STABILITY OF ADDITIVE AND A GENERALIZED QUADRATIC FUNCTIONAL EQUATIONS IN BANACH SPACES

Abstract. In this research article, we investigate the Hyers-Ulam stability of additive and generalized quadratic type functional equation

$$f(x+ay)+af(x-y)=f(x-ay)+af(x+y)$$

for $a \in \mathbb{Z} \setminus \{-1,0,1\}$ and $x, y \in X \times X$, where $f : X \times X \longrightarrow Y$, $(X, \|\cdot,\cdot\|)$ is a 2-normed space and $(Y, \|\cdot\|)$ is a Banach space.

1. Introduction

The case of approximately additive functions was solved by D.H. Hyers [9] and generalized by Th.M. Rassias [18]. During the last decades, the stability problems of several functional equations for functions from normed space to Banach space have been extensively investigated by a number of authors, [3, 2, 6, 7, 8, 11, 13, 18]. The terminology generalized Hyers-Ulam stability originates from these historical backgrounds.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is related to a symmetric biadditive function [1, 20] . It is well known that a function f is a solution of (1.1) if and only if there exists a unique symmetric biadditive function B such that $f(x)=B(x,x)$, for each x (see [1]). The biadditive function B is given by

$$B(x,y)=\frac{1}{4}(f(x+y)-f(x-y)). \tag{1.2}$$

Thus, we call the Equation (1.1) quadratic functional equation and every solution of the quadratic Equation (1.1) is said to be a quadratic function.

A stability problem for the quadratic functional equation (1.1) was solved by a lot of authors [4, 10]. Further, Jun and Lee [14] proved the generalized Hyers-Ulam stability of the pexiderized quadratic Equation (1.1).

In this paper, we investigate the Hyers-Ulam Stability of the following functional equations,

$$f(x+2y)+2f(x-y)=f(x-2y)+2f(x+y), \tag{1.3}$$

$$f(x+ay)+af(x-y)=f(x-ay)+af(x+y), \tag{1.4}$$

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for any fixed integer a with $a \neq -1, 0, 1$, introduced by Jun and Kim [12], for $f : X \times X \longrightarrow Y$, where $(X, \|\cdot,\cdot\|)$ is a 2-normed space and $(Y, \|\cdot\|)$ is a Banach space.

Theorem 1.1. [12] *(i) A function* $f : E_1 \longrightarrow E_2$ *satisfies the functional Equation* (1.3) *if and only if (ii)* $f : E_1 \longrightarrow E_2$ *satisfies the functional Equation* (1.4).
Further more, every solution of functional Equations (1.3) *and* (1.4) *has the form* $f(x) = B(x, x) + A(x) + f(0)$, *for each* $x \in E_1$, *where* $B : E_1 \times E_2 \longrightarrow E_2$ *is symmetric biadditive, and* $A : E_1 \longrightarrow E_2$ *is additive.*

In the 1960s, S. Gähler introduced the concept of linear 2-normed spaces.

Definition 1.2. *Let* X *be a linear space over* $\mathbb{R}$ *with* $\dim X > 1$ *and let* $\|\cdot,\cdot\| : X \times X \longrightarrow \mathbb{R}$ *be a function satisfying the following properties:*

(1) $\|x, y\| = 0$ *if and only if* x *and* y *are linearly dependent,*
(2) $\|x, y\| = \|y, x\|$,
(3) $\|ax, y\| = |a|\|x, y\|$,
(4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$

for each $x, y, z \in X$ *and* $a \in \mathbb{R}$. *Then the function* $\|\cdot,\cdot\|$ *is called a* 2*-norm on* X *and* $(X, \|\cdot,\cdot\|)$ *is called a* 2*-normed space..*

We introduce a basic property of 2-normed spaces as follows. Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space, $x \in X$ and $\|x, y\| = 0$ for each $y \in X$. Suppose $x \neq 0$, since $\dim X > 1$, choose $y \in X$ such that $\{x, y\}$ is linearly independent so we have $\|x, y\| \neq 0$, which is a contradiction. Therefore, we have the following lemma.

Lemma 1.3. *Let* $(X, \|\cdot,\cdot\|)$ *be a* 2*-normed space. If* $x \in X$ *and* $\|x, y\| = 0$, *for each* $y \in X$, *then* $x = 0$.

Remark 1.4. *Let* $(X, \|\cdot,\cdot\|)$ *be a* 2*-normed space. Note that the conditions* (2) *and* (4) *imply that*

$$\|x + y, z\| \leq \|x, z\| + \|y, z\|$$

for each $x, y, z \in X$. *Putting* $w = x + y$, *we get* $\|w, z\| \leq \|x, z\| + \|w - x, z\|$ *for each* $x, y, z \in X$. *So* $\|w, z\| - \|x, z\| \leq \|w - x, z\|$ *for each* $x, z, w \in X$. *Replacing* w *by* x *and* x *by* w *in the above inequality, we get* $\|x, z\| - \|w, z\| \leq \|x - w, z\|$ *for each* $x, z, w \in X$. *Thus, we have*

$$|\|x, z\| - \|y, z\|| \leq \|x - y, z\| \tag{1.5}$$

for each $x, y, z \in X$. *Hence the function* $x \to \|x, y\|$ *is continuous from* X *into* $\mathbb{R}$ *for* $y \in X$.

In the $1960s$, S. Gähler and A. White introduced the concept of 2-Banach space.

Definition 1.5. *A sequence* $\{y_n\}$ *in a* 2*-normed space* X *is called a* 2*-Cauchy sequence if*

$$\lim_{m,n\to\infty} \|y_n - y_m, z\| = 0$$

for each $z \in X$

Definition 1.6. *A sequence* $\{z_n\}$ *in a* 2*-normed space* X *is called a* 2*-convergent sequence if there is an* $z \in X$ *such that*

$$\lim_{n\to\infty} \|z_n - z, w\| = 0$$

for each $w \in X$. *If* $\{z_n\}$ *converges to* z, *we write* $\lim_{n\to\infty} z_n = z$.

Definition 1.7. *A* 2*-normed space* $(X, \|\cdot,\cdot\|)$ *is a* 2*-Banach space if every* 2*-Cauchy sequence in* X *is* 2*-convergent in* X.

Following shows that $\|\cdot,\cdot\|$ is continuous in each component.

Lemma 1.8. *For a convergent sequence* $\{z_n\}$ *in a* 2*-normed space* X,

$$\lim_{n\to\infty} \|z_n, y\| = \|\lim_{n\to\infty} z_n, y\|$$

for each $y \in X$.

Proof. Since $\{z_n\}$ is a 2-convergent sequence in the 2-normed space X, there is an $x \in X$ such that $\lim_{n\to\infty} \|z_n - x, y\| = 0$ for each $y \in X$. By (1.5), we have

$$\lim_{n\to\infty} |\|z_n, y\| - \|x, y\|| \le \lim_{n\to\infty} \|z_n - x, y\| = 0$$

for each $y \in X$. Hence

$$\lim_{n\to\infty} \|z_n, y\| = \|x, y\| = \|\lim_{n\to\infty} z_n, y\|$$

for each $y \in X$. □

2. STABILITY OF A FUNCTIONAL EQUATION FOR FUNCTIONS $f : X \times X \longrightarrow Y$, WHERE $(X, \|\cdot,\cdot\|)$ IS A 2-NORMED SPACE AND $(Y, \|\cdot\|)$ IS A BANACH SPACE.

