Compactness properties of pseudo-differential operators related with the coupled fractional Fourier transform

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Abstract

In this paper, we introduce the characterization of compactness of the coupled fractional Fourier transform (CFrFT). Few results on compactness of pseudo-differential operators (P.D.O) connected with CFrFT are investigated.

1 Introduction

In 1965, Kohn-Nirenberg and Hörmander [1] were the ones who first introduced the pseudo-differential calculus, and later authors expanded on it, primarily in a local context, to examine local regularity and local solvability of PDEs. The term "pseudo-differential operators" [2, 3, 4, 5] has a fairly broad definition and covers such topics as harmonic analysis, partial differential equation, geometry, mathematical physics, microlocal analysis, time-frequency analysis, imaging, computations, and quantum mechanics. In mathematics, natural sciences, medicine, scientific computing, and engineering, current trends and novel applications are highlighted. The emphasis is on contemporary developments in different branches of engineering, mathematical sciences, the natural sciences, medicine, scientific computers.

Pseudo-differential operators on $\mathbb{R}_+$ are standard or conventional generalizations of partial differential operators or ordinary differential operators and singular integrals.

Many faculties, scientists, Ph.D students and researchers of other field developed

the theory of pseudo-differential operators with the help of different types of integral operators like Fourier transforms (see [6, 7]), Hankel transform (see [8, 9, 10]), Fourier Bessel Transform on $\mathbb{R}_+$ (see [11, 12]), Weinstein transform (see [13]), Laguerre hypergroups (see [14]) and Jacobi differential operators (see [15]), Gyrator transform (see [16]).
The pseudo-differential operators $L(x,y,D'_{x,y})$ and $\mathscr{L}(x,y,D'_{x,y})$ related to $\mathscr{F}_{\theta_1,\theta_2}$ have been defined and discussed some estimations and some inequalities in [17].
In this manuscript, it is to be proved that the operator $L(s,t,D_{s,t}) - \mathscr{L}(s,t,D_{s,t})$ is a compact operator in $L^2(\mathbb{R}^2)$ related to the symbol $l(s,t,u,v)$. If we consider three symbols $l(s,t,u,v)$, $m(s,t,u,v)$ and $n(s,t,u,v) = l(s,t,u,v)m(s,t,u,v)$ and $L(s,t,D_{s,t})$, $M(s,t,D_{s,t})$, $N(s,t,D_{s,t})$ the connected P.D.O. respectively. Then, it is to be proved that $L(s,t,D_{s,t})M(s,t,D_{s,t}) - N(s,t,D_{s,t}) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a also compact operator in this paper.

2 Preliminaries

The fractional Fourier transform was developed in 1980 by Namias [18] as a means of determining the solutions to certain differential equations that sometimes arise in quantum physics. McBride and Kerr [19] further refined his findings by creating an operational calculus for the fractional Fourier transform. Fractional Fourier transform has drawn increased attention in recent years due to its many applications in the fields of image processing, signal analysis, and optics. This transformation is crucial for resolving a number of issues in signal processing, optics, and quantum physics [18, 20, 21, 22, 23, 24, 25, 26, 27, 28]. A variety of mathematical analytic fields have examined the fractional Fourier transform, which is a generalisation of the ordinary Fourier transform. These areas include wavelets [29, 30], pseudo-differential operators [31], and generalised functions [27, 32, 33, 42]. The well-known Fourier transform of a function $\phi \in L_1(\mathbb{R})$, represented by $\widehat{\phi}$, is described as The Fourier transform of a function $phi \in L_1(\mathbb{R})$ is defined by

$$\widehat{\phi}(\eta) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\eta\zeta}\phi(\zeta)d\zeta$$

so that its inverse is given by

$$\phi(\zeta) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\eta\zeta}\widehat{\phi}(\eta)d\eta$$

provided the integrals exist.
We recall the one-dimensional fractional Fourier transform [34, 35, 36, 37] of a function $\phi \in L_1(\mathbb{R})$ with parametre θ, denoted by $(\mathscr{F}_\theta\phi)(\eta) = \widehat{\phi}_\theta(\eta)$ is given in $L_1(\mathbb{R})$ as follows:

$$(\mathscr{F}_\theta\phi)(\eta) = \widehat{\phi}_\theta(\eta) = \int_{\mathbb{R}} K_\theta(\zeta,\eta)\phi(\zeta)d\zeta \tag{1}$$

where the kernel $K_\theta(\zeta,\eta)$ is given by

$$K_\theta(\zeta,\eta)=\begin{cases} C_\theta e^{\frac{i(\zeta^2+\eta^2)\cot\theta}{2}-i\zeta\eta\csc\theta}, & \theta\neq n\pi, n\in\mathbb{Z}\\ \frac{1}{\sqrt{2\pi}}e^{-i\zeta\eta}, & \theta=\frac{\pi}{2}\\ \delta(\zeta-\eta), & \theta=2n\pi\\ \delta(\zeta+\eta), & \theta=(2n+1)\pi \end{cases}$$

$C_\theta=\sqrt{\frac{1-i\cot\theta}{2\pi}}$ and studied some properties of this transform.

