Positive periodic solutions to Liénard equation with indefinite weights

Abstract—The Liénard equation is not only of significant theoretical interest but also plays a fundamental role in physical modeling. In particular, Liénard equations with indefinite weights introduce additional complexities that render their analysis more challenging. This paper establishes sufficient conditions for the existence of positive periodic solutions to a class of generalized Liénard equations with sign-changing coefficients. Our approach is primarily based on the Krasnoselskiĭ's-Guo fixed point theorem, combined with the positivity properties of the associated Green function.

Keywords—Periodic solution; Liénard equation; Indefinite singularity; Krasnoselskiĭ's-Guo fixed point theorem.

1 Introduction

Singular differential equations have wide applications in mechanics, electronics, modern biology, and many other fields [1, 9]. Among them, the study of the Liénard equation is of great theoretical value and plays an important role in physical modeling. For example, the bubble physics model proposed by Professor Torres in his monograph [11], as well as the Micro-electro-mechanical systems (MEMS) mass-spring model, both demonstrate the modeling significance of this equation in concrete problems. Therefore, research on the Liénard equation has long attracted considerable attention.

The origin of the Liénard equation can be traced back to the 1920s. In 1926, van der Pol [5] proposed the equation

$$x'' + \mu(x^2 - 1)x' + x = 0, \quad \mu \neq 0,$$

in his study of isochronous oscillations in triode circuits. This equation shares the same structural form as the one later introduced by the French engineer Liénard [10] in 1928, who generalized it to the more comprehensive form

$$x'' + f(x)x' + g(x) = 0,$$

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which is now referred to as the classical Liénard equation.

With the development of this subject, singular Liénard equations have received continuous attention, and many existence results for periodic solutions have been obtained using a variety of techniques, including topological degree theory, fixed point theorems, and the method of upper and lower solutions [8, 13, 15–18]. More recently, in 2021, Xin and Cheng [14] considered the existence of positive periodic solutions for generalized singular ϕ-Laplacian type Liénard equations of the form

$$(\phi(x'))' + f(t,x)x' + \frac{b(t)}{x^\rho} = h(t)x^m,$$

where ρ is a positive constant and m is a constant. By applying the method of upper and lower solutions, they established existence results. Cheng et al. [4] investigated the following Lennard Jones potential

$$x'' = \frac{a(t)}{x^\rho} - \frac{b(t)}{x^\delta},$$

where ρ and δ are two positive constants, and $a(t)$, $b(t) \in L^1(\mathbb{R}/T\mathbb{Z})$ are T-periodic sign-changing functions, By applying the method of Krasnoselskiĭ's-Guo fixed point theorem, they proved the existence of periodic solutions for this equation.

Motivated by the above works, the present paper investigates the existence of periodic solutions for the following Liénard equation with indefinite weight

$$x'' = \frac{g(t)}{x^\mu} - \frac{h(t)}{x^\lambda} + p(t). \tag{1.1}$$

where $\mu, \lambda > 0$ are constants, and $g(t), h(t), p(t) \in L^1(\mathbb{R}/T\mathbb{Z})$ are T-periodic sign-changing functions. The sign-changing nature of the coefficients makes the analysis more intricate. Our approach is mainly based on the Krasnoselskiĭ's-Guo fixed point theorem, combined with the positivity of the associated Green function, to establish existence results for periodic solutions.

2 Preliminary Lemmas

In this section, we will primarily concentrate on the Krasnoselskiĭ's-Guo fixed point theorem, which will be used in the proofs of our theorems below.

Lemma 2.1. *(Krasnoselskiĭ's-Guo fixed point theorem [6]) Let X be a Banach space and $\mathcal{K}$ a cone in X. Assume that Ω_1 and Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Let*

$$\Phi : \mathcal{K} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{K}$$

be a completely continuous operator such that one of the following conditions holds:

(i) $\|\Phi x\| \geq \|x\|$ for $x \in \mathcal{K} \cap \partial\Omega_1$ and $\|\Phi x\| \leq \|x\|$ for $x \in \mathcal{K} \cap \partial\Omega_2$;

(ii) $\|\Phi x\| \leq \|x\|$ for $x \in \mathcal{K} \cap \partial\Omega_1$ and $\|\Phi x\| \geq \|x\|$ for $x \in \mathcal{K} \cap \partial\Omega_2$.