Throughout this section, $(X, \|\cdot,\cdot\|)$ is a 2-normed space and $(Y, \|\cdot\|)$ is a Banach space. For a function $f : X \times X \longrightarrow Y$, we consider the functional equation

$$\begin{aligned} f(x_1 + ay_1, x_2 + ay_2) + af(x_1 - y_1, x_2 - y_2) =& f(x_1 - ay_1, x_2 - ay_2) \\ & + af(x_1 + y_1, x_2 + y_2) \end{aligned} \tag{2.1}$$

Define $D_f : X \times X \times X \times X \longrightarrow Y$ by

$$\begin{aligned} D_f\big((x_1, x_2), (y_1, y_2)\big) =& f(x_1 + ay_1, x_2 + ay_2) + af(x_1 - y_1, x_2 - y_2) \\ & - f(x_1 - ay_1, x_2 - ay_2) - af(x_1 + y_1, x_2 + y_2) \end{aligned}$$

for each $x_1, x_2, y_1, y_2 \in X$, $a \neq -1, 0, 1$.

Theorem 2.1. *Let* $\varepsilon \ge 0$, $0 < p < 1$ *and* $f : X \times X \longrightarrow Y$ *be a mapping satisfying*

$$\big\|D_f\big((x_1, x_2), (y_1, y_2)\big)\big\| \le \varepsilon(\|x_1, x_2\|^p + \|y_1, y_2\|^p) \tag{2.2}$$

for each $x, y, z \in X$. *Then there exists a unique quadratic mapping* $Q : X \longrightarrow X$ *which satisfies* (2.1) *and*

$$\Big\|\frac{f(x_1, x_2) + f(-x_1, -x_2)}{2} - Q(x_1, x_2) - f(0, 0)\Big\| \le \varepsilon\|x_1, x_2\|^p \frac{|a|^{2p} + 2|a| + 1}{2^{2p}(|a|^2 - |a|^{2p})} \tag{2.3}$$

for each $x_1, x_2, y_1, y_2 \in X$.

Proof. Let $f_1 : X \times X \longrightarrow Y$ be a function defined by

$$f_1(x_1, x_2) = \frac{1}{2}[f(x_1, x_2) + f(-x_1, x_2)] - f(0, 0) \tag{2.4}$$

for each $x_1, x_2, y_1, y_2 \in X$. Then $f_1(0, 0) = 0$ and $f_1(x_1, x_2) = f_1(-x_1, -x_2)$, for each $x_1, x_2, y_1, y_2 \in X$. Also

$$\big\|D_{f_1}\big((x_1, x_2), (y_1, y_2)\big)\big\| \le \varepsilon(\|x_1, x_2\|^p + \|y_1, y_2\|^p) \tag{2.5}$$

for each $x_1, x_2, y_1, y_2 \in X$. Letting $(y_1, y_2) = (x_1, x_2)$ in (2.2), we get

$$\|f_1((a+1)x_1, (a+1)x_2) - f_1((a-1)x_1, (a-1)x_2) - af_1(2x_1, 2x_2)\| \leq 2\varepsilon\|x_1, x_2\|^p \tag{2.6}$$

for each $x_1, x_2, y_1, y_2 \in X$. Replacing (x_1, x_2) by (ay_1, ay_2) in (2.6), we get

$$\|f_1(2ay_1, 2ay_2) + af_1((a-1)y_1, (a-1)y_2) - af_1((a+1)y_1, (a+1)y_2)\| \leq \varepsilon(|a|^{2p}+1)\|y_1, y_2\|^p \tag{2.7}$$

for each $y_1, y_2 \in X$. Replacing (y_1, y_2) by (x_1, x_2) in (2.7), we get

$$\begin{aligned} &\|f_1(2ax_1, 2ax_2) + af_1((a-1)x_1, (a-1)x_2) - af_1((a+1)x_1, (a+1)x_2)\| \\ &\leq \varepsilon(|a|^{2p} + 1)\|x_1, x_2\|^p \end{aligned} \tag{2.8}$$

for each $x_1, x_2 \in X$. Multiplying $|a|$ on both sides of (2.7) and adding to (2.8), we get

$$\|f_1(2ax_1, 2ax_2) - a^2 f_1(2x_1, 2x_2)\| \leq 2|a|\varepsilon\|x_1, x_2\|^p + \varepsilon[|a|^{2p} + 1]\|x_1, x_2\|^p \tag{2.9}$$

for each $x_1, x_2 \in X$. Replacing (x_1, x_2) by $(\frac{x_1}{2}, \frac{x_2}{2})$ in (2.9), we get

$$\|f_1(ax_1, ax_2) - a^2 f_1(x_1, x_2), z\| \leq 2 \cdot 2^{-2p}|a|\varepsilon\|x_1, x_2\|^p + \varepsilon 2^{-2p}[|a|^{2p} + 1]\|x_1, x_2\|^p$$

for each $x_1, x_2 \in X$. Therefore

$$\left\|\frac{f_1(ax_1, ax_2)}{a^2} - f_1(x_1, x_2)\right\| \leq \frac{2 \cdot 2^{-2p}\varepsilon}{|a|}\|x_1, x_2\|^p + \varepsilon\frac{2^{-p}}{|a|^2}[|a|^{2p} + 1]\|x_1, x_2\|^p \tag{2.10}$$

for each $x_1, x_2 \in X$. Replacing (x_1, x_2) by (ax_1, ax_2) in (2.10), we get

$$\left\|\frac{f_1(a^2x_1, a^2x_2)}{a^2} - f_1(ax_1, ax_2)\right\| \leq \frac{2 \cdot \varepsilon \cdot 2^{-2p}|a|^{2p}}{|a|}\|x_1, x_2\|^p + \frac{\varepsilon \cdot 2^{-2p}}{|a|^2}\left[|a|^{4p} + |a|^{2p}\right]\|x_1, x_2\|^p \tag{2.11}$$

for each $x_1, x_2 \in X$. By (2.10) and (2.11), we get

$$\begin{aligned} &\left\|\frac{f_1(a^2x_1, a^2x_2)}{a^4} - f_1(x_1, x_2)\right\| \\ &\leq \left\|\frac{f_1(a^2x_1, a^2x_2)}{a^4} - \frac{f_1(ax_1, ax_2)}{a^2}\right\| + \left\|\frac{f_1(ax_1, ax_2)}{a^2} - f_1(x_1, x_2)\right\| \\ &\leq \frac{1}{|a|^2}\left[\frac{2 \cdot \varepsilon \cdot 2^{-2p}|a|^{2p}}{|a|}\|x_1, x_2\|^p + \frac{\varepsilon \cdot 2^{-2p}}{|a|^2}\left[|a|^{4p} + |a|^{2p}\right]\|x_1, x_2\|^p\right] \\ &+ \frac{2 \cdot \varepsilon \cdot 2^{-2p}}{|a|}\|x_1, x_2\|^p + \frac{\varepsilon \cdot 2^{-2p}}{|a|^2}\left[|a|^{2p} + 1\right]\|x_1, x_2\|^p \\ &= \frac{2 \cdot 2^{-2p} \cdot \varepsilon}{|a|}\left[1 + \frac{|a|^{2p}}{|a|^2}\right]\|x_1, x_2\|^p \\ &+ \frac{\varepsilon \cdot 2^{-2p}}{|a|}\left[\left(1 + |a|^{2p}\right) + \left(|a|^{4p} + |a|^{2p}\right)\frac{1}{|a|^2}\right]\|x_1, x_2\|^p \end{aligned}$$