The corresponding inversion formula of $(\mathscr{F}_\theta\phi)(\eta)$ is defined in the following ways

$$\phi(\zeta)=\int_{\mathbb{R}}\overline{K_\theta(\zeta,\eta)}(\mathscr{F}_\theta\varphi)(\eta)d\eta \tag{2}$$

$$\overline{K_\theta}(\zeta,\eta)=\overline{C_\theta}e^{\frac{-i(\zeta^2+\eta^2)\cot\theta}{2}+i\zeta\eta\csc\theta}$$

and $\overline{C_\theta}=\sqrt{\frac{1+i\cot\theta}{2\pi}}=C_{-\theta}$.

Hence, $\overline{K_\theta}(\zeta,\eta)=K_{-\theta}(\zeta,\eta)$.

It implies that the inverse of a FrFT with the parameter θ is the FrFT with the parameter $-\theta$.

Exploiting the tensor product of n copies of the one-dimensional fractional Fourier transform each of order $\theta_p,\ p=\ 1,\ 2,\ 3,...,n$ [38], the fractional Fourier transform has been extended to the higher-dimensional transform.

We assume that $\theta=(\theta_1,\theta_2)$, $\mathbf{x}=(x,\eta)$, $\mathbf{y}=(y,\zeta)$, $K_\theta(\mathbf{x},\mathbf{y})=K_{\theta_1}(x,\eta).K_{\theta_2}(y,\zeta)=K_{\theta_1,\theta_2}(x,y,\eta,\zeta)$, where $K_{\theta_1}(x,\eta)$ and $\mathscr{K}_{\theta_2}(y,\zeta)$ defined as above.

The two-dimensional fractional Fourier transform [17, 39, 40, 41] is defined as follows:

$$\begin{aligned}[\mathscr{F}_\theta\phi](\eta,\zeta)=[\mathscr{F}_{\theta_1,\alpha_1}\phi](\eta,\zeta) &= \int_{\mathbb{R}}\int_{\mathbb{R}}K_\theta(\mathbf{x},\mathbf{y})\phi(x,y)dxdy\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}K_{\theta_1}(x,\eta)K_{\theta_2}(y,\zeta)\phi(x,y)dxdy\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}K_{\theta_1,\theta_2}(x,y,\eta,\zeta)\phi(x,y)dxdy. \end{aligned} \tag{3}$$

The corresponding inversion formula of (3) is defined as follows:

$$\phi(x,y)=\int_{\mathbb{R}}\int_{\mathbb{R}}\overline{K_{\theta_1,\theta_2}(x,y,\eta,\zeta)}[\mathscr{F}_{\theta_1,\theta_1}\phi](\eta,\zeta)d\eta d\zeta. \tag{4}$$

It is easy to observe that for $\theta_1=\theta_2=\frac{\pi}{2}$, the two-dimensional fractional Fourier transform $\mathscr{F}_{\theta_1,\theta_2}$ becomes a classical two-dimensional Fourier transform.

Definition 1. *A tempered distribution ϕ belongs to the Sobolev type spce $\mathscr{H}^s(\mathbb{R}\times\mathbb{R})$, and $s\in\mathbb{R}$ if its coupled fractional Fourier transform $\mathscr{F}_{\theta_1,\theta_2}\phi$ corresponding to a locally integrable function $(\mathscr{F}_{\theta_1,\theta_2}\phi)(\xi,\eta)$ over $\mathbb{R}\times\mathbb{R}$ such that*

$${}^{(\theta_1,\theta_2)}\|\phi\|_s=\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\{(1+|\xi|^2)(1+|\eta|^2)\}^{\frac{s}{2}}(\mathscr{F}_{\theta_1,\theta_2}\phi)(\xi,\eta)\right|^2d\eta d\xi\right)^{\frac{1}{2}}<\infty. \tag{5}$$

This space is complete with respect to the norm ${}^{(\theta_1,\theta_2)}\|\phi\|_s$.

Definition 2. *The space $\mathscr{S}(\mathbb{R}\times\mathbb{R})$ is the collection of all complex valued infinitely differentiable functions $\phi(\xi,\eta)\in\mathbb{R}\times\mathbb{R}$ for every choice of $l_1,\ l_2,\ m_1,\ m_2\in\mathbb{N}_0$ which for*

$$\Gamma^{l_1,\ l_2}_{m_1,\ m_2}(\phi)=\sup_{(x,y)\in\mathbb{R}\times\mathbb{R}}\left|x^{l_1}y^{l_2}\frac{\partial^{m_1}}{\partial x^{m_1}}\frac{\partial^{m_2}}{\partial y^{m_2}}\phi(x,y)\right|<\infty. \tag{6}$$

The dual of $\mathscr{S}(\mathbb{R}\times\mathbb{R})$ is denoted by $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$.
If φ is a locally integrable and polynomial growth function on $\mathbb{R}\times\mathbb{R}$, then φ generates a distribution in $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$ as follows:

$$\langle\varphi,\ \phi\rangle=\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(\xi,\eta)\phi(\xi,\eta)d\xi d\eta,\quad\forall\phi\in\mathscr{S}(\mathbb{R}\times\mathbb{R}). \tag{7}$$

The elements of $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$ are known as tempered distributions.