Then Φ has a fixed point in the set $\mathcal{K} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

To write the periodic problem as an equivalent fixed point problem, we will use the notion of the Green function. Additionally, the Green function is a crucial mathematical tool and can be found in many papers(such as [2,3,7,12]).

Lemma 2.2. *(see [7, Lemma 2.4]) If* $M > 0$ *is such that* $M \neq \frac{2k\pi}{T}$ *for any natural* k*, then for any* $e \in L^1(\mathbb{R}/T\mathbb{Z})$ *the equation*

$$x'' + M^2 x = e(t)$$

has a unique T*-periodic solution given by*

$$x(t) = \int_0^T G_1(t,s)e(s)ds,$$

where the Green function $G_1(t,s)$ *has the following form*

$$G_1(t,s) = \begin{cases} \dfrac{\cos M(t-s-\frac{T}{2})}{2M\sin\frac{MT}{2}}, & 0 \le s \le t \le T, \\ \dfrac{\cos M(t-s+\frac{T}{2})}{2M\sin\frac{MT}{2}}, & 0 \le t < s \le T. \end{cases}$$

In addition, if $M < \frac{\pi}{T}$*, then* $G_1(t,s) > 0$ *for all* $(t,s) \in [0,T] \times [0,T]$ *and* $\int_0^T G_1(t,s)M^2 ds \equiv 1$.

Lemma 2.3. *(see [12, Corollary 2.2]) If* $M > 0$*, then for each* $e \in L^1(\mathbb{R}/T\mathbb{Z})$ *the equation*

$$-x'' + M^2 x = e(t)$$

has a unique T*-periodic solution provided by*

$$x(t) = \int_0^T G_2(t,s)e(s)ds,$$

where the Green function $G_2(t,s)$ *has the following form*

$$G_2(t,s) = \begin{cases} \dfrac{\exp(-M(s-t)) + \exp(M(s-t-T))}{2M(1-\exp(-MT))}, & 0 \le t < s \le T, \\ \dfrac{\exp(-M(s-t+T)) + \exp(M(s-t))}{2M(1-\exp(-MT))}, & 0 \le s \le t \le T. \end{cases}$$

Besides, $G_2(t,s) > 0$ *for* $(t,s) \in [0,T] \times [0,T]$ *and* $\int_0^T G_2(t,s)M^2 ds \equiv 1$.

3 Definitions and Main proofs.

Before presenting our main theorems and proofs, we first introduce some definitions that will be used throughout this paper. To begin with, define

$$\begin{aligned} &\mathcal{A}_1 := \min_{0 \le s,t \le T} G_1(t,s) = \frac{1}{2M}\cot\frac{MT}{2}, \ \mathcal{B}_1 := \max_{0 \le s,t \le T} G_1(t,s) = \frac{1}{2M\sin\frac{MT}{2}}, \\ &\mathcal{A}_2 := \min_{0 \le s,t \le T} G_2(t,s) = \frac{\exp(-\frac{MT}{2})}{M(1-\exp(-MT))}, \ \mathcal{B}_2 := \max_{0 \le s,t \le T} G_2(t,s) = \frac{1+\exp(-MT)}{2M(1-\exp(-MT))}, \\ &\sigma_1 := \frac{\mathcal{A}_1}{\mathcal{B}_1}, \ \sigma_2 := \frac{\mathcal{A}_2}{\mathcal{B}_2}. \end{aligned} \tag{3.1}$$

It is clear that $0 < \mathcal{A}_i \leq \mathcal{B}_i$ and $0 < \sigma_i \leq 1$, $i = 1, 2$ from Lemmas 2.2 and 2.3. Next, define

$$\mathcal{K}_i := \{x \in C_T : \min_{t\in\mathbb{R}} x(t) \geq \sigma_i \|x\|\}, \quad i = 1, 2,$$

where $C_T := \{x \in C(\mathbb{R}, \mathbb{R}) : x(t+T) = x(t), \text{ for all } t \in \mathbb{R}\}$ and $\|x\| := \max_{t\in\mathbb{R}} |x(t)|$. It is easy to verify that $\mathcal{K}_i$ is a cone in C_T. Finally, for given periodic functions $h(t)$ and $e(t)$, we denote

$$h^+(t) := \max\{0, h(t)\},\ h^-(t) := -\min\{0, h(t)\},\ \bar{h} := \frac{1}{T}\int_0^T h(t)dt,\ e^* := \max_{t\in\mathbb{R}} e(t),\ e_* := \min_{t\in\mathbb{R}} e(t).$$

Now, we divide the main results into two cases. The main theorems and proofs are given below.

