}$, $\tilde{\delta}$ are given as (3.4), (3.5), (3.6) ,(3.7).

Proof. Using Binet' s formula

$$W_n=S_1\alpha^n+S_2\beta^n+S_3\gamma^n+S_4,$$

where S_1, S_2, S_3, S_4 are given in (1.3) we get

$$\begin{aligned}
\sum_{n=0}^{\infty} HW \frac{x^n}{n!} &= \sum_{n=0}^{\infty} HW_n \frac{x^n}{n!} + j \sum_{n=0}^{\infty} HW_{n+1} \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} (S_1\alpha^n + S_2\beta^n + S_3\gamma^n + S_4) \frac{x^n}{n!} + j \sum_{n=0}^{\infty} (S_1\alpha^{n+1} + S_2\beta^{n+1} + S_3\gamma^{n+1} + S_4) \frac{x^n}{n!} \\
&= (S_1 e^{\alpha x} + S_2 e^{\beta x} + S_3 e^{\gamma x} + S_4 e^{x}) + j(S_1 \alpha e^{\alpha x} + S_2 \beta e^{\beta x} + S_3 \gamma e^{\gamma x} + S_4 e^{x}) \\
&= S_1 e^{\alpha x}(1 + j\alpha) + S_2 e^{\beta x}(1 + j\beta) + S_3 e^{\gamma x}(1 + j\gamma) + S_4 e^{x}(1 + j) \\
&= S_1 e^{\alpha x}\tilde{\alpha} + S_2 e^{\beta x}\tilde{\beta} + S_3 e^{\gamma x}\tilde{\gamma} + S_4 e^{x}\widetilde{1}. \ \square
\end{aligned}$$

Proof: Note that we have

$$\sum_{n=0}^{\infty} HW_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (HW_n + jHW_{n+1}) \frac{x^n}{n!}.$$

From the previous lemma, we derive the following outcomes as particular instances.

Corollary 6. *Exponential generating function of hiperbolic Adrien and hiperbolic Adrien-Lucas numbers are given by:*

a):

$$\begin{aligned}
\sum_{n=0}^{\infty} \tilde{A}_n \frac{x^n}{n!} &= \Big(\frac{(2\alpha^2 + \alpha + 1)}{4\alpha^2 + 3\alpha - 1} e^{\alpha x} + \frac{(2\beta^2 + \beta + 1)}{4\beta^2 + 3\beta - 1} e^{\beta x} + \frac{(2\gamma^2 + \gamma + 1)}{4\gamma^2 + 3\gamma - 1} e^{\gamma x} - \frac{1}{3} e^{x}\Big) \\
&\quad + j\Big(\frac{(2\alpha^2 + \alpha + 1)}{4\alpha^2 + 3\alpha - 1} e^{\alpha x} + \frac{(2\beta^2 + \beta + 1)}{4\beta^2 + 3\beta - 1} e^{\beta x} + \frac{(2\gamma^2 + \gamma + 1)}{4\gamma^2 + 3\gamma - 1} e^{\gamma x} - \frac{1}{3} e^{x}\Big) \\
&= \frac{(2\alpha^2 + \alpha + 1)\widetilde{\alpha}}{4\alpha^2 + 3\alpha - 1} e^{\alpha x} + \frac{(2\beta^2 + \beta + 1)\widetilde{\beta}}{4\beta^2 + 3\beta - 1} e^{\beta x} + \frac{(2\gamma^2 + \gamma + 1)\widetilde{\gamma}}{4\gamma^2 + 3\gamma - 1} e^{\gamma x} - \widetilde{1} e^{x}
\end{aligned}$$

b):

$$\begin{aligned}
\sum_{n=0}^{\infty} \tilde{B}_n \frac{x^n}{n!} &= e^{\alpha x} + e^{\beta x} + e^{\gamma x} + e^{x} + j(\alpha e^{\alpha x} + \beta e^{\beta x} + \gamma e^{\gamma x} + e^{x}) \\
&= e^{\alpha x}\tilde{\alpha} + e^{\beta x}\tilde{\beta} + e^{\gamma x}\tilde{\gamma} + e^{x}\tilde{1}
\end{aligned}$$

4. Obtaining Binet Formula From Generating Function

Next, by the using generating function for HW_n find Binet formula of hyperbolic generalized Adrien number $\{HW_n\}$.