Theorem 1. *Let $K_{\theta_1,\theta_2}(x,y,\eta,\zeta)$ be the kernel of the two-dimensional fractional Fourier transform. Then, for all $\varphi(x,y)\in\mathscr{S}(\mathbb{R}\times\mathbb{R})$, we have*
(i) $D^r_{x,y}K_{\theta_1,\theta_2}(x,y,\eta,\zeta)=\{i(\eta\csc\theta_1+\zeta\csc\theta_2)\}^rK_{\theta_1,\theta_2}(x,y,\eta,\zeta)$,
(ii) $\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(x,y)D^r_{x,y}K_{\theta_1,\theta_2}(x,y,\eta,\zeta)dxdy=\int_{\mathbb{R}}\int_{\mathbb{R}}K_{\theta_1,\theta_2}(x,y,\eta,\zeta)(D'_{x,y})^r\varphi(x,y)dxdy$,
(iii) $\mathscr{F}_{\theta_1,\theta_2}\{(D'_{x,y})^r\varphi(x,y)\}(\eta,\zeta)=\{i(\eta\csc\theta_1+\zeta\csc\theta_2)\}^r(\mathscr{F}_{\theta_1,\theta_2}\varphi(x,y))(\eta,\zeta)$,
for all $r\in\mathbb{N}$, where $D_{x,y}=[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+i(x\cot\theta_1+y\cot\theta_2)]$
and $D'_{x,y}=-[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}-i(x\cot\theta_1+y\cot\theta_2)]$.

Proof. See [39]. □

2.1 | Symbol Classes

In this section, the symbol classes Λ is discussed in [17]. Let $l(s,t,u,v)$ be a complex valued function defined for $s,t,u\neq 0,v\neq 0\in\mathbb{R}$. The function $l(s,t,u,v)\in\mathbb{C}^\infty(\mathbb{R}\times\mathbb{R}\times\mathbb{R}-\{0\}\times\mathbb{R}-\{0\})$ is said to be an element of the class Λ if and only if $l(s,t,t_1u,t_2v)=l(s,t,u,v)$ for $t_1>0,\ t_2>0$, and assume also that

$$\lim_{(|s|,|t|)\to(\infty,\infty)}l(s,t,u,v)=l(\infty,\infty,u,v)$$

exists for $u\neq 0,v\neq 0\in\mathbb{R}$ and $l(\infty,\infty,u,v)$ is a mapping $\mathbb{C}^\infty$-function.

Now we define $l'(s,t,u,v)=l(s,t,u,v)-l(\infty,\infty,u,v)$, and assume the estimates

$$(1+s^2+t^2)^p\left|\frac{\partial^k}{\partial s^k}\frac{\partial^l}{\partial t^l}\frac{\partial^m}{\partial u^m}\frac{\partial^n}{\partial v^n}l'(s,t,u,v)\right|\leq\mathbb{C}_{p,k,l,m,n},\quad\forall s,\ t,\ u\neq 0,\ v\neq 0\in\mathbb{R} \tag{8}$$

here p=1,2,3,.....,k, l, m, n are natural numbers.

Theorem 2. *(i) We get*
$|l(\infty,\infty,\xi,\zeta)-l(\infty,\infty,\delta,\eta)|\leq\mathbb{C}\big((|\xi-\delta|+|\zeta-\eta|)/(|\xi|+|\zeta|+|\delta|+|\eta|)\big)$,

$\forall \xi,\ \zeta,\ \delta,\ \eta$ *arbitray in* $\mathbb{R}-\{0\}$.
(ii) The estimates $(1+x^2csc^2\theta_1+y^2csc^2\theta_2)^p|\mathscr{F}_{\theta_1,\theta_2}(l')(x,y,\xi,\zeta)| \leq \mathbb{M}_p$
$\forall x,\ y,\ \xi \neq 0,\ \zeta \neq 0 \in \mathbb{R},\ p=1,\ 2,\ 3,\ 4,\ 5...;$
(iii) $(1+x^2csc^2\theta_1+y^2csc^2\theta_2)^p|\mathscr{F}_{\theta_1,\theta_2}(l')(x,y,\xi,\zeta)-\mathscr{F}_{\theta_1,\theta_2}(l')(x,y,\delta,\eta)|$
$\leq \mathbb{M}_p(|\xi-\delta|+|\zeta-\eta|)(|\xi|+|\zeta|+|\delta|+|\eta|)^{-1},\ \ \forall \xi,\ \zeta,\ \delta,\ \eta \in \mathbb{R}-\{0\},$
$\forall x,y \in \mathbb{R},\ p=1,2,...to\ \infty$ *being*
$\mathscr{F}_{\theta_1,\theta_2}(l')(x,y,\xi,\zeta)=\int_{\mathbb{R}}\int_{\mathbb{R}}K_{\theta_1,\theta_2}(t,u,x,y)l'(t,u,\xi,\zeta)dtdu,$
$\forall x,y,\xi \neq 0,\zeta \neq 0 \in \mathbb{R}$ *are verified.*

Proof. (i) Similar proof of Theorem 1 (a)[7].