3.1 The case $\mu > \lambda$

Theorem 3.1. *Assume that there exist four constants $N \in (0, \frac{\pi}{T})$, $a, b, c \in (0, 1)$ and $a + b + c = 1$ such that*

$$\overline{h^-} < \sigma_1^{1+\lambda}\overline{h^+},\ \overline{p^+} \leq \sigma_1\overline{p^-},\ (\mathcal{A}_1 T\overline{g^+})^{\frac{1}{1+\mu}} > \frac{1}{\sigma_1}\max\left\{\left(\frac{\|g^-\|_\infty}{aN^2}\right)^{\frac{1}{1+\mu}}, \left(\frac{\|h^+\|_\infty}{bN^2}\right)^{\frac{1}{1+\lambda}}, \frac{\|p^-\|_\infty}{cN^2}\right\}. \quad (3.2)$$

If $\mu > \lambda$, then equation (1.1) *admits at least one positive T-periodic solution.*

Proof. We can write equation (1.1) as

$$x'' + N^2 x = \frac{g(t)}{x^\mu} - \frac{h(t)}{x^\lambda} + N^2 x + p(t), \quad (3.3)$$

a T-periodic solution of equation (3.3) is just a fixed point of the map Φ defined by

$$(\Phi x)(t) := \int_0^T G_1(t, s)\left(\frac{g(s)}{x(s)^\mu} - \frac{h(s)}{x(s)^\lambda} + N^2 x(s) + p(s)\right)ds, \quad (3.4)$$

and we know that $G_1(t, s) > 0$ for all $(t, s) \in [0, T] \times [0, T]$ from Lemma 2.2.

Now we define two open sets

$$\Omega_1 := \{x \in C_T : \|x\| < r_1\} \text{ and } \Omega_2 := \{x \in C_T : \|x\| < R_1\}.$$

Note that Φ is well-defined in $\mathcal{K}_1 \cap (\overline{\Omega}_2 \backslash \Omega_1)$ and is a completely continuous operator via Ascoli-Arzelà Theorem.

By (3.2), the two positive constants r_1 and R_1 can be fixed such that

$$R_1 > r_1 = (\mathcal{A}_1 T\overline{g^+})^{\frac{1}{1+\mu}} > \frac{1}{\sigma_1}\max\left\{\left(\frac{\|g^-\|_\infty}{aN^2}\right)^{\frac{1}{1+\mu}}, \left(\frac{\|h^+\|_\infty}{bN^2}\right)^{\frac{1}{1+\lambda}}, \frac{\|p^-\|_\infty}{cN^2}\right\}.$$

First, we claim that $\Phi(\mathcal{K}_1 \cap (\overline{\Omega}_2 \setminus \Omega_1)) \subset \mathcal{K}_1$. In fact, for any $x \in \mathcal{K}_1 \cap (\overline{\Omega}_2 \setminus \Omega_1)$, we arrive at

$$\sigma_1 r_1 \leq x(t) \leq R_1, \text{ for all } t \in \mathbb{R}.$$

Since $r_1 > \frac{1}{\sigma_1} \max\left\{ \left(\frac{\|g^-\|_\infty}{aN^2} \right)^{\frac{1}{1+\mu}}, \left(\frac{\|h^+\|_\infty}{bN^2} \right)^{\frac{1}{1+\lambda}}, \frac{\|p^-\|_\infty}{cN^2} \right\}$, and $a+b+c=1$ we obtain

$$\begin{aligned}\frac{g(t)}{x^\mu} - \frac{h(t)}{x^\lambda} + N^2x + p(t) &= \frac{g^+(t)}{x^\mu} - \frac{g^-(t)}{x^\mu} - \frac{h^+(t)}{x^\lambda} + \frac{h^-(t)}{x^\lambda} + N^2x + p^+(t) - p^-(t) \\ &\geq -\frac{g^-(t)}{x^\mu} - \frac{h^+(t)}{x^\lambda} - p^-(t) + N^2x \\ &= -\frac{g^-(t)}{x^\mu} - \frac{h^+(t)}{x^\lambda} - p^-(t) + (a+b+c)N^2x \\ &\geq -\frac{\|g^-\|_\infty}{(\sigma_1 r_1)^\mu} + aN^2(\sigma_1 r_1) - \frac{\|h^+\|_\infty}{(\sigma_1 r_1)^\lambda} + bN^2(\sigma_1 r_1) - \|p^-\|_\infty + cN^2(\sigma_1 r_1) \\ &\geq 0,\end{aligned} \tag{3.5}$$

