Sufficient Conditions for *n*-th Root of *x* Being an Irrational Number for all $n \in \mathbf{N}$, $n \geq 2$

Abstract

The challenging question "Given a number $x \in \mathbf{N}$, $x \geq 2$: For which $n \in \mathbf{N}$, $n \geq 2$ the value of $\sqrt[n]{x}$ is a rational number?" has been solved completely if the prime factorization of *x* is available. However, prime factorization may be impossible in practice, in particular for extremely large numbers. Therefore, in this paper we present various sufficient conditions which allow us to prove that "$\sqrt[n]{x} \notin \mathbf{Q}$, $\forall n \in \mathbf{N}$, $n \geq 2$" just based on the knowledge of only one or two factors of the prime factorization of *x*.

Keywords: *Number theory, n-th roots of real numbers, irrational n-th roots, sufficient conditions for irrational roots*

1. Introduction

An important question studied in Number Theory is whether $\sqrt[n]{x}$ is a rational or an irrational number (for $x \in \mathbf{R}$). In this paper we want to solve the challenging problem "Finding sufficient conditions such that $\sqrt[n]{x} \notin \mathbf{Q}$, $\forall n \in \mathbf{N}$, $n \geq 2$ " for arbitrary $x \in \mathbf{R}$.

Our earlier publications, e.g. [1], have demonstrated that it is not necessary to solve the problem for $x \in \mathbf{R}$. Instead, it is sufficient to solve the problem just for arbitrary $x \in \mathbf{N}$, $x \geq 2$. For details regarding this simplification of the more general problem the reader may consult [1]. (*Remark:* In the sequel, let $\mathbf{N}_{\geq 2}$ denote the set $\mathbf{N}_{\geq 2} := \{ y \in \mathbf{N} \mid y \geq 2 \}$).

Therefore, in this contribution, we will restrict our search for sufficient conditions such that $\sqrt[n]{x} \notin \mathbf{Q}$, $\forall n \in \mathbf{N}_{\geq 2}$ holds, to values of *x*, $x \in \mathbf{N}_{\geq 2}$. As suggested in [1], again we make use of the (unique) prime factorization [2] of *x*, which we assume to be given by

$$x = p_1^{k\_1} \bullet p_2^{k\_2} \bullet\bullet\bullet\ p_m^{k\_m} \qquad (1)$$

where $m \in \mathbf{N}$, p_i prime numbers being pairwise different, k_i, $\in \mathbf{N}$, $i \in \{1, 2, \ldots, m\}$.

Remark: k_i to be read as k_i .

Prime factorization or integer factorization has been studied very intensively by mathematicians during the past [3, 4]. In particular, different factorization algorithms have been compared in terms of their efficiency and complexity [5, 6] and their required computational effort [7].

Assuming that the prime factorization of x is given by eq. (1), then let CD(x) denote

$$CD(x) = \{c \in \mathbf{N}_{\geq 2} \mid \forall\, i \in \{1, 2, \ldots, m\}, \exists \mu(i) \in \mathbf{N}: c \bullet \mu(i) = k_i\} \quad (2)$$

So, CD(x) denotes the set of common divisors of $\{k_1, k_2, \ldots, k_m\}$.

It has been proven in [1] that

$$\sqrt[n]{x} \in \mathbf{Q} \Leftrightarrow n \in CD(x).$$

And, therefore $\sqrt[n]{x} \notin \mathbf{Q}, \forall n \in \mathbf{N}_{\geq 2} \Leftrightarrow CD(x) = \emptyset$.

It is well-known that it may be practically impossible to determine the complete prime factorization of x for extremely large values of x. In Computer Science, this interesting fact has been used to specify public-key cryptosystems such as the RSA algorithm [8, 9], where RSA is named according to the initials of the co-authors' surnames, i.e. *R*ivest, *S*hamir, *A*dleman [8]. The basic idea of RSA is : if we know two very large prime numbers p and q we construct a "key" $k=p \bullet q$. Then, if k possesses a sufficiently large number of digits – using currently available computing power – the attacker will be unable to factorize k into two prime numbers (assuming he/she has not any knowledge regarding neither p nor q). So, if the number of digits of k is kept sufficiently large, RSA can provide an elegant means for secure communication via insecure networks such as the Internet.