(ii) The proof is introduced in [17].
(iii)
It can be easily proved from (ii), [17]. □

2.2 | Pseudo-Differential Operator $L(x,y,D'_{x,y})$ related to $\mathscr{F}_{\theta_1,\theta_2}$

Let $l(x,y,\xi,\zeta)=l'(x,y,\xi,\zeta)+l(\infty,\infty,\xi,\zeta)$ be a symbol, and, as previously,

$$\mathscr{F}_{\theta_1,\theta_2}(l')(x,y,\xi,\zeta)=\int_{\mathbb{R}}\int_{\mathbb{R}}K_{\theta_1,\theta_2}(t,u,x,y)l'(t,u,\xi,\zeta)dtdu,\ \ \forall x,y,\xi \neq 0,\zeta \neq 0 \in \mathbb{R}.$$

Let us define from [17], for any $\phi \in S^(\mathbb{R}\times\mathbb{R})$ and $x,y \in \mathbb{R}$, a function $\mu(x,y)=(L(x,y,D'_{x,y})\phi)(x,y)$, by

$$(L(x,y,D'_{x,y})\phi)(x,y)=\int_{\mathbb{R}}\int_{\mathbb{R}}K_{\theta_1,\theta_2}(t,u,x,y)G_{\theta_1,\theta_2}(t,u)dtdu, \tag{9}$$

where the function $G_{\theta_1,\theta_2}(t,u)$ is given by

$$G_{\theta_1,\theta_2}(t,u)=l(\infty,\infty,t,u)\widehat{\phi}_{\theta_1,\theta_2}(t,u)+\int_{\mathbb{R}}\int_{\mathbb{R}}\widehat{l'}_{\theta_1,\theta_2}(t-\xi,u-\eta,t,u)\widehat{\phi}_{\theta_1,\theta_2}(\xi,\eta)d\xi d\eta. \tag{10}$$

3 The pseudo-differential operator $\mathscr{L}(x,y,D'_{x,y})$

In this section, we discuss the pseudo-differential operator $\mathscr{L}(x,y,D'_{x,y})$ from [17]. We consider a symbol $l(x,y,\xi,\zeta)$. We introduce an operator $\mathscr{L}(x,y,D'_{x,y})$ of $\mathscr{S}(\mathbb{R}\times\mathbb{R})$ in $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$ by means of the formula

$$[\mathscr{L}(x,y,D'_{x,y})\phi](x,y)=\int_{\mathbb{R}}\int_{\mathbb{R}}K_{\theta_1,\theta_2}(t,u,x,y)\mathscr{H}_{\theta_1,\theta_2}(t,u)dtdu,$$

where, for $\phi \in \mathscr{S}$, the function $\mathscr{H}_{\theta_1,\theta_2}(t,u)$ is defined by the relation

$$\begin{aligned}\mathscr{H}_{\theta_1,\theta_2}(t,u) \ &= \ l(\infty,\infty,t,u)\widehat{\phi}_{\theta_1,\theta_2}(t,u)\\ &+\overline{C_{\theta_1}C_{\theta_2}}\int_{\mathbb{R}}\int_{\mathbb{R}}e^{i(t\lambda_1-\lambda_1^2)cot\theta_1+i(u\lambda_2-\lambda_2^2)cot\theta_2}\\ &\times\widehat{l'}_{\alpha_1,\theta_2}(t-\lambda_1,u-\lambda_2,t,u)\widehat{\phi}_{\theta_1,\theta_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2,\end{aligned}$$

$\forall \phi \in \mathscr{S}$ and $t \neq 0,\ u \neq 0 \in \mathbb{R}$.

4 Some findings about compactness

In 1972, S. Zaidman introduced the citerion of compactness on the Schwartz Space [7]. He also studied the some compactness properties of pseudo-differential operator involving Fourier transform using the well-known criterion of relative compactness of M.Riesz.

We find some results on compactness of pseudo-differential operator involving coupled fractional Fourier transform in this section. In this section, we consider three symbols $l(s,t,u,v),\ m(s,t,u,v)$ and $n(s,t,u,v) = l(s,t,u,v)m(s,t,u,v)$ and $L(s,t,D_{s,t}),\ M(s,t,D_{s,t}),\ N(s,t,D_{s,t})$ the connected P.D.O. respectively.

Theorem 3. *Let $L(s,t,D_{s,t})$ and $\mathscr{L}(s,t,D_{s,t})$ be the pseudo-differential operator related to the symbol $l(s,t,u,v)$. Then the operator $L(s,t,D_{s,t}) - \mathscr{L}(s,t,D_{s,t})$ is a compact operator in $L^2(\mathbb{R}^2)$.*

Proof. We consider $\mathscr{J} = L - \mathscr{L}$. Let $\mathscr{E}$ be a bounded subset of $L^2(\mathbb{R}^2)$. We will want to show that the set $\mathscr{J}(\mathscr{E})$ is relatively compact in $L^2(\mathbb{R}^2)$; or that

$$\mathscr{F}_{\theta_1,\theta_2}[\mathscr{J}(\mathscr{E})] = \{\mathscr{F}_{\theta_1,\theta_2}(\mathscr{J}\psi) : \psi \in \mathscr{E}\}$$

is relatively compact in $L^2(\mathbb{R}^2)$.