Theorem 7. *(Binet formula of hyperbolic generalized Adrien numbers)*

$$HW_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \tag{4.1}$$

where

$$
\begin{aligned}
p_1 &= HW_0\alpha^3 + (HW_1 - 3HW_0)\alpha^2 + (HW_2 + HW_1 + HW_0)\alpha + (HW_3 + HW_2 + HW_1), \\
p_2 &= HW_0\beta^3 + (HW_1 - 3HW_0)\beta^2 + (HW_2 + HW_1 + HW_0)\beta + (HW_3 + HW_2 + HW_1), \\
p_3 &= HW_0\gamma^3 + (HW_1 - 3HW_0)\gamma^2 + (HW_2 + HW_1 + HW_0)\gamma + (HW_3 + HW_2 + HW_1), \\
p_4 &= HW_0\delta^3 + (HW_1 - 3HW_0)\delta^2 + (HW_2 + HW_1 + HW_0)\delta + (HW_3 + HW_2 + HW_1).
\end{aligned}
$$

Proof. Let

$$h(x) = 1 - 3x + x^2 + x^4.$$

Then for some α, β, γ and δ we write

$$h(x) = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x),$$

i.e.,

$$1 - 3x + x^2 = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x), \tag{4.2}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ and $\frac{1}{\delta}$ are the roots of $h(x)$. This gives α, β, γ and δ as the roots of

$$h(\frac{1}{x}) = 1 - \frac{3}{x} + \frac{1}{x^2} + \frac{1}{x^4} = 0.$$

This implies $x^4 - 3x^3 + x^2 + u = 0$. Now, it follows that

$$\sum_{n=0}^{\infty} HWx^n = \frac{HW_0 + (HW_1 - 3HW_0)x + (HW_2 - 3HW_1 + HW_0)x^2 + (HW_3 - 3HW_2 + HW_1)x^3}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)}.$$

Then we write

$$
\begin{aligned}
& \frac{HW_0 + (HW_1 - 3HW_0)x + (HW_2 - 3HW_1 + HW_0)x^2 + (HW_3 - 3HW_2 + HW_1)x^3}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)} \\
= & \frac{B_1}{(1 - \alpha x)} + \frac{B_2}{(1 - \beta x)} + \frac{B_3}{(1 - \gamma x)} + \frac{B_4}{(1 - \delta x)}.
\end{aligned} \tag{4.3}
$$

So

$$
\begin{aligned}
& HW_0 + (HW_1 - 3HW_0)x + (HW_2 - 3HW_1 + HW_0)x^2 + (HW_3 - 3HW_2 + HW_1)x^3 \\
= & B_1(1 - \beta x)(1 - \gamma x)(1 - \delta x) + B_2(1 - \alpha x)(1 - \gamma x)(1 - \delta x) \\
& + B_3(1 - \alpha x)(1 - \beta x)(1 - \delta x) + B_3(1 - \alpha x)(1 - \beta x)(1 - \gamma x).
\end{aligned}
$$

If we consider $x = \frac{1}{\alpha}$, we get $HW_0 + (HW_1 - 3HW_0)\frac{1}{\alpha} + (HW_2 - 3HW_1 + HW_0)\frac{1}{\alpha^2} + (HW_3 - 3HW_2 + HW_1)\frac{1}{\alpha^3}$ $= B_1(1 - \frac{\beta}{\alpha})(1 - \frac{\gamma}{\alpha})(1 - \frac{\delta}{\alpha})$.

This gives

$$\begin{aligned} B_1 &= \frac{\alpha^3(HW_0 + (HW_1 - 3HW_0)\frac{1}{\alpha} + (HW_2 - 3HW_1 + HW_0)\frac{1}{\alpha^2} + (HW_3 - 6HW_2 + HW_1)\frac{1}{\alpha^3})}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} \\ &= \frac{HW_0\alpha^3 + (HW_1 - HW_0)\alpha^2 + (HW_2 - 3HW_1 + HW_0)\alpha + (HW_3 - 3HW_2 + HW_1)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} B_2 &= \frac{HW_0\beta^3 + (HW_1 - 3HW_0)\beta^2 + (HW_2 - 3HW_1 + HW_0)\beta + (HW_3 - 3HW_2 + HW_1)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\ B_3 &= \frac{HW_0\gamma^3 + (HW_1 - 3HW_0)\gamma^2 + (HW_2 - 3HW_1 + HW_0)\gamma + (HW_3 - 3HW_2 + HW_1)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\ B_4 &= \frac{HW_0\delta^3 + (HW_1 - 3HW_0)\delta^2 + (HW_2 - 3HW_1 + HW_0)\delta + (HW_3 - 3HW_2 + HW_1)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}. \end{aligned}$$

Thus (4.3) can be written as

$$\sum_{n=0}^{\infty} HWx^n = B_1(1-\alpha x)^{-1} + B_2(1-\beta x)^{-1} + B_3(1-\gamma x)^{-1} + B_4(1-\delta x)^{-1}.$$

This gives

$$\begin{aligned} \sum_{n=0}^{\infty} HWx^n &= B_1\sum_{n=0}^{\infty}\alpha^n x^n + B_2\sum_{n=0}^{\infty}\beta^n x^n + B_3\sum_{n=0}^{\infty}\gamma^n x^n + B_4\sum_{n=0}^{\infty}\delta^n x^n \\ &= \sum_{n=0}^{\infty}(B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n)x^n. \end{aligned}$$

Therefore, equating coefficients from both sides of the above expression yields the following formulation.

$$HW = B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n.$$

and then we get (4.1). □

We can get an identitiy related to hyperbolic Adrien numbers given below.