for each $x_1, x_2 \in X$. By using induction on n, we get

(2.12)

$$\begin{aligned}\Big\|\frac{f_1(a^n x_1, a^n x_2)}{a^{2n}} - f_1(x_1, x_2)\Big\| &\le \frac{2\cdot 2^{-2p}\cdot \varepsilon\|x_1, x_2\|^p}{|a|}\sum_{j=0}^{n-1}\frac{|a|^{2pj}}{|a|^{2j}} \\ &+ \frac{\varepsilon\cdot 2^{-2p}}{|a|^2}\sum_{j=0}^{n-1}\big[|a|^{2pj} + |a|^{2p(j+1)}\big]\frac{1}{|a|^{2j}}\|x_1, x_2\|^p \\ &= \frac{2\cdot 2^{-2p}\cdot \varepsilon\|x_1, x_2\|^p}{|a|}\sum_{j=0}^{n-1}|a|^{2pj-2j} \\ &+ \frac{\varepsilon\cdot 2^{-2p}}{|a|^2}\sum_{j=0}^{n-1}\big[|a|^{2pj-2j} + |a|^{2pj-2j+2p}\big]\|x_1, x_2\|^p \\ &= \frac{2\cdot 2^{-2p}\cdot \varepsilon\|x_1, x_2\|^p}{|a|}\Big[\frac{1-|a|^{2(p-1)n}}{1-|a|^{2(p-1)}}\Big] \\ &+ \frac{\varepsilon\cdot 2^{-2p}}{|a|^2}\Big[\frac{1-|a|^{2(p-1)n}}{1-|a|^{2(p-1)}} + \frac{|a|^{2p}\big(1-|a|^{2(p-1)n}\big)}{1-|a|^{2(p-1)}}\Big]\|x_1, x_2\|^p\end{aligned}$$

(2.13)

for each $x_1, x_2 \in X$. For $m, n \in \mathbb{N}$, we have

$$
\begin{aligned}
&\left\| \frac{f_1(a^m x_1, a^m x_2)}{a^{2m}} - \frac{f_1(a^n x_1, a^n x_2)}{a^{2n}} \right\| \\
&= \left\| \frac{f_1(a^{m+n-n} x_1, a^{m+n-n} x_2)}{a^{2(m+n-n)}} - \frac{f_1(a^n x_1, a^n x_2)}{a^{2n}} \right\| \\
&= \frac{1}{|a|^{2n}} \left\| \frac{f_1(a^{m-n} \cdot a^n x_1, a^{m-n} \cdot a^n x_2)}{a^{2(m-n)}} - f_1(a^n x_1, a^n x_2) \right\| \\
&\leq \frac{1}{|a|^{2n}} \frac{2 \cdot 2^{-2p} \cdot \varepsilon}{|a|} \left\| a^n x_1, a^n x_2 \right\|^p \sum_{j=0}^{m-n-1} |a|^{2(p-1)j} \\
&+ \frac{1}{|a|^{2n}} \frac{2^{-2p} \cdot \varepsilon}{|a|^2} \left\| a^n x_1, a^n x_2 \right\|^p \sum_{j=0}^{m-n-1} \left[|a|^{2(p-1)j} + |a|^{2(p-1)j+2p} \right] \\
&= \frac{2 \cdot 2^{-2p} \cdot \varepsilon}{|a|} |a|^{2(p-1)n} \|x_1, x_2\|^p \sum_{j=0}^{m-n-1} |a|^{2(p-1)j} \\
&+ \frac{2^{-2p} \cdot \varepsilon}{|a|^2} |a|^{2(p-1)n} \|x_1, x_2\|^p \sum_{j=0}^{m-n-1} \left[|a|^{2(p-1)j} + |a|^{2(p-1)j+2p} \right] \\
&= \frac{2 \cdot 2^{-2p} \cdot \varepsilon}{|a|} \|x_1, x_2\|^p \sum_{j=0}^{m-n-1} |a|^{2(p-1)(n+j)} \\
&+ \frac{2^{-2p} \cdot \varepsilon}{|a|^2} \|x_1, x_2\|^p \sum_{j=0}^{m-n-1} \left[|a|^{2(p-1)(n+j)} + |a|^{2(p-1)(n+j)+2p} \right] \\
&= \frac{2 \cdot 2^{-2p} \cdot \varepsilon}{|a|} \|x_1, x_2\|^p \frac{|a|^{2(p-1)n} \left(1 - |a|^{2(p-1)(m-n)}\right)}{1 - |a|^{2(p-1)}} \\
&+ \frac{2^{-2p} \cdot \varepsilon}{|a|^2} \|x_1, x_2\|^p \\
&\left[|a|^{2(p-1)n} \left(\frac{1 - |a|^{2(p-1)(m-n)}}{1 - |a|^{2(p-1)}} \right) + \frac{|a|^{2(p-1)n+p} \left(1 - |a|^{2(p-1)(m-n)}\right)}{1 - |a|^{2(p-1)}} \right] \\
&\longrightarrow 0 \ as \ n \to \infty, \ 0 < p < 1
\end{aligned}
$$

for each $x_1, x_2 \in X$. Therefore $\left\{ \frac{f_1(a^n x_1, a^n x_2)}{a^{2n}} \right\}$ is a Cauchy sequence in Y, for each $x_1, x_2 \in X$. Since Y is a Banach space, $\left\{ \frac{f_1(a^n x_1, a^n x_2)}{a^{2n}} \right\}$ converges, for each $x_1, x_2 \in X$. Define $Q : X \times X \longrightarrow Y$ as

$$
Q(x_1, x_2) := \lim_{n \to \infty} \frac{f_1\big(a^n x_1, a^n x_2\big)}{a^{2n}}
$$

for each $x_1, x_2 \in X$. Also, by (2.12), we have

$$\begin{aligned}
&\lim_{n\to\infty}\Big\|\frac{f_1\big(a^n x_1, a^n x_2\big)}{a^{2n}} - f_1(x)\Big\| \\
&\leq \frac{2\cdot 2^{-2p}\cdot\varepsilon}{|a|}\|x_1, x_2\|^p \frac{1}{1-|a|^{2(p-1)}} \\
&+ \frac{2^{-2p}\cdot\varepsilon}{|a|^2}\|x_1, x_2\|^p \Big[\frac{1}{1-|a|^{2(p-1)}} + \frac{|a|^{2p}}{1-|a|^{2(p-1)}}\Big] \\
&= 2\cdot 2^{-2p}\cdot\varepsilon\|x_1, x_2\|^p \frac{|a|}{|a|^2-|a|^{2p}} + 2^{-2p}\cdot\varepsilon\|x_1, x_2\|^p \Big[\frac{1}{|a|^2-|a|^{2p}} + \frac{|a|^{2p}}{|a|^2-|a|^{2p}}\Big] \\
&= 2^{-2p}\cdot\varepsilon\|x_1, x_2\|^p \frac{|a|^{2p}+2|a|+1}{|a|^2-|a|^{2p}}
\end{aligned}$$

for each $x_1, x_2 \in X$. Therefore

$$\Big\|\frac{f_1(x_1, x_2) + f_1(-x_1, -x_2)}{2} - Q(x_1, x_2) - f(0,0)\Big\| \leq \varepsilon\|x_1, x_2\|^p \frac{|a|^{2p}+2|a|+1}{2^{2p}(|a|^2-|a|^{2p})}$$

for each $x_1, x_2 \in X$. Next, we show that $D_Q\big((x_1, x_2), (y_1, y_2)\big) = 0$.