We have

$$\|[L(s,t,D_{s,t}) - \mathscr{L}(s,t,D_{s,t})]\psi\|_{\mathscr{H}^1(R^2)} \leq C'_{\theta_1,\theta_2}\|\psi\|_{\mathscr{H}^1(R^2)}.$$

It implies that the set $\{\mathscr{J}\psi : \psi \in \mathscr{E}\}$ is bounded in $\mathscr{H}^1(R^2)$. Hence the set $\{\mathscr{F}_{\theta_1,\theta_2}[\mathscr{J}\psi] : \psi \in \mathscr{E}\}$ is also bounded w.r.to the norm ${}^{\theta_1,\theta_2}\|.\|$. In addition to, we want to show that for every $\sigma_1,\ \sigma_2$,

$$\lim_{(\sigma_1,\sigma_2)\to(0,0)} \int_{-\sigma_1}^{\sigma_1}\int_{-\sigma_2}^{\sigma_2}\left|[\mathscr{F}_{\theta_1,\theta_2}(\mathscr{J}\psi)](\tau'+\sigma_1,\tau''+\sigma_2) - [\mathscr{F}_{\theta_1,\theta_2}(\mathscr{J}\psi)](\tau',\tau'')\right| = 0$$

uniformly for $\psi \in \mathscr{E}$.

Lemma 1. *Let $l(s,t,u,v) = l'(s,t,u,v) + l(\infty,\infty,u,v)$ such that $l(\infty,\infty,u,v) = 0$. Then*

$$\lim_{(|\delta'|,|\delta''|)\to(0,0)} \int_{-\rho_1}^{\rho_1}\int_{-\rho_2}^{\rho_2} |(\widehat{L}_{\theta_1,\theta_2}\psi)(\tau'+\delta',\tau''+\delta'') - (\widehat{L}_{\theta_1,\theta_2}\psi)(\tau',\tau'')|^2 d\tau'\tau'' = 0$$

uniformly for $\psi \in \mathscr{E} \cap \mathscr{S}$.

Proof. We have

$$(\widehat{L}_{\theta_1,\theta_2}\psi)(\tau',\tau'') = \int\int K_{\theta_1,\theta_2}(s,t,\tau',\tau'')l(s,t,\tau',\tau'')\psi(s,t)dsdt$$

and

$$\begin{aligned}&(\widehat{L}_{\theta_1,\theta_2}\psi)(\tau'+\delta',\tau''+\delta'')\\ =\;& \int\int K_{\theta_1,\theta_2}(s,t,\tau'+\delta',\tau''+\delta'')l(s,t,\tau'+\delta',\tau''+\delta'')\psi(s,t)dsdt.\end{aligned}$$

It implies that

$$\begin{aligned}&(\widehat{L}_{\theta_1,\theta_2}\psi)(\tau'+\delta',\tau''+\delta'') - (\widehat{L}_{\theta_1,\theta_2}\psi)(\tau',\tau'')\\ =\;& \int\int K_{\theta_1,\theta_2}(s,t,\tau'+\delta',\tau''+\delta'')l(s,t,\tau'+\delta',\tau''+\delta'')\psi(s,t)dsdt\\ &-\int\int K_{\theta_1,\theta_2}(s,t,\tau',\tau'')l(s,t,\tau',\tau'')\psi(s,t)dsdt\\ =\;& \int\int [K_{\theta_1,\theta_2}(s,t,\tau'+\delta',\tau''+\delta'') - K_{\theta_1,\theta_2}(s,t,\tau',\tau'')]\\ &\times l(s,t,\tau'+\delta',\tau''+\delta'')\psi(s,t)dsdt\\ &+\int\int K_{\theta_1,\theta_2}(s,t,\tau',\tau'')[l(s,t,\tau'+\delta',\tau''+\delta'') - l(s,t,\tau',\tau'')]\psi(s,t)dsdt\\ =\;& \mathscr{Z}_1(\tau',\tau'',\delta',\delta'') + \mathscr{Z}_2(\tau',\tau'',\delta',\delta'') \quad \text{(say)}.\end{aligned}$$

Firstly, we obtain the difference

$$\begin{aligned}&K_{\theta_1,\theta_2}(s,t,\tau'+\delta',\tau''+\delta'') - K_{\theta_1,\theta_2}(s,t,\tau',\tau'')\\ =\;& \sqrt{\frac{1-i\cot\theta_1}{2\pi}}e^{-i\{s^2+(\tau'+\delta')^2\}\cot\theta_1+is(\tau'+\delta')\csc\theta_1}\sqrt{\frac{1-i\cot\theta_2}{2\pi}}\\ &\times e^{-i\{t^2+(\tau''+\delta'')^2\}\cot\theta_2+is(\tau''+\delta'')\csc\theta_2}\\ &-\sqrt{\frac{1-i\cot\theta_1}{2\pi}}e^{-i\{s^2+\tau'^2\}\cot\theta_1+is\tau'\csc\theta_1}\sqrt{\frac{1-i\cot\theta_2}{2\pi}}\\ &\times e^{-i\{t^2+\tau''^2\}\cot\theta_2+it\tau''\csc\theta_2}.\end{aligned}$$

Taking mod on both sides, we get

$$\begin{aligned}&\left|K_{\theta_1,\theta_2}(s,t,\tau'+\delta',\tau''+\delta'') - K_{\theta_1,\theta_2}(s,t,\tau',\tau'')\right|\\ &\leq 2\left|\sqrt{\frac{1-i\cot\theta_1}{2\pi}}\sqrt{\frac{1-i\cot\theta_2}{2\pi}}\right|\\ &= C_{\pi,\theta_1,\theta_2} \quad \text{(say)}.\end{aligned}$$