Theorem 8. *For all integers m, n the following identities hold:*

$HW_{m+n} = A_{m-2}HW_{n+3} + (-A_{m-3} - A_{m-5})HW_{n+2} + (-A_{m-4})HW_{n+1} - A_{m-3}HW_n.$

Proof. First we assume that $m, n \geq 0$ then (8) can be proved by mathematical induction on m. If $m = 0$ we get

$$HW_n = A_{-2}HW_{n+3} + (-A_{-3} - A_{-5})HW_{n+2} + (-A_{-4})HW_{n+1} - A_{-3}HW_n.$$

which is true since $A_{-2}=0$, $A_{-3}=-1$, $A_{-4}=0$, $A_{-5}=1$. Assume that the equality holds for $m\leq k$. For $m=k+1$, we get

$$\begin{aligned}
HW_{k+1+n} &= 3HW_{n+k}-HW_{n+k-1}-HW_{n+k-3}, \\
& 3(A_{k-2}HW_{n+3}+(-A_{k-3}-A_{k-5})HW_{n+2}+(-A_{k-4})HW_{n+1}-A_{k-3}HW_{n}) \\
& -(A_{k-3}HW_{n+3}+(-A_{k-4}-A_{k-6})HW_{n+2}+(-A_{-5})HW_{n+1}-A_{k-4}HW_{n}) \\
& -(A_{k-5}HW_{n+3}+(-A_{k-6}-A_{k-8})HW_{n+2}+(-A_{k-6})HW_{n+1}-A_{k-6}HW_{n}).
\end{aligned}$$

Consequently, by mathematical induction on m, this proves Theorem (8).

The other cases of m,n can be proved smilarly for all integers m,n. □

Taking $HW_n=HA_n$ or $HW_n=HB_n$ in above Theorem, respectively, we get:

Corollary 9.

$$\begin{aligned}
HA_{m+n} &= A_{m-2}HA_{n+3}+(-A_{m-3}-A_{m-5})HA_{n+2}+(-A_{m-4})HA_{n+1}-A_{m-3}HA_{n}, \\
HB_{m+n} &= A_{m-2}HB_{n+3}+(-A_{m-3}-A_{m-5})HB_{n+2}+(-A_{m-4})HB_{n+1}-A_{m-3}HB_{n}.
\end{aligned}$$

5. Simson's Formulas

This section introduces Simson's formula as applied to the hyperbolic generalized Adrien numbers. This is a special case of [26,Theorem 4.1].

Theorem 10. *For all integers n, we have*

$$\begin{vmatrix}
HW_{n+3} & HW_{n+2} & HW_{n+1} & HW_{n} \\
HW_{n+2} & HW_{n+1} & HW_{n} & HW_{n-1} \\
HW_{n+1} & HW_{n} & HW_{n-1} & HW_{n-2} \\
HW_{n} & HW_{n-1} & HW_{n-2} & HW_{n-3}
\end{vmatrix} = (HW_0+HW_1+2HW_2-HW_3)(-HW_3^3+5HW_2^3+HW_1^3+$$
$$HW_0^3-(HW_0+3HW_1-7HW_2)HW_3^2$$

$$+(3HW_0-4HW_1-14HW_3)HW_2^2+(2HW_0+HW_2-6HW_3)HW_1^2-(HW_1+2HW_3)HW_0^2+13HW_1HW_2HW_3+$$
$$HW_0HW_2HW_3+5HW_0HW_1HW_3-7HW_0HW_1HW_2).$$

Proof. Take $r=3,s=-1,t=0,u=-1$. □

Corollary 11. *For all integers n, the Simson's formulas of hyperbolic generalized Adrien number and hyperbolic generalized Adrien-Lucas numbers are given as:*

$$\begin{vmatrix} \tilde{A}_{n+3} & \tilde{A}_{n+2} & \tilde{A}_{n+1} & \tilde{A}_{n} \\ \tilde{A}_{n+2} & \tilde{A}_{n+1} & \tilde{A}_{n} & \tilde{A}_{n-1} \\ \tilde{A}_{n+1} & \tilde{A}_{n} & \tilde{A}_{n-1} & \tilde{A}_{n-2} \\ \tilde{A}_{n} & \tilde{A}_{n-1} & \tilde{A}_{n-2} & \tilde{A}_{n-3} \end{vmatrix} = 3+3j,$$

$$\begin{vmatrix} \tilde{B}_{n+3} & \tilde{B}_{n+2} & \tilde{B}_{n+1} & \tilde{B}_{n} \\ \tilde{B}_{n+2} & \tilde{B}_{n+1} & \tilde{B}_{n} & \tilde{B}_{n-1} \\ \tilde{B}_{n+1} & \tilde{B}_{n} & \tilde{B}_{n-1} & \tilde{B}_{n-2} \\ \tilde{B}_{n} & \tilde{B}_{n-1} & \tilde{B}_{n-2} & \tilde{B}_{n-3} \end{vmatrix} = -2349-2349j,$$

respectively.

