UNDER PEER REVIEW

A necessary condition for determining the validity of twin prime pairs

Original Research Article

Abstract

The distribution of prime numbers in natural number sequences is extremely irregular, appearing densely at times and far apart at times. Two prime numbers with a spacing of 2 are called twin prime numbers, and where such prime pairs appear in natural number sequences is one of the key issues that urgently needs to be addressed regarding the distribution of prime numbers. Although some progress has been made in finding twin prime numbers with the improvement of the Eratosthenian sieve method and other computing algorithms, there is still a lack of theoretical and efficient methods for determining twin prime pairs. The main aim of this paper is to explore the establishment of a necessary condition for determining whether any set of adjacent odd numbers is a twin prime pair. Firstly, based on Wilson's theorem, we derive a congruence equation for the $n-$th power of 2 over a given modulus of $(2n+1)$. Then, a novel necessary condition is obtained for judging the validity of twin prime pairs by using the Chinese remainder theorem. Finally, some computational examples are provided to demonstrate the effectiveness of the proposed method.

Keywords: Twin prime pairs; Necessary condition; Congruence relationship; n-th power of 2

2010 Mathematics Subject Classification: 11A41;11N36

1 Introduction

Let $\mathcal{P}$ denote the set of all prime numbers. If both p and $(p+2)$ belong to the set $\mathcal{P}$, then $\{p, p+2\}$ is called a twin prime pair. Whether there are infinitely many twin primes remains an open problem, despite breakthroughs in the bounded gap between adjacent prime numbers (1; 2; 3; 4). However, there is ample numerical support for an even stronger statement of the Hardy-Littlewood twin prime conjecture (5) which asserts that

$$\pi_2(n) \sim C_2 \frac{n}{\log^2 n} \quad \text{as} \quad n \to \infty, \tag{1.1}$$

where

$$\pi_2(n) = \sum_{\substack{p\in\mathcal{P},\ (p+2)\in\mathcal{P} \\ p\le n}} 1, \quad C_2 = 2\prod_{p>2}\left\{1 - \frac{1}{(p-1)^2}\right\}. \tag{1.2}$$

It can be seen that research on twin prime numbers mainly focuses on infinity and distribution patterns, while in terms of determining the conditions for twin prime pairs, it still relies on determining the prime properties of individual prime numbers (6; 7; 8; 9). To find twin prime numbers, the first step is to have a method to determine whether a number is prime. The basic judgment of prime numbers can be achieved by checking whether the number can be divided by any integer smaller than its square root. If not, then the number is prime. This is a simple but inefficient method that is suitable for smaller numbers. For larger numbers, more efficient algorithms such as the Eratosthenian sieve can be used (10; 11; 12; 13). In addition, some studies focus on the distribution of twin primes and the search for larger twin prime pairs (14; 15; 16; 17).

Generally, the basic strategy for finding twin prime numbers is to first identify all prime numbers within a certain range, and then check if each pair of adjacent prime numbers differs by 2. If such pairs are found, then they are twin prime numbers. Although some progress has been made in finding twin prime numbers with the improvement of computing power and algorithms (18; 19; 20; 21), there is still a lack of simple and efficient methods for determining twin prime pairs.

The rest of this paper is organized as follows: Section 2 gives some lemmas to prepare for the proof of main results. In Section 3, we derive a necessary condition for judging the validity of twin prime pairs. Section 4 provides some numerical calculation examples based on the proposed method to demonstrate its effectiveness.

2 Lemmas

This section derives the proofs of two Lemmas in preparation for the theorem proof in the following section.