Now estimating, we get

$$
\begin{aligned}
& |\mathscr{Z}_1(\tau',\tau'',\delta',\delta'')| \\
\leq\ & C_{\pi,\theta_1,\theta_2}\int\int |l(s,t,\tau'+\delta',\tau''+\delta'')||\psi(s,t)|dsdt \\
\leq\ & C_{\pi,\theta_1,\theta_2}\left(\int\int |l(s,t,\tau'+\delta',\tau''+\delta'')|^2 dsdt\right)^{\frac{1}{2}}\left(\int\int |\psi(s,t)|^2 dsdt\right)^{\frac{1}{2}} \\
=\ & C'_{\pi,\theta_1,\theta_2}{}^{(0,0)}||\psi||_0 \qquad (11)
\end{aligned}
$$

because $|l(s,t,\tau',\tau'')|\in L^2(\mathbb{R}^4)$ uniformly with respect to $\tau',\ \tau''\in\mathbb{R}-\{0\}$,

$$
\begin{aligned}
& |\mathscr{Z}_2(\tau',\tau'',\delta',\delta'')| \leq C_{\pi,\theta_1,\theta_2}{}^{(0,0)}||\psi||_0 \\
& \times\left(\int\int |l(s,t,\tau'+\delta',\tau''+\delta'')-l(s,t,\tau',\tau'')|^2 dsdt\right)^{\frac{1}{2}}. \qquad (12)
\end{aligned}
$$

Now, we recall that it follows

$$
(1+|s|^2+|t|^2)^n|l(s,t,\tau',\tau'')|\leq C_n \qquad (13)
$$

and

$$
\begin{aligned}
& (1+|s|^2+|t|^2)^n|l(s,t,\tau'+\delta',\tau''+\delta'')-l(s,t,\tau',\tau'')| \\
& \leq C_n\frac{|\delta'|+|\delta''|}{|\tau'|+|\tau''|+|\tau'+\delta'|+|\tau''+\delta''|},\quad n=1,\ 2,\ 3,........\text{to }\infty. \qquad (14)
\end{aligned}
$$

and therefore we have for every fixed r_1 & r_2, we have

$$
\begin{aligned}
& \int_{-r_1}^{r_1}\int_{-r_2}^{r_2}\left|(\widehat{L}_{\theta_1,\theta_2}\psi)(\tau'+\delta',\tau''+\delta'')-(\widehat{L}_{\theta_1,\theta_2}\psi)(\tau',\tau'')\right|^2 d\tau' d\tau'' \\
& = \int_{-r_1}^{r_1}\int_{-r_2}^{r_2}\left|\mathscr{Z}_1(\tau',\tau'',\delta',\delta'')+\mathscr{Z}_2(\tau',\tau'',\delta',\delta'')\right|^2 d\tau' d\tau'' \\
& \leq 2\int_{-r_1}^{r_1}\int_{-r_2}^{r_2}\left|\mathscr{Z}_1(\tau',\tau'',\delta',\delta'')\right|^2 d\tau' d\tau'' + 2\int_{-r_1}^{r_1}\int_{-r_2}^{r_2}\left|\mathscr{Z}_2(\tau',\tau'',\delta',\delta'')\right|^2 \\
& \times d\tau' d\tau'' \\
& \leq 2(C'_{\pi,\theta_1,\theta_2})^2||\psi||_0^2 4r_1r_2 + 2\int_{-\rho_1}^{\rho_1}\int_{-\rho_2}^{\rho_2}\left|\mathscr{Z}_2(\tau',\tau'',\delta',\delta'')\right|^2 d\tau' d\tau'' \\
& +2\int_{\rho_1\leq|\tau'|\leq r_1}\int_{\rho_2\leq|\tau''|\leq r_2}\left|\mathscr{Z}_2(\tau',\tau'',\delta',\delta'')\right|^2 d\tau' d\tau'', \qquad (15)
\end{aligned}
$$

$\forall\rho_1\geq 0,\ \rho_2\geq 0,\ \rho_1\leq r_1,\ \rho_2\leq r_2$.
For $|\tau'|\leq\rho_1,\ |\tau''|\leq\rho_2$, we estimate $\mathscr{Z}_2(\tau',\tau'',\delta',\delta'')$ in the following way (using

(12) and (13));

$$\begin{aligned}|\mathscr{L}_2(\tau',\tau'',\delta',\delta'')| &\leq 2^{\frac{1}{2}} C_{\pi,\theta_1,\theta_2}{}^{(0,0)} ||\psi||_0 \\ &\times\left(\int\int |l(s,t,\tau'+\delta',\tau''+\delta'')|^2+|l(s,t,\tau',\tau'')|^2 dsdt\right)^{\frac{1}{2}} \\ &\leq 2^{\frac{3}{2}} C_{\pi,\theta_1,\theta_2}{}^{(0,0)} ||\psi||_0 C_n \left(\int\int (1+|s|^2+|t|^2)^{-2n} dsdt\right)^{\frac{1}{2}} \\ &= C_{\pi,\theta_1,\theta_2,n}{}^{(0,0)} ||\psi||_0, \quad \text{where n is sufficiently large.} \quad (16)\end{aligned}$$