for all $t \in \mathbb{R}$. It follows from (3.1) and (3.5) that

$$\begin{aligned}\min_{t\in\mathbb{R}} (\Phi x)(t) &= \min_{t\in\mathbb{R}} \int_0^T G_1(t,s) \left(\frac{g(s)}{x^\mu} - \frac{h(s)}{x^\lambda} + N^2x + p(s) \right) ds \\ &\geq \mathcal{A}_1 \int_0^T \left(\frac{g(s)}{x^\mu} - \frac{h(s)}{x^\lambda} + N^2x + p(s) \right) ds \\ &= \sigma_1 \mathcal{B}_1 \int_0^T \left(\frac{g(s)}{x^\mu} - \frac{h(s)}{x^\lambda} + N^2x + p(s) \right) ds \\ &\geq \sigma_1 \max_{s,t\in[0,T]} \int_0^T G_1(t,s) \left(\frac{g(s)}{x^\mu} - \frac{h(s)}{x^\lambda} + N^2x + p(s) \right) ds \\ &\geq \sigma_1 \|\Phi x\|,\end{aligned} \tag{3.6}$$

which implies $\Phi(\mathcal{K}_1 \cap (\overline{\Omega}_2 \setminus \Omega_1)) \subset \mathcal{K}_1$.

Next, we prove that

$$\|\Phi x\| \geq \|x\|, \text{ for } x \in \mathcal{K}_1 \cap \partial\Omega_1. \tag{3.7}$$

In fact, for any $x \in \mathcal{K}_1 \cap \partial\Omega_1$, it is clear $\|x\| = r_1$ and

$$\sigma_1 r_1 \leq x(t) \leq r_1, \quad \text{for all } t \in \mathbb{R}.$$

Then

$$\begin{aligned}(\Phi x)(t) &= \int_0^T G_1(t,s) \left(\frac{g(s)}{x^\mu} - \frac{h(s)}{x^\lambda} + N^2x + p(s) \right) ds \\ &= \int_0^T G_1(t,s) \left(\frac{g^+(s)}{x^\mu} - \frac{g^-(s)}{x^\mu} - \frac{h^+(s)}{x^\lambda} + \frac{h^-(s)}{x^\lambda} + N^2x + p^+(s) - p^-(s) \right) ds \\ &\geq \int_0^T G_1(t,s) \frac{g^+(s)}{x^\mu} ds \\ &\geq \frac{\mathcal{A}_1 T \overline{g^+}}{r_1^\mu} = r_1,\end{aligned} \tag{3.8}$$

since $r_1 = (\mathcal{A}_1 T \overline{g^+})^{\frac{1}{1+\mu}}$ from definition of r_1. Hence, (3.7) is satisfied.

Finally, we prove that

$$\|\Phi x\| \leq \|x\|, \ \text{ for } x \in \mathcal{K}_1 \cap \partial\Omega_2. \tag{3.9}$$

In fact, for any $x \in \mathcal{K}_1 \cap \partial\Omega_2$, it is clear $\|x\| = R_1$ and

$$\sigma_1 R_1 \leq x(t) \leq R_1, \quad \text{for all } \ t \in \mathbb{R}.$$

It follows from (3.1), $\int_0^T G_1(t,s)M^2 ds \equiv 1$ and $\overline{p^+} \leq \sigma_1 \overline{p^-}$ that