$$\begin{aligned}
\big\|D_Q\big((x_1, x_2), (y_1, y_2)\big)\big\| &= \lim_{n\to\infty}\frac{1}{a^{2n}}\big\|Df_1\big((a^n x_1, a^n x_2), (a^n y_1, a^n y_2)\big)\big\| \\
&\leq \lim_{n\to\infty}\frac{\varepsilon}{a^{2n}}\big[\|a^n x_1, a^n x_2\|^p + \|a^n y_1, a^n y_2\|^p\big] \\
&= \lim_{n\to\infty}\varepsilon|a|^{2(p-1)n}\big[\|x_1, x_2\|^p + \|y_1, y_2\|^p\big] \\
&= 0
\end{aligned}$$

for each $x_1, x_2, y_1, y_2 \in X$. Therefore $D_Q\big((x_1, x_2), (y_1, y_2)\big) = 0$. Next, we prove the uniqueness of Q. Let $Q' : X \times X \longrightarrow Y$ be another quadratic function satisfying (2.1) and (2.3). Since Q, Q' are quadratic,

$$\begin{aligned}
&\|Q(x_1, x_2) - Q'(x_1, x_2)\| \\
&= \frac{1}{a^{2n}}\big\|Q(a^n x_1, a^n x_2) - Q'(a^n x_1, a^n x_2)\big\| \\
&\leq \frac{1}{a^{2n}}\big[\big\|Q(a^n x_1, a^n x_2) - f_1(a^n x_1, a^n x_2)\big\| + \big\|f_1(a^n x_1, a^n x_2) - Q'(a^n x_1, a^n x_2)\big\|\big] \\
&= \frac{2}{a^{2n}}\varepsilon\|a^n x_1, a^n x_2\|^p \frac{|a|^{2p}+2|a|+1}{2^{2p}\big(|a|^2-|a|^{2p}\big)} \\
&= 2|a|^{2(p-1)n}\varepsilon\|x_1, x_2\|^p \frac{|a|^{2p}+2|a|+1}{2^{2p}\big(|a|^2-|a|^{2p}\big)} \\
&\longrightarrow 0 \ as \ n \to \infty
\end{aligned}$$

for each $x_1, x_2 \in X$. Therefore $Q(x_1, x_2) = Q'(x_1, x_2)$, for each $x_1, x_2 \in X$. □

Theorem 2.2. *Let $\varepsilon \geq 0, p > 1$ and let $f : X \times X \longrightarrow Y$ be a function satisfying*

$$\big\|D_f\big((x_1, x_2), (y_1, y_2)\big)\big\| \leq \varepsilon\big[\|x_1, x_2\|^p + \|y_1, y_2\|^p\big] \tag{2.14}$$

for each $x_1, x_2, y_1, y_2 \in X$. *Then there exists a unique quadratic function* $Q : X \times X \longrightarrow Y$ *satisfying* (2.1) *and*

$$\Big\| \frac{f(x_1,x_2)+f(-x_1,-x_2)}{2} - Q(x_1,x_2) - f(0,0) \Big\| \leq \varepsilon \|x_1,x_2\|^p \frac{|a|^{2p}+2|a|+1}{2^{2p}\big(|a|^2-|a|^{2p}\big)} \tag{2.15}$$

for each $x_1, x_2 \in X$.

Proof. By (2.9) of Theorem 2.1, we have

$$\big\| f_1(2ax_1, 2ax_2) - a^2 f_1(2x_1, 2x_2) \big\| \leq \big[2|a| + |a|^{2p} + 1\big] \varepsilon \|x_1, x_2\|^p \tag{2.16}$$

for each $x_1, x_2 \in X$. Replacing x_1, x_2 by $\left(\frac{x_1}{2a}, \frac{x_2}{2a}\right)$ in (2.16), we get

$$\Big\| f_1(x_1,x_2) - a^2 f_1\Big(\frac{x_1}{a}, \frac{x_2}{a}\Big) \Big\| \leq 2|a|\varepsilon 2^{-2p}\|x_1,x_2\|^p + \varepsilon \cdot 2^{-2p}\big[|a|^{-2p}+1\big]\|x_1,x_2\|^p \tag{2.17}$$

for each $x_1, x_2 \in X$. Replacing x_1, x_2 by $\left(\frac{x_1}{a}, \frac{x_2}{a}\right)$ in (2.17), we get

$$\Big\| f_1\Big(\frac{x_1}{a}, \frac{x_2}{a}\Big) - a^2 f_1\Big(\frac{x_1}{a^2}, \frac{x_2}{a^2}\Big) \Big\| \leq 2|a|\varepsilon 2^{-2p}|a|^{-4p}\|x_1,x_2\|^p + \varepsilon \cdot 2^{-2p}\big[|a|^{-2p}+|a|^{-4p}\big]\|x_1,x_2\|^p \tag{2.18}$$

for each $x_1, x_2 \in X$. Now, by (2.17) and (2.18), we get

$$\begin{aligned}
&\Big\| f_1(x_1,x_2) - a^4 f_1\Big(\frac{x_1}{a^2}, \frac{x_2}{a^2}\Big) \Big\| \\
&\leq \Big\| f_1(x_1,x_2) - a^2 f_1\Big(\frac{x_1}{a}, \frac{x_2}{a}\Big) \Big\| \\
&+ |a|^2 \Big\| f_1\Big(\frac{x_1}{a}, \frac{x_2}{a}\Big) - a^2 f_1\Big(\frac{x_1}{a^2}, \frac{x_2}{a^2}\Big) \Big\| \\
&\leq 2|a|\varepsilon 2^{-2p}|a|^{-2p}\|x_1,x_2\|^p + 2^{-2p}\varepsilon\big[|a|^{-2p+1}\big]\|x_1,x_2\|^p \\
&+ |a|^2 \cdot 2|a|\varepsilon 2^{-2p}|a|^{-4p}\|x_1,x_2\|^p + |a|^2 2^{-2p}\varepsilon\big[|a|^{-4p}+|a|^{-2p}\big]\|x_1,x_2\|^p \\
&= 2|a|\varepsilon 2^{-2p}\|x_1,x_2\|^p\big[|a|^{-2p}+|a|^{-4p}\cdot|a|^2\big] \\
&+ 2^{-2p}\varepsilon\|x_1,x_2\|^p\big[\big(1+|a|^{-2p}\big)+\big(|a|^{-2p}+|a|^{-4p}\big)|a|^2\big]
\end{aligned}$$