Lemma 2.1. *Let $n \in \mathcal{N}^+$ be an arbitrary positive integer. $(2n+1)$ is a prime if and only if*

$$(n!)^2 \equiv (-1)^{n-1} \pmod{2n+1}. \tag{2.1}$$

Proof. Note that

$$\begin{aligned}(2n-1)! &= 1\cdot 2\cdot 3\cdot 4\cdots(n-1)n(n+1)\cdots(2n-1)\\ &\equiv 1\cdot 2\cdot 3\cdot 4\cdots(n-1)n\cdot(-(2n+1-(n+1)))\cdots\\ &\quad (-(2n+1-(2n-1))) \pmod{2n+1}\\ &\equiv 1\cdot 2\cdot 3\cdot 4\cdots(n-1)n\cdot(-n)\cdot(-(n-1))\cdots(-2) \pmod{2n+1}\\ &\equiv (-1)^{n-1}(n!)^2 \pmod{2n+1}. \end{aligned} \tag{2.2}$$

From Wilson's theorem, $(2n+1)$ is a prime if and only if

$$(2n)! \equiv -1 \pmod{2n+1}, \tag{2.3}$$

which is equivalent to

$$(2n-1)! \equiv 1 \pmod{2n+1}. \tag{2.4}$$

From Eq.(2.2) and Eq. (2.4), we have Eq. (2.1). □

Lemma 2.2. *Let $n \in \mathcal{N}^+$ be an arbitrary positive integer. If $(2n+1)$ is a prime, then*

$$2^n \equiv (-1)^{\lambda_n} \pmod{2n+1}, \tag{2.5}$$

where $\lambda_n = n + \lfloor n/2 \rfloor$.

Proof. Let $k \in \mathcal{N}, k_1 \in \mathcal{N}, k_2 \in \mathcal{N}$. Due to the fact that

$$\begin{aligned}(2n-1)! &= \prod_{k\in[1,2n-1]} k\\ &= \prod_{k_1\in[1,n]}(2k_1-1)\cdot\prod_{k_2\in[1,n-1]}2k_2\\ &= 2^{n-1}(n-1)!\cdot\prod_{k_1\in[1,n]}(2k_1-1), \end{aligned} \tag{2.6}$$

we have

$$(2n-1)! \equiv 2^{n-1}(n-1)!\cdot\prod_{k_1\in[1,n]}(2k_1-1) \pmod{2n+1}. \tag{2.7}$$

Then, it is considered in two cases.

- Case i : n is an odd number.
  From Eq.(2.7), we obtain

$$\begin{aligned}(2n-1)! \equiv\; & 2^{n-1}(n-1)!\cdot\prod_{\substack{\bar{k}_1\in[1,\lfloor n/2\rfloor+1]\\ \bar{k}_1\in\mathcal{N}}}(2\bar{k}_1-1)\cdot\\ & \prod_{\substack{\hat{k}_1\in[\lfloor n/2\rfloor+2,n]\\ \hat{k}_1\in\mathcal{N}}}(2\hat{k}_1-1) \pmod{2n+1}\end{aligned}$$

$$
\begin{aligned}
&\equiv 2^{n-1}(n-1)! \cdot \prod_{\substack{\bar{k}_1 \in [1, \lfloor n/2 \rfloor + 1] \\ \bar{k}_1 \in \mathcal{N}}} (2\bar{k}_1 - 1) \cdot \\
&\quad \prod_{\substack{\hat{k}_1 \in [\lfloor n/2 \rfloor + 2, n] \\ \hat{k}_1 \in \mathcal{N}}} (-1) \left(2n + 1 - \left(2\hat{k}_1 - 1\right)\right) \pmod{2n+1} \\
&\equiv 2^{n-1}(n-1)! \cdot \prod_{\substack{\bar{k}_1 \in [1, \lfloor n/2 \rfloor + 1] \\ \bar{k}_1 \in \mathcal{N}}} (2\bar{k}_1 - 1) \cdot \\
&\quad \prod_{\substack{\hat{k}_1 \in [\lfloor n/2 \rfloor + 2, n] \\ \hat{k}_1 \in \mathcal{N}}} (-1) \cdot 2 \left(n - \hat{k}_1 + 1\right) \pmod{2n+1} \\
&\equiv 2^{n-1}(n-1)! \cdot \left(\prod_{\substack{\bar{k}'_1 \in [1, \lfloor n/2 \rfloor] \\ \bar{k}'_1 \in \mathcal{N}}} (2\bar{k}'_1 - 1) \right) \cdot n \cdot \\
&\quad \prod_{\substack{\hat{k}'_1 \in [1, \lfloor n/2 \rfloor] \\ \hat{k}'_1 \in \mathcal{N}}} (-1) \cdot 2\hat{k}'_1 \pmod{2n+1} \\
&\equiv (-1)^{\lfloor n/2 \rfloor} \cdot 2^{n-1} \left((n-1)!\right)^2 \cdot n \pmod{2n+1} \\
&\equiv (-1)^{\lfloor n/2 \rfloor} \cdot 2^{n-1} \left((n-1)!\right)^2 \cdot n \cdot (-2n) \pmod{2n+1} \\
&\equiv (-1)^{\lfloor n/2 \rfloor + 1} \cdot 2^{n} \left(n!\right)^2 \pmod{2n+1}. \qquad (2.8)
\end{aligned}
$$