6. Linear Sums

This section presents the summation formulas for the hyperbolic generalized Adrien numbers, encompassing both positive and negative subscripts. We then proceed to introduce the summation formulas for the generalized Adrien numbers.

Theorem 12. *For the generalized Adrien numbers with positive and negative subscript, we have the following formulas:*

(a): $\sum\limits_{k=0}^{n} W_k = \frac{1}{3}(-(n+3)W_{n+3} + (2n+7)W_{n+2} + (n+2)W_{n+1} + (n+4)W_n$
$+3W_3 - 7W_2 - 2W_1 - W_0)$.

(b): $\sum\limits_{k=0}^{n} W_{2k} = \frac{1}{3}(-(n+2)W_{2n+2} + (2n+5)W_{2n+1} + (n+3)W_{2n} + (n+2)W_{2n-1}$
$+2W_3 - 4W_2 - 3W_1)$.

(c): $\sum\limits_{k=0}^{n} W_{2k+1} = \frac{1}{3}(-(n+1)W_{2n+2} + (2n+5)W_{2n+1} + (n+2)W_{2n} + (n+2)W_{2n-1}$
$+2W_3 - 5W_2 - 2W_0)$.

(d): $\sum\limits_{k=1}^{n} W_{-k} = \frac{1}{3}(-(n+1)W_{-n+3} + (2n+1)W_{-n+2} + (n+2)W_{-n+1} + (n+3)W_{-n}+$
$W_3 - W_2 - 2W_1 - 3W_0)$.

(e): $\sum\limits_{k=1}^{n} W_{-2k} = \frac{1}{3}(-(n+2)W_{-2n+2} + (2n+3)W_{-2n+1} + (n+4)W_{-2n} + (n+2)W_{-2n-1}$
$+2W_3 - 4W_2 - W_1 - 4W_0)$.

(f): $\sum\limits_{k=1}^{n} W_{-2k+1} = \frac{1}{3}(-(n+3)W_{-2n+2} + 2(n+3)W_{-2n+1} + (n+2)W_{-2n} + (n+2)W_{-2n-1}$
$+2W_3 - 3W_2 - 4W_1 - 2W_0)$.

Proof. For the proof, see Soykan [28]. □

As a first special case of the above theorem, we have the following summation formulas for hyperbolic numbers.

THEOREM 13. *For the hyperbolic numbers, we have the following formulas:*

(a): $\sum_{k=0}^{n} HW_k = \frac{1}{3}(-(n+3)HW_{n+3} + (2n+7)HW_{n+2} + (n+2)HW_{n+1} + (n+4)HW_n$
$+3HW_3 - 7HW_2 - 2HW_1 - HW_0)$.

(b): $\sum_{k=0}^{n} HW_{2k} = \frac{1}{3}(-(n+2)HW_{2n+2} + (2n+5)HW_{2n+1} + (n+3)HW_{2n} + (n+2)HW_{2n-1}$
$+2HW_3 - 4HW_2 - 3HW_1)$.

(c): $\sum_{k=0}^{n} HW_{2k+1} = \frac{1}{3}(-(n+1)HW_{2n+2} + (2n+5)HW_{2n+1} + (n+2)HW_{2n} + (n+2)HW_{2n-1}$
$+2HW_3 - 5HW_2 - 2HW_0)$.

(d): $\sum_{k=1}^{n} HW_{-k} = \frac{1}{3}(-(n+1)HW_{-n+3} + (2n+1)HW_{-n+2} + (n+2)HW_{-n+1} + (n+3)HW_{-n}$
$+HW_3 - HW_2 - 2HW_1 - 3HW_0)$.

(e): $\sum_{k=1}^{n} HW_{-2k} = \frac{1}{3}(-(n+2)HW_{-2n+2} + (2n+3)HW_{-2n+1} + (n+4)HW_{-2n} + (n+2)HW_{-2n-1}$
$+2HW_3 - 4HW_2 - HW_1 - 4HW_0)$.

(f): $\sum_{k=1}^{n} HW_{-2k+1} = \frac{1}{3}(-(n+3)HW_{-2n+2} + 2(n+3)HW_{-2n+1} + (n+2)HW_{-2n} + (n+2)HW_{-2n-1}$
$+2HW_3 - 3HW_2 - 4HW_1 - 2HW_0)$.

Proof.

(a): Use Theorem [12] (a), (b), (c), (d), (e), (f) using (3.1), we get

$$\sum_{k=0}^{n} HW_k = HW_k + jHW_{k+1}.$$

Theorem 12 then (a), (b), (c), (d), (e), (f) the proof is easily attainable, respectively. □

(b):

As a first special case of the above theorem, we have the following summation formulas for hyperbolic Adrien numbers.

THEOREM 14. *For $n \geq 0$, hyperbolic generalized Adrien numbers have the following properties:*

(a): $\sum_{k=0}^{n} \tilde{A}_k = \frac{1}{3}(-(n+3)\tilde{A}_{n+3} + (2n+7)\tilde{A}_{n+2} + (n+2)\tilde{A}_{n+1} + (n+4)\tilde{A}_n + 1)$.