for each $x_1, x_2 \in X$. By using induction on n, we get

$$
\begin{aligned}
&\Big\| f_1(x_1,x_2) - a^{2n} f_1\Big(\frac{x_1}{a^n}, \frac{x_2}{a^n}\Big)\Big\| \\
&\le 2|a|\varepsilon 2^{-2p}\|x_1,x_2\|^p \sum_{j=0}^{n-1} |a|^{-2p(j+1)} \cdot |a|^{2j} \\
&+ 2^{-2p}\varepsilon\|x_1,x_2\|^p \sum_{j=0}^{n-1} \big[|a|^{-2pj} + |a|^{-2p(j+1)}\big]|a|^{2j} \\
&= 2|a|\varepsilon 2^{-2p}\|x_1,x_2\|^p \sum_{j=0}^{n-1} |a|^{2(-p+1)j-2p} \\
&+ 2^{-2p}\varepsilon\|x_1,x_2\|^p \sum_{j=0}^{n-1} \big[|a|^{2(-p+1)j} + |a|^{2(-p+1)j-2p}\big] \\
&= 2|a|\varepsilon 2^{-2p}\|x_1,x_2\|^p \frac{|a|^{-2p}\big(1-|a|^{2(-p+1)n}\big)}{1-|a|^{2(-p+1)}} \\
&+ 2^{-2p}\varepsilon\|x_1,x_2\|^p \Big[\frac{1-|a|^{2(-p+1)n}}{1-|a|^{2(-p+1)}} + \frac{|a|^{-2p}\big(1-|a|^{2(-p+1)n}\big)}{1-|a|^{2(-p+1)}}\Big] \qquad (2.19)
\end{aligned}
$$

for each $x_1, x_2 \in X$. For $m, n \in \mathbb{N}$, we have

$$\begin{aligned}
&\Big\| a^{2m} f_1\Big(\frac{x_1}{a^m}, \frac{x_2}{a^m}\Big) - a^{2n} f_1\Big(\frac{x_1}{a^n}, \frac{x_2}{a^n}\Big)\Big\| \\
&= \Big\| a^{2(m+n-n)} f_1\Big(\frac{x_1}{a^{m+n-n}}, \frac{x_2}{a^{m+n-n}}\Big) - a^{2n} f_1\Big(\frac{x_1}{a^n}, \frac{x_2}{a^n}\Big)\Big\| \\
&= |a|^{2n}\Big\| a^{2(m-n)} f_1\Big(\frac{x_1}{a^{m-n}\cdot a^n}, \frac{x_2}{a^{m-n}\cdot a^n}\Big) - f_1\Big(\frac{x_1}{a^n}, \frac{x_2}{a^n}\Big)\Big\| \\
&\leq |a|^{2n} \cdot 2|a|\varepsilon 2^{-2p}\Big\|\frac{x_1}{a^n}, \frac{x_2}{a^n}\Big\|^p \sum_{j=0}^{m-n-1} |a|^{2(-p+1)j-2p} \\
&+ |a|^{2n} 2^{-2p} \cdot \varepsilon \Big\|\frac{x_1}{a^n}, \frac{x_2}{a^n}\Big\|^p \sum_{j=0}^{m-n-1} \big[|a|^{2(-p+1)j} + |a|^{2(-p+1)j-2p}\big] \\
&= 2|a|\varepsilon 2^{-2p}\|x_1, x_2\|^p |a|^{2(-p+1)n} \sum_{j=0}^{m-n-1} |a|^{2(-p+1)j-2p} \\
&+ \varepsilon 2^{-2p}\|x_1, x_2\|^p |a|^{2(-p+1)n} \sum_{j=0}^{m-n-1} \big[|a|^{2(-p+1)j} + |a|^{2(-p+1)j-2p}\big] \\
&= 2|a|\varepsilon 2^{-2p}\|x_1, x_2\|^p \sum_{j=0}^{m-n-1} |a|^{2(-p+1)(n+j)-2p} \\
&+ \varepsilon 2^{-2p}\|x_1, x_2\|^p \sum_{j=0}^{m-n-1} \big[|a|^{2(-p+1)(n+j)} + |a|^{2(-p+1)(n+j)-2p}\big] \\
&= 2|a|\varepsilon 2^{-2p}\|x_1, x_2\|^p \frac{|a|^{2(-p+1)n-p}\big(1 - |a|^{2(-p+1)(m-n)}\big)}{1 - |a|^{2(-p+1)}} \\
&+ \varepsilon 2^{-2p}\|x_1, x_2\|^p \\
&\Big[\frac{|a|^{2(-p+1)n}\big(1 - |a|^{2(-p+1)(m-n)}\big)}{1 - |a|^{2(-p+1)}} + \frac{|a|^{2(-p+1)n-p}\big(1 - |a|^{2(-p+1)(m-n)}\big)}{1 - |a|^{2(-p+1)}}\Big] \\
&\longrightarrow 0 \; as \; n \to \infty, \; p > 1
\end{aligned}$$

for each $x_1, x_2 \in X$. Therefore $\Big\{a^{2n} f_1\Big(\frac{x_1}{a^n}, \frac{x_2}{a^n}\Big)\Big\}$ is a Cauchy sequence in Y, for each $x_1, x_2 \in X$. Since Y is a Banach space, $\Big\{a^{2n} f_1\Big(\frac{x_1}{a^n}, \frac{x_2}{a^n}\Big)\Big\}$ converges, for each $x_1, x_2 \in X$. Define $Q : X \times X \longrightarrow Y$ as

$$Q(x_1, x_2) := \lim_{n\to\infty} a^{2n} f_1\Big(\frac{x_1}{a^n}, \frac{x_2}{a^n}\Big)$$

for each $x_1, x_2 \in X$. By (2.19), we get

$$\begin{aligned}
\lim_{n\to\infty} \Big\| f_1(x_1, x_2) - a^{2n} f_1\Big(\frac{x_1}{a^n}, \frac{x_2}{a^n}\Big)\Big\| &\leq 2|a|\varepsilon 2^{-2p}\|x_1, x_2\|^p \frac{|a|^{-2p}}{1 - |a|^{2(-p+1)}} \\
&+ \varepsilon 2^{-2p}\|x_1, x_2\|^p \Big[\frac{1}{1 - |a|^{2(-p+1)}} + \frac{|a|^{-2p}}{1 - |a|^{2(-p+1)}}\Big] \\
&= \varepsilon 2^{-2p}\|x_1, x_2\|^p \frac{|a|^{2p} + 2|a| + 1}{|a|^{2p} - |a|^2}
\end{aligned}$$

for each $x_1, x_2 \in X$. Therefore

$$\left\| \frac{f(x_1,x_2)+f(-x_1,-x_2)}{2} - Q(x_1,x_2) - f(0,0) \right\| \leq \varepsilon \|x_1,x_2\|^p \frac{|a|^{2p}+2|a|+1}{2^{2p}\big(|a|^{2p}-|a|^2\big)}$$

for each $x_1, x_2 \in X$. The further part of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.3. *Let $\varepsilon \geq 0, 0 < p < \frac{1}{2}$ and let $f : X \times X \longrightarrow Y$ be a function satisfying*

$$\big\|D_f\big((x_1,x_2),(y_1,y_2)\big)\big\| \leq \varepsilon\big[\|x_1,x_2\|^p + \|y_1,y_2\|^p\big] \tag{2.20}$$

for each $x_1, x_2, y_1, y_2 \in X$. Then there exists a unique additive function $A : X \times X \longrightarrow Y$ satisfying (2.1) *and*

$$\left\| \frac{f(x_1,x_2)+f(-x_1,-x_2)}{2} - A(x_1,x_2) \right\| \leq \frac{\varepsilon\|x_1,x_2\|^p}{2(|a|-|a|^{2p})} \tag{2.21}$$

for each $x_1, x_2 \in X$.