$$\begin{aligned}
(\Phi x)(t) &= \int_0^T G_1(t,s)\left(\frac{g(s)}{x^\mu} - \frac{h(s)}{x^\lambda} + N^2 x + p(s)\right) ds \\
&= \int_0^T G_1(t,s)\left(\frac{g^+(s)}{x^\mu} - \frac{g^-(s)}{x^\mu} - \frac{h^+(s)}{x^\lambda} + \frac{h^-(s)}{x^\lambda} + N^2 x + p^+(s) - p^-(s)\right) ds \\
&\leq \frac{\mathcal{B}_1 T\overline{g^+}}{(\sigma_1 R_1)^\mu} - \frac{\mathcal{A}_1 T\overline{g^-}}{R_1^\mu} - \frac{\mathcal{A}_1 T\overline{h^+}}{R_1^\lambda} + \frac{\mathcal{B}_1 T\overline{h^-}}{(\sigma_1 R_1)^\lambda} + \mathcal{B}_1 T\overline{p^+} - \mathcal{A}_1 T\overline{p^-} + R_1 \qquad (3.10)\\
&\leq \frac{\mathcal{B}_1 T\overline{g^+}}{(\sigma_1 R_1)^\mu} - \frac{\mathcal{A}_1 T\overline{g^-}}{R_1^\mu} - \frac{\mathcal{A}_1 T\overline{h^+}}{R_1^\lambda} + \frac{\mathcal{B}_1 T\overline{h^-}}{(\sigma_1 R_1)^\lambda} + R_1 \\
&\leq R_1,
\end{aligned}$$

where $\frac{\mathcal{B}_1 T\overline{g^+}}{(\sigma_1 R_1)^\mu} - \frac{\mathcal{A}_1 T\overline{g^-}}{R_1^\mu} - \frac{\mathcal{A}_1 T\overline{h^+}}{R_1^\lambda} + \frac{\mathcal{B}_1 T\overline{h^-}}{(\sigma_1 R_1)^\lambda} + R_1 \leq R_1$ holds, i.e.,

$$\overline{g^+} - \sigma_1^{1+\mu}\overline{g^-} \leq \left(\sigma_1^{1+\lambda}\overline{h^+} - \overline{h^-}\right)(\sigma_1 R_1))^{\mu-\lambda}$$

for sufficiently large R_1 and $\overline{h^-} < \sigma_1^{1+\lambda}\overline{h^+}$. This implies that (3.9) holds. The proof is finished. □

In the Theorem 3.1, we know that $0 < N < \frac{\pi}{T}$. Next we will give a result similar to Theorem 3.1, in the case without any restriction on $N > 0$.

Theorem 3.2. *Assume that there exist four constants $N > 0$, $a, b, c \in (0,1)$ and $a + b + c = 1$ such that*

$$\overline{h^+} < \sigma_2^{1+\lambda}\overline{h^-}, \ \ \overline{p^-} \leq \sigma_2 \overline{p^+}, \ \ (\mathcal{A}_2 T\overline{g^-})^{\frac{1}{1+\mu}} > \frac{1}{\sigma_2}\max\left\{\left(\frac{\|g^+\|_\infty}{aN^2}\right)^{\frac{1}{1+\mu}}, \left(\frac{\|h^-\|_\infty}{bN^2}\right)^{\frac{1}{1+\lambda}}, \frac{\|p^+\|_\infty}{cN^2}\right\}.$$

If $\mu > \lambda$, then equation (1.1) *admits at least one positive T-periodic solution.*

Proof. We can write equation (1.1) as

$$-x'' + N^2 x = -\frac{g(t)}{x^\mu} + \frac{h(t)}{x^\lambda} + N^2 x - p(t), \tag{3.11}$$

a T-periodic solution of equation (3.11) is just a fixed point of the map Ψ defined by

$$(\Psi x)(t) := \int_0^T G_2(t,s)\left(-\frac{g(s)}{x(s)^\mu} + \frac{h(s)}{x(s)^\lambda} + N^2 x(s) - p(s)\right) ds,$$

and we know that $G_2(t,s) > 0$ for all $(t,s) \in [0,T] \times [0,T]$ from Lemma 2.3. From this point, the proof follows the same steps as Theorem 3.1.

3.2 The case $\mu < \lambda$

Theorem 3.3. *Assume that there exist four constants $N \in (0, \frac{\pi}{T})$, $a, b, c \in (0,1)$ and $a+b+c=1$ such that*

$$(\mathcal{A}_1 T\overline{g^+})^{\frac{1}{1+\mu}} > \left(\frac{\sigma_1^{1+\lambda}\overline{h^+} - \overline{h^-}}{\left(\overline{g^+} - \sigma_1^{1+\mu}\overline{g^-}\right)\sigma_1^{\lambda-\mu}}\right)^{\frac{1}{\lambda-\mu}} > \frac{1}{\sigma_1}\max\left\{\left(\frac{\|g^-\|_\infty}{aN^2}\right)^{\frac{1}{1+\mu}}, \left(\frac{\|h^+\|_\infty}{bN^2}\right)^{\frac{1}{1+\lambda}}, \frac{\|p^-\|_\infty}{cN^2}\right\},$$