(b): $\sum_{k=0}^{n} \tilde{A}_{2k} = \frac{1}{3}(-(n+2)\tilde{A}_{2n+2} + (2n+5)\tilde{A}_{2n+1} + (n+3)\tilde{A}_{2n} + (n+2)\tilde{A}_{2n-1} + j + 1)$.

(c): $\sum_{k=0}^{n} \tilde{A}_{2k+1} = \frac{1}{3}(-(n+1)\tilde{A}_{2n+2} + (2n+5)\tilde{A}_{2n+1} + (n+2)\tilde{A}_{2n} + (n+2)\tilde{A}_{2n-1} + 1)$.

(d): $\sum_{k=1}^{n} \tilde{A}_{-k} = \frac{1}{3}(-(n+1)\tilde{A}_{-n+3} + (2n+1)\tilde{A}_{-n+2} + (n+2)\tilde{A}_{-n+1} + (n+3)\tilde{A}_{-n} + 4j + 3)$.

(e): $\sum_{k=1}^{n} \tilde{A}_{-2k} = \frac{1}{3}(-(n+2)\tilde{A}_{-2n+2} + (2n+3)\tilde{A}_{-2n+1} + (n+4)\tilde{A}_{-2n} + (n+2)\tilde{A}_{-2n-1} + 3j + 3)$.

(f): $\sum_{k=1}^{n} \tilde{A}_{-2k+1} = \frac{1}{3}(-(n+3)\tilde{A}_{-2n+2} + 2(n+3)\tilde{A}_{-2n+1} + (n+2)\tilde{A}_{-2n} + (n+2)\tilde{A}_{-2n-1} + 4j + 3)$.

In the following, we derive the ordinary generating functions corresponding to certain special cases of the hyperbolic generalized Adrien numbers.

THEOREM 15. *The ordinary generating functions of the sequences $\widetilde{W}_{2n}$, $\widetilde{W}_{2n+1}$ are given as follows:*

(a): $\sum_{n=0}^{\infty} HW_{2n}x^n = \dfrac{3x^2HW_3 + (x^3 - 8x^2 + x)HW_2 - 3x^3HW_1 + (x^3 + 2x^2 - 7x + 1)HW_0}{x^4 + 2x^3 + 3x^2 - 7x + 1}$

(b): $\sum_{n=0}^{\infty} HW_{2n+1}x^n = \dfrac{(x^3 + x^2 + x)HW_3 - (3x^3 + 3x^2)HW_2 + (x^3 + 2x^2 - 7x + 1)HW_1 - 3x^2HW_0}{x^4 + 2x^3 + 3x^2 - 7x + 1}$

From the last Theorem, we have the following Corollary which gives sum formula of hyperbolic Adrien numbers (Take $HW_n = \tilde{A}_n$ with

$\tilde{A}_0 = j,\ \tilde{A}_1 = 1 + 3j,\ \tilde{A}_2 = 3 + 8j,\ \tilde{A}_3 = 8 + 21j$)

COROLLARY 16. *For $n \geq 0$ hyperbolic Adrien numbers have the following properties.*

(a): $\sum_{n=0}^{\infty} \tilde{A}_{2n}x^n = \frac{j\left(x^3+2x^2-7x+1\right)-3x^3(3j+1)+3x^2(21j+8)+(8j+3)\left(x^3-8x^2+x\right)}{x^4+2x^3+3x^2-7x+1}.$

(b): $\sum_{n=0}^{\infty} \tilde{A}_{2n+1}x^n = -\frac{(8j+3)\left(3x^3+3x^2\right)-(3j+1)\left(x^3+2x^2-7x+1\right)-(21j+8)\left(x^3+x^2+x\right)+3jx^2}{x^4+2x^3+3x^2-7x+1}.$

7. Matrices related with Hyperbolic Generalized Adrien Numbers

This part of the study introduces matrix identities that arise in connection with the hyperbolic Adrien numbers.

By using the $\{A_n\}$ which is defined by the fourth-order recurrence relation as follows:

$$A_n = 3A_{n-1} - A_{n-2} - A_{n-4}\ ,$$

with the initial conditions

$$A_0 = 0,\ A_1 = 1,\ A_2 = 3,\ A_3 = 8. \tag{7.1}$$

We define the square matrix M of order 4 as

$$M = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $detM = 1$. Then, we give the following Lemma.