Proof. Let $f_2 : X \times X \longrightarrow Y$ be a function defined by $f_2(x_1,x_2) := \frac{1}{2}\big[f(x_1,x_2) - f(-x_1,-x_2)\big]$, for each $x_1, x_2 \in X$. Then $f_2(0,0) = 0$.
$f_2(-x_1,-x_2) = -f_2(x_1,x_2)$. Also

$$\big\|Df_2\big((x_1,x_2),(y_1,y_2)\big)\big\| \leq \varepsilon\big[\|x_1,x_2\|^p + \|y_1,y_2\|^p\big] \tag{2.22}$$

for each $x_1, x_2, y_1, y_2 \in X$. Putting $(x_1,x_2) = (0,0)$ in (2.22), we get

$$\big\|f_2(ay_1,ay_2) + f_2(ay_1,ay_2) - 2af_2(y_1,y_2)\big\| \leq \varepsilon\|y_1,y_2\|^p$$

for each $y_1, y_2 \in X$. Therefore

$$\big\|2f_2(ay_1,ay_2) - 2af_2(y_1,y_2)\big\| \leq \varepsilon\|y_1,y_2\|^p$$

for each $y_1, y_2 \in X$. Therefore

$$\big\|f_2(ay_1,ay_2) - af_2(y_1,y_2)\big\| \leq \frac{\varepsilon\|y_1,y_2\|^p}{2} \tag{2.23}$$

for each $y_1, y_2 \in X$. Replacing (y_1,y_2) by (x_1,x_2) in (2.23), we get

$$\big\|f_2(ax_1,ax_2) - af_2(x_1,x_2)\big\| \leq \frac{\varepsilon\|x_1,x_2\|^p}{2} \tag{2.24}$$

for each $x_1, x_2 \in X$. Therefore

$$\left\|\frac{f_2(ax_1,ax_2)}{a} - f_2(x_1,x_2)\right\| \leq \frac{\varepsilon\|x_1,x_2\|^p}{2|a|} \tag{2.25}$$

for each $x_1, x_2 \in X$. Replacing (x_1,x_2) by (ax_1,ax_2) in (2.25), we get

$$\left\|\frac{f_2(a^2x_1,a^2x_2)}{a} - f_2(ax_1,ax_2)\right\| \leq \frac{\varepsilon|a|^{2p}}{2|a|}\|x_1,x_2\|^p \tag{2.26}$$

for each $x_1, x_2 \in X$. By (2.25) and (2.26), we get

$$\begin{aligned}\Big\|\frac{f_2(a^2x_1,a^2x_2)}{a^2}-f_2(x_1,x_2)\Big\| &\le \Big\|\frac{f_2(a^2x_1,a^2x_2)}{a^2}-\frac{f_2(ax_1,ax_2)}{a}\Big\|\\ &+\Big\|\frac{f_2(ax_1,ax_2)}{a}-f_2(x_1,x_2)\Big\|\\ &\le \frac{\varepsilon|a|^{2p}}{2|a|^2}\|x_1,x_2\|^p+\frac{\varepsilon}{2|a|}\|x_1,x_2\|^p\\ &=\frac{\varepsilon}{2|a|}\Big[1+\frac{|a|^{2p}}{|a|}\Big]\|x_1,x_2\|^p\end{aligned}$$

for each $x_1, x_2 \in X$. By using induction on n, we get

$$\begin{aligned}\Big\|\frac{f_2(a^nx_1,a^nx_2)}{a^n}-f_2(x_1,x_2)\Big\| &\le \frac{\varepsilon\|x_1,x_2\|^p}{2|a|}\sum_{j=0}^{n-1}\frac{|a|^{2pj}}{|a|^j}\\ &=\frac{\varepsilon\|x_1,x_2\|^p}{2|a|}\sum_{j=0}^{n-1}|a|^{(2p-1)j}\\ &=\frac{\varepsilon\|x_1,x_2\|^p}{2|a|}\Big[\frac{1-|a|^{(2p-1)n}}{1-|a|^{(2p-1)}}\Big] \qquad (2.27)\end{aligned}$$

for each $x_1, x_2 \in X$. For $m, n \in \mathbb{N}$,

$$\begin{aligned}&\Big\|\frac{f_2(a^mx_1,a^mx_2)}{a^m}-\frac{f_2(a^nx_1,a^nx_2)}{a^n}\Big\|\\ &=\Big\|\frac{f_2(a^{m+n-n}x_1,a^{m+n-n}x_2)}{a^{m+n-n}}-\frac{f_2(a^nx_1,a^nx_2)}{a^n}\Big\|\\ &=\frac{1}{|a|^n}\Big\|\frac{f_2(a^{m-n}\cdot a^nx_1,a^{m-n}\cdot a^nx_2)}{a^{m-n}}-f_2(a^nx_1,a^nx_2)\Big\|\\ &\le\frac{1}{|a|^n}\frac{\varepsilon}{2|a|}\big\|a^nx_1,a^nx_2\big\|^p\sum_{j=0}^{m-n-1}|a|^{(2p-1)j}\\ &=\frac{\varepsilon}{2|a|}\|x_1,x_2\|^p|a|^{(2p-1)n}\sum_{j=0}^{m-n-1}|a|^{(2p-1)j}\\ &=\frac{\varepsilon}{2|a|}\|x_1,x_2\|^p\sum_{j=0}^{m-n-1}|a|^{(2p-1)(n+j)}\\ &=\frac{\varepsilon\|x_1,x_2\|^p}{2|a|}\frac{|a|^{(2p-1)n}\big(1-|a|^{(2p-1)(m-n)}\big)}{1-|a|^{(2p-1)}}\\ &\longrightarrow 0\ as\ n\to\infty,\ 0<p<\frac{1}{2}\end{aligned}$$

for each $x_1, x_2 \in X$. Therefore $\Big\{\frac{f_2(a^nx_1,a^nx_2)}{a^n}\Big\}$ is a Cauchy sequence in Y, for each $x_1, x_2 \in X$. Since Y is a Banach space, $\Big\{\frac{f_2(a^nx_1,a^nx_2)}{a^n}\Big\}$ converges, for each $x_1, x_2 \in X$. Define $Q : X \times X \longrightarrow Y$ as

$$A(x_1,x_2):=\lim_{n\to\infty}\frac{f_2(a^nx_1,a^nx_2)}{a^n}$$

for each $x_1, x_2 \in X$. By (2.27), we get

$$\lim_{n\to\infty} \Big\| \frac{f_2(a^n x_1, a^n x_2)}{a^n} - f_2(x_1, x_2) \Big\| \le \frac{\varepsilon \|x_1, x_2\|^p}{2|a|} \cdot \frac{1}{1 - |a|^{p-1}}$$

for each $x_1, x_2 \in X$. Therefore

$$\Big\| \frac{f(x_1, x_2) - f(-x_1, -x_2)}{2} - A(x_1, x_2) \Big\| \le \frac{\varepsilon \|x_1, x_2\|^p}{2\big(|a| - |a|^{2p}\big)}$$

for each $x_1, x_2 \in X$. Next, we show that A satisfies (2.1).