$\overline{h^-} < \sigma_1^{1+\lambda}\overline{h^+}$, and $\sigma_1\overline{p^-} \geq \overline{p^+}$. If $\mu < \lambda$, then equation (1.1) *admits at least one positive T-periodic solution.*

Now we define two open sets

$$\Omega_3 := \{x \in C_T^1 : \|x\| < r_2\} \text{ and } \Omega_4 := \{x \in C_T^1 : \|x\| < R_2\},$$

where r_2 and R_2 are two constants and

$$R_2 = (\mathcal{A}_2 T\overline{g^+})^{\frac{1}{1+\mu}} > \left(\frac{\sigma_1^{1+\lambda}\overline{h^+} - \overline{h^-}}{\left(\overline{g^+} - \sigma_1^{1+\mu}\overline{g^-}\right)\sigma_1^{\lambda-\mu}}\right)^{\frac{1}{\lambda-\mu}} > r_2 > \frac{1}{\sigma_2}\max\left\{\left(\frac{\|g^-\|_\infty}{aN^2}\right)^{\frac{1}{1+\mu}}, \left(\frac{\|h^+\|_\infty}{bN^2}\right)^{\frac{1}{1+\lambda}}, \frac{\|p^-\|_\infty}{cN^2}\right\}.$$

By an analogous reasoning as in the proof of Theorem 3.1, we get that $\Phi : \mathcal{K}_1 \cap (\overline{\Omega}_4 \setminus \Omega_3) \in \mathcal{K}_1$, where Φ is defined in (3.4).

Next, we prove that

$$\|\Phi x\| \leq \|x\|, \text{ for } x \in \mathcal{K}_1 \cap \partial\Omega_3. \tag{3.12}$$

In fact, for any $x \in \mathcal{K}_1 \cap \partial\Omega_3$, it is clear $\|x\| = r_2$ and

$$\sigma_1 r_2 \leq x(t) \leq r_2, \quad \text{for all } t \in \mathbb{R}.$$

It follows from (3.1), $\int_0^T G_1(t,s)M^2 ds \equiv 1$ and $\overline{p^+} \leq \sigma_1\overline{p^-}$ that

$$\begin{aligned}
(\Phi x)(t) &= \int_0^T G_1(t,s)\left(\frac{g(s)}{x^\mu} - \frac{h(s)}{x^\lambda} + N^2 x + p(s)\right) ds \\
&= \int_0^T G_1(t,s)\left(\frac{g^+(s)}{x^\mu} - \frac{g^-(s)}{x^\mu} - \frac{h^+(s)}{x^\lambda} + \frac{h^-(s)}{x^\lambda} + N^2 x + p^+(s) - p^-(s)\right) ds \\
&\leq \frac{\mathcal{B}_1 T\overline{g^+}}{(\sigma_1 r_1)^\mu} - \frac{\mathcal{A}_1 T\overline{g^-}}{r_1^\mu} - \frac{\mathcal{A}_1 T\overline{h^+}}{r_1^\lambda} + \frac{\mathcal{B}_1 T\overline{h^-}}{(\sigma_1 r_1)^\lambda} + \mathcal{B}_1 T\overline{p^+} - \mathcal{A}_1 T\overline{p^-} + r_1 \\
&\leq \frac{\mathcal{B}_1 T\overline{g^+}}{(\sigma_1 r_1)^\mu} - \frac{\mathcal{A}_1 T\overline{g^-}}{r_1^\mu} - \frac{\mathcal{A}_1 T\overline{h^+}}{r_1^\lambda} + \frac{\mathcal{B}_1 T\overline{h^-}}{(\sigma_1 r_1)^\lambda} + r_1 \\
&\leq r_1,
\end{aligned} \tag{3.13}$$

where $\frac{\mathcal{B}_1 T\overline{g^+}}{(\sigma_1 r_1)^\mu} - \frac{\mathcal{A}_1 T\overline{g^-}}{r_1^\mu} - \frac{\mathcal{A}_1 T\overline{h^+}}{r_1^\lambda} + \frac{\mathcal{B}_1 T\overline{h^-}}{(\sigma_1 r_1)^\lambda} \leq 0$ holds, i.e.,