LEMMA 17. *For $n \geq 0$ the following identity is true*

$$\begin{pmatrix} HW_{n+3} \\ HW_{n+2} \\ HW_{n+1} \\ HW_n \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}. \tag{7.2}$$

Proof. First, we prove the assertion for the case $n \geq 0$. Lemma 17 can be given by mathematical induction on n. If $n = 0$ we get

$$\begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}$$

which is true. We assume that (7.2) is true for $n = k$. Thus the following identity is true.

$$\begin{pmatrix} HW_{k+3} \\ HW_{k+2} \\ HW_{k+1} \\ HW_k \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}.$$

For $n = k + 1$, we get

$$\begin{aligned} \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix} &= \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} HW_{k+3} \\ HW_{k+2} \\ HW_{k+1} \\ HW_k \end{pmatrix} \\ &= \begin{pmatrix} HW_{k+4} \\ HW_{k+3} \\ HW_{k+2} \\ HW_{k+1} \end{pmatrix}. \end{aligned}$$

Consequently, by mathematical induction on n, the proof is completed. □

Note that

$$A^n = \begin{pmatrix} A_{n+1} & -A_n - A_{n-2} & -A_{n-1} & -A_n \\ A_n & -A_{n-1} - A_{n-3} & -A_{n-2} & -A_{n-1} \\ A_{n-1} & -A_{n-2} - A_{n-4} & -A_{n-3} & -A_{n-2} \\ A_{n-2} & -A_{n-3} - A_{n-5} & -A_{n-4} & -A_{n-3} \end{pmatrix}.$$

For the proof see [27].

We define

$$N_{HW} = \begin{pmatrix} HW_3 & HW_2 & HW_1 & HW_0 \\ HW_2 & HW_1 & HW_0 & HW_{-1} \\ HW_1 & HW_0 & HW_{-1} & HW_{-2} \\ HW_0 & HW_{-1} & HW_{-2} & HW_{-3} \end{pmatrix}, \tag{7.3}$$

$$E_{HW} = \begin{pmatrix} HW_{n+3} & HW_{n+2} & HW_{n+1} & HW_n \\ HW_{n+2} & HW_{n+1} & HW_n & HW_{n-1} \\ HW_{n+1} & HW_n & HW_{n-1} & HW_{n-2} \\ HW_n & HW_{n-1} & HW_{n-2} & HW_{n-3} \end{pmatrix}. \tag{7.4}$$

Now, we have the following theorem for N_{HW} and E_{HW}.

Theorem 18. *Using N_{HW} and E_{HW}, we get*

$$A^n N_{HW} = E_{HW}.$$

Proof. Note that we get

$$\begin{aligned} A^n N_{\widetilde{W}} &= \begin{pmatrix} A_{n+1} & -A_n - A_{n-2} & -A_{n-1} & -A_n \\ A_n & -A_{n-1} - A_{n-3} & -A_{n-2} & -A_{n-1} \\ A_{n-1} & -A_{n-2} - A_{n-4} & -A_{n-3} & -A_{n-2} \\ A_{n-2} & -A_{n-3} - A_{n-5} & -A_{n-4} & -A_{n-3} \end{pmatrix} \begin{pmatrix} HW_3 & HW_2 & HW_1 & HW_0 \\ HW_2 & HW_1 & HW_0 & HW_{-1} \\ HW_1 & HW_0 & HW_{-1} & HW_{-2} \\ HW_0 & HW_{-1} & HW_{-2} & HW_{-3} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \end{aligned}$$

where

$$
\begin{aligned}
a_{11} &= A_{n+1}HW_3+(-A_n-A_{n-2})HW_2+(-A_{n-1})HW_1+(-A_n)HW_0,\\
a_{12} &= A_{n+1}HW_2+(-A_n-A_{n-2})HW_1+(-A_{n-1})HW_0+(-A_n)HW_{-1},\\
a_{13} &= A_{n+1}HW_1+(-A_n-A_{n-2})HW_0+(-A_{n-1})HW_{-1}+(-A_n)HW_{-2},\\
a_{14} &= A_{n+1}HW_0+(-A_n-A_{n-2})HW_{-1}+(-A_{n-1})HW_{-2}+(-A_n)HW_{-3},\\
a_{21} &= A_nHW_3+(-A_{n-1}-A_{n-3})HW_2+(-A_{n-2})HW_1+(-A_{n-1})HW_0,\\
a_{22} &= A_nHW_2+(-A_{n-1}-A_{n-3})HW_1+(-A_{n-2})HW_0+(-A_{n-1})HW_{-1},\\
a_{23} &= A_nHW_1+(-A_{n-1}-A_{n-3})HW_0+(-A_{n-2})HW_{-1}+(-A_{n-1})HW_{-2},\\
a_{24} &= A_nHW_0+(-A_{n-1}-A_{n-3})HW_{-1}+(-A_{n-2})HW_{-2}+(-A_{n-1})HW_{-3},\\
a_{31} &= A_{n-1}HW_3+(-A_{n-2}-A_{n-4})HW_2+(-A_{n-3})HW_1+(-A_{n-2})HW_0,\\
a_{32} &= A_{n-1}HW_2+(-A_{n-2}-A_{n-4})HW_1+(-A_{n-3})HW_0+(-A_{n-2})HW_{-1},\\
a_{33} &= A_{n-1}HW_1+(-A_{n-2}-A_{n-4})HW_0+(-A_{n-3})HW_{-1}+(-A_{n-2})HW_{-2},\\
a_{34} &= A_{n-1}HW_0+(-A_{n-2}-A_{n-4})HW_{-1}+(-A_{n-3})HW_{-2}+(-A_{n-2})HW_{-3},\\
a_{41} &= A_{n-2}HW_3+(-A_{n-3}-A_{n-5})HW_2+(-A_{n-4})HW_1+(-A_{n-3})HW_0,\\
a_{42} &= A_{n-2}HW_2+(-A_{n-3}-A_{n-5})HW_1+(-A_{n-4})HW_0+(-A_{n-3})HW_{-1},\\
a_{43} &= A_{n-2}HW_1+(-A_{n-3}-A_{n-5})HW_0+(-A_{n-4})HW_{-1}+(-A_{n-3})HW_{-2},\\
a_{44} &= A_{n-2}HW_0+(-A_{n-3}-A_{n-5})HW_{-1}+(-A_{n-4})HW_{-2}+(-A_{n-3})HW_{-3}.
\end{aligned}
$$