$$\begin{aligned} \big\|D_A\big((x_1, x_2), (y_1, y_2)\big)\big\| &= \lim_{n\to\infty} \frac{1}{|a|^n} \big\| Df_2\big((a^n x_1, a^n x_2), (a^n y_1, a^n y_2)\big) \big\| \\ &\le \lim_{n\to\infty} \frac{\varepsilon}{|a|^n} \big[\big\|a^n x_1, a^n x_2\big\|^p + \big\|a^n y_1, a^n y_2\big\|^p \big] \\ &= \lim_{n\to\infty} \varepsilon \big[|a|^{(2p-1)n} \|x_1, x_2\|^p + |a|^{(2p-1)n} \|y_1, y_2\|^p \big] \\ &= 0 \end{aligned}$$

for each $x_1, x_2, y_1, y_2 \in X$. Therefore $D_A\big((x_1, x_2), (y_1, y_2)\big) = 0$. So, A satisfies (2.1). Next, we show the uniqueness of A. Let $A' : X \times X \longrightarrow Y$ be another additive function satisfying (2.1) and (2.2). Since A and A' are additive,
$A(a^n x_1, a^n x_2) = |a|^n A(x_1, x_2)$,
$A'(a^n x_1, a^n x_2) = |a|^n A'(x_1, x_2)$, for each $x_1, x_2 \in X$.

$$\begin{aligned} &\|A(x_1, x_2) - A'(x_1, x_2)\| \\ &= \frac{1}{|a|^n} \|A(a^n x_1, a^n x_2) - A'(a^n x_1, a^n x_2)\| \\ &\le \frac{1}{|a|^n} \big[\big\|A(a^n x_1, a^n x_2) - f_2(a^n x_1, a^n x_2)\big\| + \big\|f_2(a^n x_1, a^n x_2) - A'(a^n x_1, a^n x_2)\big\| \big] \\ &\le \frac{1}{|a|^n} \frac{2\varepsilon \big\|a^n x_1, a^n x_2\big\|^p}{2\big(|a| - |a|^{2p}\big)} \\ &= |a|^{(2p-1)n} \frac{\varepsilon \|x_1, x_2\|^p}{|a| - |a|^{2p}} \\ &\longrightarrow 0 \ as \ n \to \infty \end{aligned}$$

for each $x_1, x_2 \in X$. Therefore $A(x_1, x_2) = A'(x_1, x_2)$, for each $x_1, x_2 \in X$. □

Theorem 2.4. *Let $\varepsilon \ge 0, p > \frac{1}{2}$ and let $f : X \times X \longrightarrow Y$ be a function satisfying*

$$\big\|D_f\big((x_1, x_2), (y_1, y_2)\big)\big\| \le \varepsilon \big[\|x_1, x_2\|^p + \|y_1, y_2\|^p \big] \tag{2.28}$$

for each $x_1, x_2, y_1, y_2 \in X$. Then there exists a unique additive function $A : X \times X \longrightarrow Y$ satisfying (2.1) *and*

$$\Big\| \frac{f(x_1, x_2) + f(-x_1, -x_2)}{2} - A(x_1, x_2) \Big\| \le \frac{\varepsilon \|x_1, x_2\|^p}{2\big(|a|^{2p} - |a|\big)} \tag{2.29}$$

for each $x_1, x_2 \in X$.

Proof. By (2.24) of Theorem 2.3, we get

$$\big\| f_2(a x_1, a x_2) - a f_2(x_1, x_2) \big\| \le \frac{\varepsilon \|x_1, x_2\|^p}{2} \tag{2.30}$$

for each $x_1, x_2 \in X$. Replacing (x_1, x_2) by $\left(\frac{x_1}{a}, \frac{x_2}{a}\right)$ in (2.30), we get

$$\left\| f_2(x_1, x_2) - a f_2\left(\frac{x_1}{a}, \frac{x_2}{a}\right) \right\| \le \frac{\varepsilon |a|^{-2p}}{2} \|x_1, x_2\|^p \tag{2.31}$$

for each $x_1, x_2 \in X$. Replacing (x_1, x_2) by $\left(\frac{x_1}{a}, \frac{x_2}{a}\right)$ in (2.31), we get

$$\left\| f_2\left(\frac{x_1}{a}, \frac{x_2}{a}\right) - a f_2\left(\frac{x_1}{a^2}, \frac{x_2}{a^2}\right) \right\| \le \frac{\varepsilon |a|^{-4p}}{2} \|x_1, x_2\|^p \tag{2.32}$$

for each $x_1, x_2 \in X$. By (2.31) and (2.32), we get

$$\begin{aligned} \left\| f_2(x_1, x_2) - a^2 f_2\left(\frac{x_1}{a^2}, \frac{x_2}{a^2}\right) \right\| &\le \left\| f_2(x_1, x_2) - a f_2\left(\frac{x_1}{a}, \frac{x_2}{a}\right) \right\| \\ &\quad + \left\| a f_2\left(\frac{x_1}{a}, \frac{x_2}{a}\right) - a^2 f_2\left(\frac{x_1}{a}, \frac{x_2}{a}\right) \right\| \\ &\le \frac{\varepsilon |a|^{-2p} \|x_1, x_2\|^p}{2} + \frac{\varepsilon |a| |a|^{-4p} \|x_1, x_2\|^p}{2} \\ &= \frac{\varepsilon \|x_1, x_2\|^p}{2} \left[|a|^{-2p} + |a|^{-4p} |a| \right] \end{aligned}$$

for each $x_1, x_2 \in X$. By using induction on n, we get

$$\begin{aligned} \left\| f_2(x_1, x_2) - a^n f_2\left(\frac{x_1}{a^n}, \frac{x_2}{a^n}\right) \right\| &\le \frac{\varepsilon \|x_1, x_2\|^p}{2} \sum_{j=0}^{n-1} |a|^{-2p(j+1)} \cdot |a|^j \\ &= \frac{\varepsilon \|x_1, x_2\|^p}{2} \sum_{j=0}^{n-1} |a|^{(-2p+1)j-2p} \\ &= \frac{\varepsilon \|x_1, x_2\|^p}{2} \frac{|a|^{-2p}\left(1 - |a|^{(-2p+1)n}\right)}{1 - |a|^{(-2p+1)}} \end{aligned} \tag{2.33}$$