- Case ii : n is an even number.
  Similar to Eq.(2.8), we have

$$
\begin{aligned}
(2n-1)! &\equiv 2^{n-1}(n-1)! \cdot \prod_{\substack{\bar{k}_1 \in [1, n/2] \\ \bar{k}_1 \in \mathcal{N}}} (2\bar{k}_1 - 1) \cdot \\
&\quad \prod_{\substack{\hat{k}_1 \in [n/2+1, n] \\ \hat{k}_1 \in \mathcal{N}}} (2\hat{k}_1 - 1) \pmod{2n+1} \\
&\equiv 2^{n-1}(n-1)! \cdot \left(\prod_{\substack{\bar{k}_1 \in [1, n/2] \\ \bar{k}_1 \in \mathcal{N}}} (2\bar{k}_1 - 1) \right) \cdot \\
&\quad \prod_{\substack{\hat{k}'_1 \in [1, n/2] \\ \hat{k}'_1 \in \mathcal{N}}} (-1) \cdot 2\hat{k}'_1 \pmod{2n+1} \\
&\equiv (-1)^{n/2} \cdot 2^{n-1}(n-1)! \cdot n! \pmod{2n+1} \\
&\equiv (-1)^{n/2} \cdot 2^{n-1} n! \cdot (n-1)! \cdot (-2n) \pmod{2n+1} \\
&\equiv (-1)^{n/2+1} \cdot 2^{n} \left(n!\right)^2 \pmod{2n+1}. \qquad (2.9)
\end{aligned}
$$

Note that $n/2 = \lfloor n/2 \rfloor$ while n belongs to Case ii. Based on Lemma 1, if $(2n+1)$ is a prime, we rewrite Eq.(2.8) and Eq.(2.9) as

$$\begin{aligned}(2n-1)! &\equiv (-1)^{\lfloor n/2\rfloor+1}\cdot 2^n\cdot(-1)^{n-1} \pmod{2n+1}\\ &\equiv (-1)^{n+\lfloor n/2\rfloor}\cdot 2^n \pmod{2n+1}\\ &\equiv 1 \pmod{2n+1}, \end{aligned}\tag{2.10}$$

which yields Eq.(2.5). This completes the proof.

□

3 Necessary condition for twin prime pairs

According to the lemmas in the previous section, this section provides the necessary proof process for the existence of twin prime pairs.

Theorem 3.1. *Let $n \in \mathcal{N}^+$ be an arbitrary positive integer. If the set of $\{2n-1, 2n+1\}$ is a pair of twin primes,it should satisfy the following condition.*

$$2^n \equiv \begin{cases} (-1)^{\lambda_{n-1}}\left(2n^2+3n\right) \pmod{4n^2-1}, & \text{if n is odd}\\ (-1)^{\lambda_{n-1}}\left(2n^2+n+1\right) \pmod{4n^2-1}, & \text{otherwise,}\end{cases}\tag{3.1}$$

where $\lambda_{n-1} = n-1+\lfloor (n-1)/2\rfloor$.