Using the theorem (8) the proof is done. □

By taking $W_n = A_n$ with A_0, A_1, A_2, A_3 in (7.3) and (7.4)

and $\widetilde{W}_n = \widetilde{W}B_n$ with $\widetilde{W}B_0$, $\widetilde{W}B_1$, $\widetilde{W}B_2$, $\widetilde{W}B_3$ in (7.3) and (7.4)

respectively, we get:

$$N_{\tilde{A}} = \begin{pmatrix} 8+21j & 3+8j & 1+3j & j \\ 3+8j & 1+3j & j & 0 \\ 1+3j & j & 0 & 0 \\ j & 0 & 0 & -1 \end{pmatrix},$$

$$E_{\tilde{A}} = \begin{pmatrix} \tilde{A}_{n+3} & \tilde{A}_{n+2} & \tilde{A}_{n+1} & \tilde{A}_{n} \\ \tilde{A}_{n+2} & \tilde{A}_{n+1} & \tilde{A}_{n} & \tilde{A}_{n-1} \\ \tilde{A}_{n+1} & \tilde{A}_{n} & \tilde{A}_{n-1} & \tilde{A}_{n-2} \\ \tilde{A}_{n} & \tilde{A}_{n-1} & \tilde{A}_{n-2} & \tilde{A}_{n-3} \end{pmatrix},$$

$$N_{\tilde{B}} = \begin{pmatrix} 18+43j & 7+18j & 3+7j & 4+3j \\ 7+18j & 3+7j & 4+3j & 4j \\ 3+7j & 4+3j & 4j & -2 \\ 4+3j & 4j & -2 & 9-2j \end{pmatrix},$$

$$E_{\tilde{B}} = \begin{pmatrix} \tilde{B}_{n+3} & \tilde{B}_{n+2} & \tilde{B}_{n+1} & \tilde{B}_{n} \\ \tilde{B}_{n+2} & \tilde{B}_{n+1} & \tilde{B}_{n} & \tilde{B}_{n-1} \\ \tilde{B}_{n+1} & \tilde{B}_{n} & \tilde{B}_{n-1} & \tilde{B}_{n-2} \\ \tilde{B}_{n} & \tilde{B}_{n-1} & \tilde{B}_{n-2} & \tilde{B}_{n-3} \end{pmatrix}.$$

From Theorem [18], we can write the following corollary.

Corollary 19. *The following identities are hold:*

- **a):** $A^n N_{\widetilde{W}A} = E_{\widetilde{W}A}$.
- **b):** $A^n N_{\widetilde{W}B} = E_{\widetilde{W}B}$.

Conclusion

In this paper, we introduced and analyzed the hyperbolic generalized Adrien numbers, which extend the classical Adrien and Adrien–Lucas sequences to the hyperbolic number system. By redefining these sequences within the algebra $\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\}$, for more detail see [32], we derived their Binet' formulas,generating functions and recurrence relations, demonstrating that the fundamental structure of Adrien numbers can be coherently generalized under hyperbolic arithmetic.

Moreover, we obtained several important results, including Simson's formulas, summation identities and matrix representations for the hyperbolic Adrien and Adrien-Lucas numbers. These findings not only preserve the main algebraic characteristics of the original sequences but also reveal new relationships and properties unique to the hyperbolic framework. Such results strengthen the connections between recurrence sequences and hypercomplex number theory.

Overall, this study contributes to the broader field of number theory by showing that the Adrien sequence structure remains robust when extended to non-Euclidean algebras. The proposed framework opens new directions for future research on dual, dual-hyperbolic and higher-dimensional generalizations, as well as potential applications in physics, engineering, and computational mathematics.

References

[1] Akar, M., Yüce, S., Şahin, Ş., On the Dual Hyperbolic Numbers and the Complex Hyperbolic Numbers, Journal of Computer Science & Computational Mathematics, 8(1), 1-6, 2018.

[2] Aydın, F. T., Hyperbolic Fibonacci Sequence, Universal Journal of Mathematics and Applications, 2 (2), 59-64, 2019. of Computer Science & Computational Mathematics, 8(1), 1-6, 2018.

[3] B. Behera and B. Panda, "Hyperbolic Fibonacci and Lucas Numbers "International Journal of Mathematical Archive", vol. 3, no. 12, pp. 470–475, 2012.