Due to the potential lack of the complete prime factorization of x, we want to answer the question $\sqrt[n]{x} \notin \mathbf{Q}, \forall n \in \mathbf{N}_{\geq 2}$ just based on a very rudimentary knowledge of the prime factorization of x. We are going to present a large variety of very general conditions which imply $\sqrt[n]{x} \notin \mathbf{Q}, \forall n \in \mathbf{N}_{\geq 2}$ for all natural numbers x satisfying the corresponding conditions. Pleasingly, the conditions for irrational n-th roots, which we identify, just require only one or two factors of the prime factorization of x.

Let us emphasize the (rather surprising) fact that for all approaches presented in this paper to determine whether "$\sqrt[n]{x} \notin \mathbf{Q}, \forall n \in \mathbf{N}_{\geq 2}$", at no point we need to know more than two factors of the prime factorization of x. Anyway, it is evident that the prime factorization of an extremely large natural number may lead, in principle, to an infinite number of different prime factors.

This paper generalizes our preliminary results for tests of the property "$\sqrt[n]{x} \notin \mathbf{Q}, \forall n \in \mathbf{N}_{\geq 2}$" without possessing complete knowledge of the prime factorization of x (cf. [10]).

2. Sufficient Conditions Assuming one Factor of the Prime Factorization of $x \in \mathbf{N}$ is Available

In the following we assume that – for a natural number $x \in \mathbf{N}_{\geq 2}$ – the prime factorization of x is given by:

$$x = p_1^{k\_1} \bullet p_2^{k\_2} \bullet\bullet\bullet\, p_i^{k\_i} \bullet\bullet\bullet\, p_j^{k\_j} \bullet\bullet\bullet\, p_m^{k\_m} \quad (3)$$

$m \in \mathbf{N}$, where $m \geq 1$ is assumed in Section 2 and $m \geq 2$ assumed in Section 3, p_i prime numbers, $p_i \neq p_j, \forall i \neq j$, $k_i \in \mathbf{N}$.

In this section let us suppose that only one factor of this prime factorization, namely $p_i^{k\_i}$, is available to us.

Theorem: Let $p_i^{k\_i}$ be one factor of the prime factorization (cf. eq. (3)) and k_i=1. Then $\sqrt[n]{x} \notin$ **Q,** $\forall n \in \mathbf{N}_{\geq 2}$.

Proof: k_i=1 implies that the set CD(x) of common divisors (larger than 1) of $\{k_1, k_2, \ldots k_i \ldots, k_m\}$ is empty which proves the assertion. □

Corollary: If m=1 then $x=p^1=p$, p representing a prime number. Evidently, $\sqrt[n]{p} \notin$ **Q,** $\forall n \in \mathbf{N}_{\geq 2}$.

Remark: This is a simple proof of the fact that $\sqrt[n]{p} \notin$ **Q,** $\forall n \in \mathbf{N}_{\geq 2}$ for all prime numbers p.

Another theorem, which covers an interesting case in which knowledge is available with respect to (at least) one factor of the prime factorization of x, is the following one:

Theorem: If, in the prime factorization of x, there exists (at least) one $i \in \{1, 2, \ldots, m\}$ such that k_i is an odd number this implies that for all $n \in$ **N**, n being an even number: $\sqrt[n]{x} \notin$ **Q**.

Proof: If n is an even natural number then $\exists\, \nu \in$ **N**: $n = 2 \bullet \nu \Rightarrow$ it does not exist $\mu \in$ **N** such that $n \bullet \mu$ = k_i because $n \bullet \mu = 2 \bullet \nu \bullet \mu$ is an even number and k_i is assumed to be odd. So, k_i cannot be an integer multiple of n, which would be a necessary condition for $\sqrt[n]{x} \in$ **Q**. □

Remark: If we consider the special case m=1 of the theorem, this implies that $x=p^k$, p being a prime and k an odd natural number. A direct consequence of setting ν=1 in the theorem is that, then, $\sqrt{x} \notin$ **Q**. So, we obtain a very simple sufficient condition to identify natural numbers the square roots of which are irrational.

3. Sufficient Conditions Assuming two Factors of the Prime Factorization of *x*∈N are Available

In this section let us suppose that two factors of the prime factorization of x, cf. eq. (3), namely $p_i^{k\_i}$ and $p_j^{k\_j}$, are available to us, $i \neq j$.

Theorem: Assume that k_i (resp. k_i) and k_j (resp. k_j) do not share any common divisor (larger than 1). Then $\sqrt[n]{x} \notin$ **Q**, $\forall n \in \mathbf{N}_{\geq 2}$.