Proof. If both $(2n-1)$ and $(2n+1)$ are primes, according to Lemma 2, we have

$$\begin{cases} 2^n \equiv (-1)^{\lambda_{n-1}} * 2 \pmod{2n-1},\\ 2^n \equiv (-1)^{\lambda_n} \pmod{2n+1}.\end{cases}\tag{3.2}$$

By the Chinese remainder theorem, it can be obtained

$$\begin{aligned}2^n &\equiv (-1)^{\lambda_{n-1}}\cdot 2\cdot n\cdot(2n+1)+(-1)^{\lambda_n}\cdot n\cdot(2n-1) \pmod{(2n-1)(2n+1)}\\ &\equiv (-1)^{\lambda_{n-1}}\cdot(4n^2+2n)+(-1)^{\lambda_n}\cdot(2n^2-n) \pmod{4n^2-1}.\end{aligned}\tag{3.3}$$

Since that

$$\lambda_n = \lambda_{n-1}+1+\lfloor n/2\rfloor-\lfloor (n-1)/2\rfloor,\tag{3.4}$$

it is obtained

$$\lambda_n = \begin{cases}\lambda_{n-1}+1, & \text{if n is odd}\\ \lambda_{n-1}, & \text{otherwise.}\end{cases}\tag{3.5}$$

By Using Eq.(3.5), Eq.(3.3) can be rewritten as

$$2^n \equiv \begin{cases} (-1)^{\lambda_{n-1}}\left(2n^2+3n\right) \pmod{4n^2-1}, & \text{if n is odd}\\ (-1)^{\lambda_{n-1}}\left(6n^2+n\right) \pmod{4n^2-1}, & \text{otherwise.}\end{cases}\tag{3.6}$$

Note that

$$6n^2+n \equiv 2n^2+n+1 \pmod{4n^2-1}.\tag{3.7}$$

Equation (3.6) and Eq.(3.7) give Eq.(3.1) which completes the proof. □

Note that if $\{\{2n-1, 2n+1\} : n \in \mathcal{N}\}$ is a set of prime pairs, it needs to meet the following condition:

$$\{\{2n-1, 2n+1\} : n \in \mathcal{N}\} \subset \{\{6m-1, 6m+1\} : m \in \mathcal{N}\}. \tag{3.8}$$

Thus, if letting $n = 3m$, Theorem 3.1 gives directly the following corollary.

Corollary 3.2. *Let $m \in \mathcal{N}^+$ be an arbitrary positive integer. If the set of $\{6m-1, 6m+1\}$ is a pair of twin primes, it should satisfy*

$$2^{3m} \equiv \begin{cases} (-1)^{\lambda_{3m-1}} \left(18m^2 + 9m\right) \pmod{36m^2 - 1}, & \text{if 3m is odd} \\ (-1)^{\lambda_{3m-1}} \left(18m^2 + 3m + 1\right) \pmod{36m^2 - 1}, & \text{otherwise.} \end{cases} \tag{3.9}$$

Here, $\lambda_{3m-1} = 3m - 1 + \lfloor (3m-1)/2 \rfloor$.

Theorem 3.3. *Let $n \in \mathcal{N}^+$ be an arbitrary positive integer. If the set of $\{2n-1, 2n+1\}$ is a pair of twin primes, the following condition should be satisfied.*

$$\begin{cases} 2^{n-1}\left(2^n - (-1)^{\lambda_{n-1}}\right) \equiv 1 \pmod{4n^2 - 1}, & \text{if n is odd} \\ 2^n\left(2^n - (-1)^{\lambda_{n-1}}\right) \equiv 2n + 1 \pmod{4n^2 - 1}, & \text{otherwise.} \end{cases} \tag{3.10}$$