for each $x_1, x_2 \in X$. For $m, n \in \mathbb{N}$

$$\begin{aligned} \left\| a^m f_2\left(\frac{x_1}{a^m}, \frac{x_2}{a^m}\right) - a^n f_2\left(\frac{x_1}{a^n}, \frac{x_2}{a^n}\right) \right\| &= \left\| a^{m+n-n} f_2\left(\frac{x_1}{a^{m+n-n}}, \frac{x_2}{a^{m+n-n}}\right) - a^n f_2\left(\frac{x_1}{a^n}, \frac{x_2}{a^n}\right) \right\| \\ &= |a|^n \left\| a^{m-n} f_2\left(\frac{x_1}{a^{m-n} \cdot a^n}, \frac{x_2}{a^{m-n} \cdot a^n}\right) - f_2\left(\frac{x_1}{a^n}, \frac{x_2}{a^n}\right) \right\| \\ &\le |a|^n \left\| \frac{x_1}{a^n}, \frac{x_2}{a^n} \right\|^p \cdot \frac{\varepsilon}{2} \sum_{j=0}^{m-n-1} |a|^{(-2p+1)j-2p} \\ &= \frac{\varepsilon \|x_1, x_2\|^p}{2} |a|^{(-2p+1)n} \sum_{j=0}^{m-n-1} |a|^{(-2p+1)j-2p} \\ &= \frac{\varepsilon \|x_1, x_2\|^p}{2} \sum_{j=0}^{m-n-1} |a|^{(-2p+1)(n+j)-2p} \\ &= \frac{\varepsilon \|x_1, x_2\|^p}{2} \sum_{j=0}^{m-n-1} \frac{|a|^{(-2p+1)n}\left(1 - |a|^{(-2p+1)(m-n)}\right)}{1 - |a|^{(-2p+1)}} \\ &\longrightarrow 0 \ as\ n \to \infty,\ p > \frac{1}{2} \end{aligned}$$

for each $x_1, x_2 \in X$. Therefore $\left\{ a^n f_2\left(\frac{x_1}{a^n}, \frac{x_2}{a^n}\right) \right\}$ is a Cauchy sequence in Y, for each $x_1, x_2 \in X$. Since Y is a Banach space, $\left\{ a^n f_2\left(\frac{x_1}{a^n}, \frac{x_2}{a^n}\right) \right\}$ converges, for each

$x_1, x_2 \in X$. Define $A : X \times X \longrightarrow Y$ as

$$A(x_1, x_2) := \lim_{n\to\infty} a^n f_2\Big(\frac{x_1}{a^n}, \frac{x_2}{a^n}\Big)$$

for each $x_1, x_2 \in X$. By (2.33), we get

$$\lim_{n\to\infty} \Big\|f_2(x_1, x_2) - a^n f_2\Big(\frac{x_1}{a^n}, \frac{x_2}{a^n}\Big)\Big\| \leq \frac{\varepsilon\|x_1, x_2\|^p}{2} \frac{1}{|a|^{2p} - |a|}$$

for each $x_1, x_2 \in X$. Therefore

$$\Big\|\frac{f(x_1, x_2) - f(-x_1, -x_2)}{2} - A(x_1, x_2)\Big\| \leq \frac{\varepsilon\|x_1, x_2\|^p}{2\big(|a|^{2p} - |a|\big)}$$

for each $x_1, x_2 \in X$. The further part of the proof is similar to that of the proof of Theorem 2.3. □

Theorem 2.5. *Let* $\varepsilon \geq 0, 0 < p < \frac{1}{2}$ *and let* $f : X \times X \longrightarrow Y$ *be a function such that*

$$\big\|D_f\big((x_1, x_2), (y_1, y_2)\big)\big\| \leq \varepsilon\big[\|x_1, x_2\|^p + \|y_1, y_2\|^p\big] \tag{2.34}$$

for each $x_1, x_2, y_1, y_2 \in X$. *Then there exists a unique additive function* $A : X \times X \longrightarrow Y$ *and there exists a unique quadratic function* $Q : X \times X \longrightarrow Y$ *satisfying* (2.1) *and the inequality*

(2.35)

$$\|f(x_1, x_2) - A(x_1, x_2) - Q(x_1, x_2) - f(0, 0)\| \leq \varepsilon\|x_1, x_2\|^p \left[\frac{|a|^{2p} + 2|a| + 1}{2^{2p}\big(|a|^2 - |a|^{2p}\big)} + \frac{1}{2\big(|a| - |a|^{2p}\big)}\right]$$

for each $x_1, x_2 \in X$.

Proof. Since $0 < p < \frac{1}{2} \Rightarrow p < 1$ and f satisfies (2.2). Therefore by Theorem 2.1, there exists a unique quadratic mapping $Q : X \times X \longrightarrow Y$ such that

(2.36)

$$\Big\|\frac{f(x_1, x_2) + f(-x_1, -x_2)}{2} - Q(x_1, x_2) - f(0, 0)\Big\| \leq \varepsilon\|x_1, x_2\|^p \frac{|a|^{2p} + 2|a| + 1}{2^{2p}\big(|a|^2 - |a|^{2p}\big)}$$

for each $x_1, x_2 \in X$. Since $0 < p < \frac{1}{2}$ and f satisfies (2.2), there exists a unique additive function $A : X \times X \longrightarrow Y$ such that

$$\Big\|\frac{f(x_1, x_2) - f(-x_1, -x_2)}{2} - A(x_1, x_2)\Big\| \leq \frac{\varepsilon\|x_1, x_2\|^p}{2\big(|a| - |a|^{2p}\big)} \tag{2.37}$$

for each $x_1, x_2 \in X$. Now

$$\begin{aligned}
&\|f(x_1, x_2) - Q(x_1, x_2) - A(x_1, x_2) - f(0, 0)\| \\
&= \Big\|\frac{f(x_1, x_2)}{2} + \frac{f(-x_1, -x_2)}{2} + \frac{f(x_1, x_2)}{2} - \frac{f(-x_1, -x_2)}{2} - Q(x_1, x_2) - A(x_1, x_2) - f(0, 0)\Big\| \\
&= \Big\|\frac{f(x_1, x_2) + f(-x_1, -x_2)}{2} + \frac{f(x_1, x_2) - f(-x_1, -x_2)}{2} - Q(x_1, x_2) - A(x_1, x_2) - f(0, 0)\Big\| \\
&\leq \Big\|\frac{f(x_1, x_2) + f(-x_1, -x_2)}{2} - Q(x_1, x_2) - f(0, 0)\Big\| + \Big\|\frac{f(x_1, x_2) - f(-x_1, -x_2)}{2} - A(x_1, x_2)\Big\| \\
&\leq \varepsilon\|x_1, x_2\|^p \left[\frac{|a|^{2p} + 2|a| + 1}{2^{2p}\big(|a|^2 - |a|^{2p}\big)} + \frac{1}{2\big(|a| - |a|^{2p}\big)}\right]
\end{aligned}$$

for each $x_1, x_2 \in X$. □

Theorem 2.6. *Let $\varepsilon \geq 0, p > 1$ and let $f : X \times X \longrightarrow Y$ be a function such that*

$$\big\|D_f\big((x_1,x_2),(y_1,y_2)\big)\big\| \leq \varepsilon\big[\|x_1,x_2\|^p + \|y_1,y_2\|^p\big] \tag{2.38}$$

for each $x_1, x_2, y_1, y_2 \in X$. Then there exists a unique additive function $A : X \times X \longrightarrow Y$ and there exists a unique quadratic function $Q : X \times X \longrightarrow Y$ satisfying (2.1) *and the inequality*

(2.39)

$$\|f(x_1,x_2) - Q(x_1,x_2) - A(x_1,x_2) - f(0,0)\| \leq \varepsilon\|x_1,x_2\|^p\left[\frac{|a|^{2p}+2|a|+1}{2^{2p}\big(|a|^{2p}-|a|^2\big)} + \frac{1}{2\big(|a|^{2p}-|a|\big)}\right]$$

for each $x_1, x_2 \in X$.

Proof. Proof is similar to that of the proof of Theorem 2.5. ☐

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