For $|\tau'|\geq \rho_1,\ |\tau''|\geq \rho_2$, we use the estimate (deriving from (14))

$$\begin{aligned}|\mathscr{L}_2(\tau',\tau'',\delta',\delta'')| &\leq C_n \left(\int\int (1+|s|^2+|t|^2)^{-2n} dsdt\right)^{\frac{1}{2}(0,0)} ||\psi||_0 (|\delta'|+|\delta''|) \\ &\times(|\tau'|+|\tau''|)^{-1} \qquad (17)\end{aligned}$$

and hence we obtain, using (15), (16) and (17), the inequality

$$\begin{aligned}&\int_{r_1}^{-r_1}\int_{r_2}^{-r_2} \left|(\widehat{L}_{\theta_1,\theta_2}\psi)(\tau'+\delta',\tau''+\delta'')-(\widehat{L}_{\theta_1,\theta_2}\psi)(\tau',\tau'')\right|^2 d\tau' d\tau'' \\ &\leq 8(C'_{\pi,\theta_1,\theta_2})^2 ({}^{(0,0)}||\psi||_0)^2 r_1 r_2 + (C_{\pi,\theta_1,\theta_2,n})^2 ({}^{(0,0)}||\psi||_0)^2 \int_{-r_1}^{r_1}\int_{-r_2}^{r_2} d\tau' d\tau'' \\ &+2(C'_{\pi,\theta_1,\theta_2})^2 ({}^{(0,0)}||\psi||_0)^2 \int_{\rho_1\leq|\tau'|\leq r_1}\int_{\rho_2\leq|\tau''|\leq r_2} \frac{1}{(|\tau'|+|\tau''|)^2} d\tau' d\tau'' \\ &\leq 8(C'_{\pi,\theta_1,\theta_2})^2 ({}^{(0,0)}||\psi||_0)^2 r_1 r_2 + 4(C_{\pi,\theta_1,\theta_2,n})^2 ({}^{(0,0)}||\psi||_0)^2 r_1 r_2 \\ &+2(C'_{\pi,\theta_1,\theta_2})^2 ({}^{(0,0)}||\psi||_0)^2 \frac{1}{(|\rho_1|+|\rho_2|)^2} \\ &= 2(C'_{\pi,\theta_1,\theta_2})^2 ({}^{(0,0)}||\psi||_0)^2 [4r_1 r_2 + \frac{1}{(|\rho_1|+|\rho_2|)^2}] + 4(C_{\pi,\theta_1,\theta_2,n})^2 ||\psi||_0^2 r_1 r_2.\end{aligned}$$

If $\psi\in\mathscr{E}\cap\mathscr{S}$, we take $\varepsilon>0$, and choose at first $r_0'(\varepsilon)$ and $r_0''(\varepsilon)$ such that $4(C_{\pi,\theta_1,\theta_2,n})^2({}^{(0,0)}||\psi||_0)^2 r_1 r_2 \leq \frac{\varepsilon}{2}$.
Once $r_0'(\varepsilon)$ and $r_0''(\varepsilon)$ fixed, such that $2(C'_{\pi,\theta_1,\theta_2})^2({}^{(0,0)}||\psi||_0)^2[4r_1r_2+\frac{1}{(|\rho_1|+|\rho_2|)^2}] < \frac{\varepsilon}{2}$.
We arrive hence for $|\tau'|\leq|\tau_0'|,\ |\tau''|\leq|\tau_0''|$ and $\psi\in\mathscr{E}\cap\mathscr{S}$, at the estimate

$$\begin{aligned}&\int_{r_1}^{-r_1}\int_{r_2}^{-r_2} \left|(\widehat{L}_{\theta_1,\theta_2}\psi)(\tau'+\delta',\tau''+\delta'')-(\widehat{L}_{\theta_1,\theta_2}\psi)(\tau',\tau'')\right|^2 d\tau' d\tau'' \\ &\leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

The Lemma 1 is proved. □

Lemma 2. *We have, in the case of a symbol $l(x,y,\xi,\eta)$ with $l(\infty,\infty,\xi,\eta)\equiv 0$*

$$\lim_{(|\delta'|,|\delta''|)\to(0,0)} \int_{-r_1}^{r_1}\int_{-r_2}^{r_2} |(\widehat{\mathscr{L}_{\theta_1,\theta_2}}\psi)(\tau'+\delta',\tau''+\delta'')-(\widehat{\mathscr{L}_{\theta_1,\theta_2}}\psi)(\tau',\tau'')|^2 d\tau' d\tau'' = 0$$

uniformly for $\psi \in \mathscr{E} \cap \mathscr{S}$, $\forall$ *fixed* $r_1 > 0,\ r_2 > 0$.

Proof. In fact, we have

$$(\widehat{\mathscr{L}_{\theta_1,\theta_2}}\psi)(\tau',\tau'') = \int\int \widehat{l}_{\theta_1,\theta_2}(\tau'-\eta',\tau''-\eta'',\eta',\eta'')\widehat{\psi}_{\theta_1,\theta_2}(\eta',\eta'')d\eta' d\eta''$$

and

$$\begin{aligned}&(\widehat{\mathscr{L}_{\theta_1,\theta_2}}\psi)(\tau'+\delta',\tau''+\delta'')\\ &= \int\int \widehat{l}_{\theta_1,\theta_2}(\tau'+\delta'-\eta',\tau''+\delta''-\eta'',\eta',\eta'')\widehat{\psi}_{\theta_1,\theta_2}(\eta',\eta'')d\eta' d\eta''.\end{aligned}$$