$$r_1^{\lambda-\mu}\left(\sigma_1^{\lambda-\mu}\overline{g^+} - \sigma_1^{1+\lambda}\overline{g^-}\right) \leq \sigma_1^{1+\lambda}\overline{h^+} - \overline{h^-}$$

for $\overline{h^-} < \sigma_1^{1+\lambda}\overline{h^+}$ and $r_1 < \left(\frac{\sigma_1^{1+\lambda}\overline{h^+} - \overline{h^-}}{(\overline{g^+} - \sigma_1^{1+\mu}\overline{g^-})\sigma_1^{\lambda-\mu}}\right)^{\frac{1}{\lambda-\mu}}$. This implies that (3.12) holds.

Finally, we prove that

$$\|\Phi x\| \geq \|x\|, \ \text{ for } x \in \mathcal{K}_1 \cap \partial\Omega_4. \tag{3.14}$$

In fact, for any $x \in \mathcal{K}_1 \cap \partial\Omega_4$, it is clear $\|x\| = R_1$ and

$$\sigma_1 R_1 \leq x(t) \leq R_1, \quad \text{for all } \ t \in \mathbb{R}.$$

Then

$$\begin{aligned}
(\Phi x)(t) &= \int_0^T G_1(t,s)\left(\frac{g(s)}{x^\mu} - \frac{h(s)}{x^\lambda} + N^2 x + p(s)\right) ds \\
&= \int_0^T G_1(t,s)\left(\frac{g^+(s)}{x^\mu} - \frac{g^-(s)}{x^\mu} - \frac{h^+(s)}{x^\lambda} + \frac{h^-(s)}{x^\lambda} + N^2 x + p^+(s) - p^-(s)\right) ds \\
&\geq \int_0^T G_1(t,s)\frac{g^+(s)}{x^\mu} ds \\
&\geq \frac{\mathcal{A}_1 T\overline{g^+}}{R_1^\mu} = R_1,
\end{aligned} \tag{3.15}$$

since $R_1 = (\mathcal{A}_1 T\overline{g^+})^{\frac{1}{1+\mu}}$ from definition of R_1. Hence, (3.14) is satisfied. The proof is finished.

By Theorems 3.3, we can also come to the following conclusion.

Theorem 3.4. *Assume that there exist four constants $N > 0$, $a, b, c \in (0,1)$ and $a + b + c = 1$ such that*

$$(\mathcal{A}_2 T\overline{g^-})^{\frac{1}{1+\mu}} > \left(\frac{\sigma_2^{1+\lambda}\overline{h^-} - \overline{h^+}}{\left(\overline{g^-} - \sigma_2^{1+\mu}\overline{g^+}\right)\sigma_2^{\lambda-\mu}}\right)^{\frac{1}{\lambda-\mu}} > \frac{1}{\sigma_2}\max\left\{\left(\frac{\|g^+\|_\infty}{aN^2}\right)^{\frac{1}{1+\mu}}, \left(\frac{\|h^-\|_\infty}{bN^2}\right)^{\frac{1}{1+\lambda}}, \frac{\|p^+\|_\infty}{cN^2}\right\},$$

$\overline{h^+} < \sigma_2^{1+\lambda}\overline{h^-}$ and $\overline{p^-} \leq \sigma_2\overline{p^+}$. If $\mu < \lambda$, then equation (1.1) admits at least one positive T-periodic solution.

To illustrate the conclusions, we will present a specific form of equation (1.1) as an example:

$$x'' = \frac{150\sin 2t + 740}{x^{1.2}} - \frac{1 + 0.1\sin 2t}{x^{0.5}} + 1.2\cos 2t - 0.5. \tag{3.16}$$

where $\mu = 1.2$, $\lambda = 0.5$, $g(t) = 150\sin 2t + 740$, $h(t) = 1 + 0.1\sin 2t$, $p(t) = 1.2\cos 2t - 0.5$ and $T = \pi$. Put $N = 0.5$, $a = 0.4$, $b = 0.3$ and $c = 0.3$, futher

$$\overline{h^-} = 0 < \sigma_1^{1+\lambda}\overline{h^+} \approx 0.595, \ \ \overline{p^+} \approx 0.167 \leq \sigma_1\overline{p^-} \approx 0.471,$$

$$(\mathcal{A}_1 T\overline{g^+})^{\frac{1}{1+\mu}} \approx 34.1 > \frac{1}{\sigma_1}\max\left\{\left(\frac{\|g^-\|_\infty}{aN^2}\right)^{\frac{1}{1+\mu}}, \left(\frac{\|h^+\|_\infty}{bN^2}\right)^{\frac{1}{1+\lambda}}, \frac{\|p^-\|_\infty}{cN^2}\right\} \approx 32.06.$$

Thus, by Theorem 3.1, the equation possesses at least one T-positive periodic solution.