Proof: Let CD $(\{k_i, k_j\})$ denote the set of common divisors of k_i and k_j. Then, as assumed, CD $(\{k_i, k_j\}) = \emptyset$. As CD(x) $\subseteq$ CD $(\{k_i, k_j\})$ this implies CD (x) = $\emptyset$ and, therefore, $\sqrt[n]{x} \notin$ **Q,** $\forall n \in \mathbf{N}_{\geq 2}$. □

Remark: A special case of this theorem is: k_i represents a prime number and k_j is not an integer multiple of k_i.

Example: x= 648 = $2^3 \bullet 3^4$ which could lead to the choice k_i=3 and k_j=4, where 4 is not an integer multiple of 3. So, $\sqrt[n]{648} \notin$ **Q,** $\forall n \in \mathbf{N}_{\geq 2}$. Evidently, the example covers an infinite set of additional numbers $y \in$ **N**, namely all numbers y leading to a prime factorization of

$$y = 2^3 \bullet 3^4 \bullet p_1^{k\_1} \bullet p_2^{k\_2} \bullet\bullet\bullet\, p_r^{k\_r}$$

where p_ν denote arbitrary prime numbers $\forall\ \nu \in \{1, 2, \ldots, r\}$ and $p_\nu \notin \{2, 3\}\ \forall\ \nu$; moreover, $k_\nu \in \mathbf{N}$, k_ν being arbitrary, too, for $\forall\ \nu \in \{1, 2, \ldots, r\}$.

4. Efficient Ways to Determine one or two Appropriate Factors of the Prime Factorization of $x \in$ N (x very large)

I. Determination of One Factor

We will now shortly indicate how simple it typically is – even for extremely large numbers of *x* – to determine one factor of *x*'s factorization possessing the properties as desired by us to apply our sufficient conditions. Let us therefore look at one factor $p_i^{k\_i}$ of *x*'s prime factorization with k_i=1.

Tests to be applied:

- p-TEST: Is x an integer multiple of p?

and

- p^2-TEST: Is x an integer multiple of p^2 ?

If p-TEST is positive and p^2-TEST is negative, then we have already found a factor $p_i^{k\_i}$ of *x*'s prime factorization with k_i=1 which directly allows us to make use of our sufficient conditions for $\sqrt[n]{x} \notin \mathbf{Q}, \forall n \in \mathbf{N}_{\geq 2}$.

Remark: Evidently, both tests are rather trivial for small primes. If necessary, the tests can of course be repeated for higher primes. The cases *p*=2 and *p*=5 can be even answered without any computational support in a matter of seconds. For *p*=2 the p-TEST means to answer the question “Is *x* an even number?” and the p^2-TEST means “are the last two digits of *x* an integer multiple of 4?”. And for *p*=5 the tests are comparably simple, i.e. the p-TEST for *p*=5 means to test whether the last digit of *x* is 0 or 5 AND the p^2-TEST for *p*=5 means to test whether the last two digits of *x*, i.e. (x_1,x_0), satisfy the condition $(x_1,x_0) \in \{(00), (25), (50), (75)\}$.

II. Determination of Two Factors

The primary goal of our test in this second part is – again for extremely large numbers of *x* – to determine two exponents k_r and k_s which are co-prime. To simplify the task let us limit our search for one k_r being odd and one k_s being even. Again, these very elementary and simple tests should most often allow us to quickly obtain two appropriate factors of the prime factorization of *x*.

5. Summary and conclusion

An important advantage resulting from our tests presented is that, for the sufficient conditions to be tested, it is just necessary to have knowledge of a very small part of the prime factorization of *x* (namely only one or two of the corresponding prime factors have to be known to apply the tests). This can be very valuable in cases in which the complete prime factorization may be impossible.

It is evident that already the tests presented in Section 2 are sufficient to prove for more than 40% of all natural numbers that $\sqrt[n]{x} \notin \mathbf{Q}, \forall n \in \mathbf{N}_{\geq 2}$. Therefore, we expect that if we make use of the tests in Section 3, too, then – for significantly more than 50% of all natural numbers $x \in \mathbf{N}_{\geq 2}$ – it can be proven that also $\sqrt[n]{x} \notin \mathbf{Q}$, $\forall n \in \mathbf{N}_{\geq 2}$ holds. To summarize, applying the tests presented in this paper, in most cases already a very rudimentary knowledge of the prime factorization of x allows us to determine with complete certainty whether $\sqrt[n]{x}$ is an irrational number $\forall n \in \mathbf{N}_{\geq 2}$. Evidently, this can be considered as a major step beyond the famous proof by Euclid proving that $\sqrt{2} \notin \mathbf{Q}$ [11].

- **Disclaimer (Artificial intelligence)**
  The author hereby declares that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

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