[4] Biss, D.K., Dugger, D., Isaksen, D.C., Large annihilators in Cayley-Dickson algebras, Communication in Algebra, 36 (2), 632-664, 2008.

[5] Bród, D., Liana, A., Włoch, I., Two Generalizations of Dual-Hyperbolic Balancing Numbers, Symmetry, 12(11), 1866, 2020.

[6] Cheng, H. H., Thompson, S., Dual Polynomials and Complex Dual Numbers for Analysis of Spatial Mechanisms, Proc. of ASME 24th Biennial Mechanisms Conference, Irvine, CA, August, 19-22, 1996.

[7] Cihan, A., A. Z. Azak, M. A. Güngör, M. Tosun, A Study on Dual Hyperbolic Fibonacci and Lucas Numbers, An. Şt. Univ. Ovidius Constanta, 27(1), 35–48, 2019.

[8] Cockle, J., On a New Imaginary in Algebra, Philosophical magazine, London-Dublin-Edinburgh, 3(34), 37-47, 1849.

[9] Demirci F, Soykan Y, Gaussian generalized Adrien numbers, Archives of Current Research International 25(7),469-491,(2025).

[10] Demirci F, Soykan Y, Dual hyperbolic Adrien numbers, Asian Research Journalof Mathematics.21(8).(2025).63-91.

[11] Dikmen, C. M., Altınsoy, M., On Third Order Hyperbolic Jacobsthal Numbers, Konuralp Journal of Mathematics, 10 (1), 118-126, 2022.

[12] Eren, O., Soykan, Y., Gaussian Generalized Woodall Numbers, Archives of Current Research International, 23, 8, 48-68, 2023.

[13] Fjelstad, P., Gal, S.G., n-dimensional Hyperbolic Complex Numbers, Advances in Applied Clifford Algebras, 8(1), 47-68, 1998.

[14] Göcen, M., Soykan, Y., Horadam 2^k-Ions, Konuralp Journal of Mathematics, 7(2), 492-501, 2019.

[15] Göcen, M., Dikmen, C. M.,Kaya, Y., Soykan, Y., A study on dual generalized Fibonacci matrices, 39:1,41–54, (2025).

[16] Hamilton, W.R., Elements of Quaternions, Chelsea Publishing Company, New York , 1969.

[17] Imaeda, K., Imaeda, M., Sedenions: algebra and analysis, Applied Mathematics and Computation, 115, 77-88, 2000.

[18] J. Baez, The octonions, Bull. Amer. Math. Soc. 39(2), 145-205, 2002.

[19] Kalça, F. Z., A Study on Dual Hyperbolic Generalized Pandita numbers. Archives of Current Research International 8(9).(2025).412-436.

[20] Kantor, I., Solodovnikov, A., Hypercomplex Numbers, Springer-Verlag, New York, 1989.

[21] Moreno, G., The zero divisors of the Cayley-Dickson algebras over the real numbers, Bol. Soc. Mat. Mexicana 3(4), 13-28, 1998.

[22] Soykan,Y., Tribonacci and Tribonacci-Lucas Sedenions. Mathematics 7(1), 74, 2019.

[23] Soykan, Y., Generalized Adrien Numbers, Applied Mathematic and Computer Science,7(1),37-51.

[24] Soykan, Y., Taşdemir, E., Okumuş, İ., On dual hyperbolic numbers with generalized Jacobsthal numbers components, Indian J Pure Appl Math, 54, 824–840, 2023.

[25] Soykan, Y., Gümüş, M., Göcen, M., A study on dual hyperbolic generalized Pell numbers, Malaya Journal Of Matematik, 09(03), 99-116, 2021.

[26] Soykan, Y., Simson Identity of Generalized m-step Fibonacci Numbers, Int. J. Adv. Appl. Math. and Mech. 7(2), 45-56,

[27] Soykan, Y., Properties of Generalized (r,s,t,u)-Numbers, Earthline Journal of Mathematical Sciences, 5(2), 297-327, (2021). https://doi.org/10.34198/ejms.5221.297327.

[28] Soykan, Y., Sums and Generating Functions of Special Cases of Generalized Tetranacci Polynomials, International Journal of Advances in Applied Mathematics and Mechanics, 12(4), 34-10, 2025.

[29] Yüce, S., Sayılar ve Cebir, Ankara, June 2020.

[30] T. Koshy, Fibonacci and Lucas Numbers with Applications, Wiley-Interscience, 2001.

[31] Taş, S., On Hyperbolic Jacobsthal-Lucas Sequence, Fundamental Journal of Mathematics and Applications, 5(1), 16-20, 2022.52v3, MathNT, 2021.

[32] Sobczyk, G., The Hyperbolic Number Plane, The College Mathematics Journal, 26(4), 268-280, 1995.

[33] Yılmaz B, Soykan Y. On Dual Hyperbolic Guglielmo Numbers Journal of Advances in Mathematics and Computer Science.39(4),37-61, (2024).