Remark 3.1. It should be emphasized that the existence results established in Theorems 3.1 – 3.4 apply specifically to systems with distinct singularity indices, i.e., when$\mu \neq \lambda$. The case where $\mu = \lambda = \rho$ requires separate consideration. If $\mu = \lambda = \rho$, the original equation simplifies to the form:

$$x'' = \frac{g(t) - h(t)}{x^\rho} + p(t), \quad x(0) = x(T), x'(0) = x'(T).$$

This symmetry in singularity indices fundamentally alters the analytical framework. The construction of dual-cone operators T_1 and T_2, which relies on distinct Green's functions G_1 and G_2, is no longer viable in this symmetric case. Consequently, the Krasnoselskiı̆'s-Guo fixed-point scheme employed in our proofs cannot be directly applied.Establishing existence of positive periodic solutions for the case $\mu = \lambda$ would require alternative methodologies, such as the method of upper and lower solutions or modified fixed-point theorems adapted to single-operator formulations.This clarifies the rationale for deliberately excluding the case $\mu = \lambda$ in the current theorems.

□

4 Conclusion

In this section, we primarily summarize and compare the main theorems presented in the paper, illustrating our key results in a table.

Table 1: Comparison result

Theorem	Theorem 3.1	Theorem 3.3
Equation	$x'' = \frac{g(t)}{x^\mu} - \frac{h(t)}{x^\lambda} + p(t)$	$x'' = \frac{g(t)}{x^\mu} - \frac{h(t)}{x^\lambda} + p(t)$
Cases	$u > \lambda$	$u < \lambda$
Conditions	1 $a+b+c=1,\ \ a,b,c \in (0,1),\ \ 0 < N < \frac{\pi}{T}$ 2 $\overline{h^-} < \sigma_1^{1+\lambda}\overline{h^+},\ \ \overline{p^+} \le \sigma_1 \overline{p^-},$ 3 $(\mathcal{A}_1 T\overline{g^+})^{\frac{1}{1+\mu}} >$ $\frac{1}{\sigma_1} \max\left\{ \left(\frac{\lVert g^-\rVert_\infty}{aN^2}\right)^{\frac{1}{1+\mu}}, \left(\frac{\lVert h^+\rVert_\infty}{bN^2}\right)^{\frac{1}{1+\lambda}}, \frac{\lVert p^-\rVert_\infty}{cN^2} \right\}$	1 $a+b+c=1,\ \ a,b,c \in (0,1),\ \ 0 < N < \frac{\pi}{T}$ 2 $\overline{h^-} < \sigma_1^{1+\lambda}\overline{h^+},\ \ \sigma_1 \overline{p^-} \ge \overline{p^+},$ 3 $(\mathcal{A}_1 T\overline{g^+})^{\frac{1}{1+\mu}} >$ $\left(\frac{\sigma_1^{1+\lambda}\overline{h^+} - \overline{h^-}}{\left(\overline{g^+} - \sigma_1^{1+\mu}\overline{g^-}\right)\sigma_1^{\lambda-\mu}}\right)^{\frac{1}{\lambda-\mu}} >$ $\frac{1}{\sigma_1} \max\left\{ \left(\frac{\lVert g^-\rVert_\infty}{aN^2}\right)^{\frac{1}{1+\mu}}, \left(\frac{\lVert h^+\rVert_\infty}{bN^2}\right)^{\frac{1}{1+\lambda}}, \frac{\lVert p^-\rVert_\infty}{cN^2} \right\}$
Result	equation admits at least one positive T-periodic solution	equation admits at least one positive T-periodic solution.

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Declarations

The authors declare that they have no conflict of interest concerning the publication of this manuscript.

Data Availability Statements

The data sets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

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