Proof. If both $(2n-1)$ and $(2n+1)$ are primes, according to Fermat's Little Theorem, we have

$$\begin{cases} 2^{2n} \equiv 4 \pmod{2n - 1}, \\ 2^{2n} \equiv 1 \pmod{2n + 1}. \end{cases} \tag{3.11}$$

By the Chinese remainder theorem, it can be obtained

$$\begin{aligned} 2^{2n} &\equiv 4 \cdot n \cdot (2n+1) + n \cdot (2n-1) \pmod{(2n-1)(2n+1)} \\ &\equiv 10n^2 + 3n + 2 \pmod{4n^2 - 1} \\ &\equiv 2n^2 + 3n + 2 \pmod{4n^2 - 1}. \end{aligned} \tag{3.12}$$

Based on Eq.(3.1) in Theorem 1 and Eq.(3.12), we obtain

$$\begin{cases} 2^{2n} - (-1)^{\lambda_{n-1}} \cdot 2^n \equiv 2 \pmod{4n^2 - 1}, & \text{if n is odd} \\ 2^{2n} - (-1)^{\lambda_{n-1}} \cdot 2^n \equiv 2n + 1 \pmod{4n^2 - 1}, & \text{otherwise.} \end{cases} \tag{3.13}$$

Note that $(2, 4n^2 - 1) = 1$. Reorganizing the left side of Eq.(3.13) yields Eq.(3.10). This completes the proof. □

Similarly, if letting $n = 3m$, Theorem 3.3 gives the following conclusion.

Corollary 3.4. *Let $m \in \mathcal{N}^+$ be an arbitrary positive integer. If the set of $\{6m-1, 6m+1\}$ is a pair of twin primes, it should satisfy*

$$\begin{cases} 2^{3m-1}\left(2^{3m} - (-1)^{\lambda_{3m-1}}\right) \equiv 1 \pmod{36m^2 - 1}, & \text{if 3m is odd} \\ 2^{3m}\left(2^{3m} - (-1)^{\lambda_{3m-1}}\right) \equiv 6m + 1 \pmod{36m^2 - 1}, & \text{otherwise.} \end{cases} \tag{3.14}$$

where

$$\lambda_{3m-1} = 3m - 1 + \lfloor (3m-1)/2 \rfloor. \tag{3.15}$$

Table 1: Application of Corollary 3.4

m	3m	$6m-1$	$6m+1$	$36m^2-1$	$(-1)^{\lambda_{3m-1}}$	$*f_1 \pmod{36m^2-1}$	$*f_2 \pmod{36m^2-1}$
1	3	5	7	35	-1	1	-
2	6	11	13	143	-1	-	13
3	9	17	19	323	1	1	-
4	12	23	25	575	1	-	370
5	15	29	31	899	-1	1	-
6	18	35	37	1295	-1	-	555
7	21	41	43	1763	1	1	-
8	24	47	49	2303	1	-	1036
9	27	53	55	2805	-1	1821	-
10	30	59	61	3599	-1	-	61
11	33	65	67	4355	1	3351	-
12	36	71	73	5183	1	-	73
13	39	77	79	6083	-1	1660	-
14	42	83	85	7055	-1	-	6310
15	45	89	91	8099	1	5964	-
16	48	95	97	9215	1	-	5335

$*f_1 = 2^{3m-1}\left(2^{3m} - (-1)^{\lambda_{3m-1}}\right)$, $*f_2 = 2^{3m}\left(2^{3m} - (-1)^{\lambda_{3m-1}}\right)$.

4 Examples

This section provides some numerical calculation examples to demonstrate the effectiveness of the proposed method.

Based on Corollary 2, without loss of generality, we have conducted computational verification on the congruence equation (3.14) for each $m (m \in [1, 16])$ whose results are shown in Table 1. From Table 1, it can be observed that when $m \in \{1, 2, 3, 5, 7, 10, 12\}$, the congruence equation (3.14) holds, while in other cases, its congruence relationship does not hold. We conclude that if the congruence relationship (3.14) is not met when m takes a given value, then $\{6m - 1, 6m + 1\}$ is definitely not a twin prime pair.

5 Conclusion

The paper presents an analytical method that enhances current computational and heuristic approaches by utilizing Wilson's theorem and the Chinese Remainder Theorem to derive a novel required condition for discovering twin prime pairings. The proposed congruence-based framework improves theoretical comprehension and offers a more methodical approach to twin prime pair verification. Meanwhile, its practical usefulness is reinforced by the inclusion of computational examples, which provides a basis for additional mathematical research and algorithmic advancement in prime number theory.

Acknowledgment

The author is deeply grateful to the referees for their useful suggestions.

Competing Interests

The author declares no conflict of interest.

References

[1] Banks, W.D., Freiberg, T., Turnage-Butterbaugh, C.L.: Consecutive primes in tuples. Acta Arithmetica **167**(3) (2013)

[2] Zhang, Y.: Bounded gaps between primes. Annals of Mathematics (2014)

[3] Mor, A., Gupta, S.: On existence of prime k-tuples conjecture for positive proportion of admissible k-tuples. Baghdad Science Journal **21**(3) (2024)

[4] Stadlmann, J.: On primes in arithmetic progressions and bounded gaps between many primes. Advances in Mathematics **468**(000) (2025)

[5] Green, B., Tao, T.: Linear equations in primes. Mathematika **39**(2), 367–378 (2006)

[6] Atanassov, K.T., Vassilev-Missana, M.V.: On explicit formulae for prime and twin prime numbers. Ital.j.pure Appl.math **20**(20), 103–120 (2006)

[7] Porshnev, S.V.: On the issues of prime numbers' and twin primes' distribution on the ulam spiral. In the World of Scientific Discoveries / V Mire Nauchnykh Otkryt (2013)

[8] Tessema, A.T.: Advanced mathematical formulas to calculate prime numbers. Mathematics and Computer Science **6**(6) (2021)

[9] Kowitz, K.: Twin, cousin, and sexy prime counting functions. explicit formulas. Ukrainian Mathematical Journal **76**(8), 1424–1429 (2025)

[10] Fouvry, E., Grupp, F.: Weighted sieves and twin prime type equations. Duke Mathematical Journal **58**(3) (1989)

[11] Polymath, D.H.J.: Deterministic methods to find primes. Mathematics (2010)

[12] Ren, W.: A symmetric uniform formula and sole index method for sieving (twin) primes. IEEE (2020)

[13] Mothebe, M.F.: Sieve methods and the twin prime conjecture. South East Asian Journal of Mathematics & Mathematical Sciences **20**(1) (2024)

[14] Wolf, M.: Some remarks on the distribution of twin primes. Mathematics (2001)

[15] Indlekofer, K.H., Jrai, A.: Largest known twin primes. Mathematics of Computation **65**(213), 427–428 (1996)

[16] Dinculescu, A.: On the numbers that determine the distribution of twin primes. Surveys in Mathematics and its Applications **13**(10), 171–181 (2018)

[17] Eddin, S.S., Suzuki, Y.: On the distribution of products of two primes. Journal of Number Theory **214**(000) (2020)

[18] Chaves, M.: Twin primes and a primality test by indivisibility. Mathematical Gazette **95**(533), 266–269 (2011)

[19] Karatay, M., Aylanc, A., Zkan, S.: Algorithm on finding twin prime numbers. Journal of Modern Technology & Engineering **4**(3) (2019)

[20] Prasad Rao, B.N., Rangamma, M.: A primality test and a theorem on twin primes. In: International Conference on Mathematical Analysis and Computing (2021)

[21] Iovane, G., Di Gironimo, P., Benedetto, E., DAlfonso, V.: Some properties and algorithms for twin primes. Applied Sciences **14**(17) (2024)