Now, we have

$$\begin{aligned}&(\widehat{\mathscr{L}_{\theta_1,\theta_2}}\psi)(\tau'+\delta',\tau''+\delta'') - (\widehat{\mathscr{L}_{\theta_1,\theta_2}}\psi)(\tau',\tau'')\\ =\ &\int\int \left[\widehat{l}_{\theta_1,\theta_2}(\tau'+\delta'-\eta',\tau''+\delta''-\eta'',\eta',\eta'') - \widehat{l}_{\theta_1,\theta_2}(\tau'-\eta',\tau''-\eta'',\eta',\eta'')\right]\\ &\times \widehat{\psi}_{\theta_1,\theta_2}(\eta',\eta'')d\eta' d\eta''.\end{aligned}$$

It implies that

$$\begin{aligned}&\left|(\widehat{\mathscr{L}_{\theta_1,\theta_2}}\psi)(\tau'+\delta',\tau''+\delta'') - (\widehat{\mathscr{L}_{\theta_1,\theta_2}}\psi)(\tau',\tau'')\right|^2\\ &\leq c\left(\int\int |\widehat{\psi}_{\theta_1,\theta_2}(\eta',\eta'')|^2 d\eta' d\eta''\right)\\ \times\ &\int\int |\widehat{l}_{\theta_1,\theta_2}(\tau'+\delta'-\eta',\tau''+\delta''-\eta'',\eta',\eta'') - \widehat{l}_{\theta_1,\theta_2}(\tau'-\eta',\tau''-\eta'',\eta',\eta'')|^2\\ &\times d\eta' d\eta''\\ =\ &c(^{(0,0)}||\psi||_0)^2\\ \times\ &\int\int |\widehat{l}_{\theta_1,\theta_2}(\tau'+\delta'-\eta',\tau''+\delta''-\eta'',\eta',\eta'') - \widehat{l}_{\theta_1,\theta_2}(\tau'-\eta',\tau''-\eta'',\eta',\eta'')|^2\\ &\times d\eta' d\eta''.\end{aligned} \tag{18}$$

We apply Taylor's formula; we obtain the relation

$$\begin{aligned}&\widehat{l}_{\theta_1,\theta_2}(\tau'+\delta'-\eta',\tau''+\delta''-\eta'',\eta',\eta'') - \widehat{l}_{\theta_1,\theta_2}(\tau'-\eta',\tau''-\eta'',\eta',\eta'')\\ =\ &\left(\delta',\delta'', grad\ \widehat{l}_{\theta_1,\theta_2}(\tau'+t_1\delta'-\eta',\tau''+t_2\delta''-\eta'',\eta',\eta'')\right),\\ &0 < t_1 < 1,\ 0 < t_2 < 1\end{aligned}$$

and therefore the estimate

$$\begin{aligned}&|\widehat{l}_{\theta_1,\theta_2}(\tau'+\delta'-\eta',\tau''+\delta''-\eta'',\eta',\eta'') - \widehat{l}_{\theta_1,\theta_2}(\tau'-\eta',\tau''-\eta'',\eta',\eta'')|\\ \leq\ &|\delta'||\delta''||grad\ \widehat{l}_{\theta_1,\theta_2}(\tau'+t_1\delta'-\eta',\tau''+t_2\delta''-\eta'',\eta',\eta'')|.\end{aligned}$$

Let us remember now that $\widehat{l}_{\theta_1,\theta_2}(\lambda_1,\lambda_2,\eta',\eta'') \in \mathscr{S}(\mathbb{R}^4)$ uniformly for $\eta' \neq 0,\ \eta'' \neq 0 \in \mathbb{R}$ and we get

$$\left|(1+|\lambda_1|^2+|\lambda_2|^2)^n \frac{\partial}{\partial \lambda_1}\frac{\partial}{\partial \lambda_2}\widehat{l}_{\theta_1,\theta_2}(\lambda_1,\lambda_2,\eta',\eta'')\right| \leq C'_n,\ \forall\, \lambda_1,\ \lambda_2 \in \mathbb{R},$$

which gives

$$\begin{aligned}&|grad\ \widehat{l}_{\theta_1,\theta_2}(\tau'+t_1\delta'-\eta',\tau''+t_2\delta''-\eta'',\eta',\eta'')|\\ &\leq C'_n(1+|\tau'+t_1\delta'-\eta'|^2+|\tau'+t_1\delta'-\eta'|^2)^{-n},\end{aligned}$$

$\forall\, n = 1,2,3,4,5,$..........to ∞ and by integrating with respect to η' & η'' we arrive at the result (in estimate (18)). $\square$

Lemma 3. *We have in the case* $l(\infty,\infty,\xi,\eta) \equiv 0$, *that,* $\forall\ \rho_1 > 0$ *and* $\rho_2 > 0$

$$\lim_{(|\delta'|,|\delta''|)\to(0,0)} \int_{-\rho_1}^{\rho_1}\int_{-\rho_2}^{\rho_2} \left|\widehat{L}_{\theta_1,\theta_2}\psi)(\tau'+\delta',\tau''+\delta'')-\widehat{L}_{\theta_1,\theta_2}\psi)(\tau',\tau'')\right|^2 d\tau' d\tau'' = 0$$

$$\lim_{(|\delta'|,|\delta''|)\to(0,0)} \int_{-r_1}^{r_1}\int_{-r_2}^{r_2} |(\widehat{\mathscr{L}_{\theta_1,\theta_2}}\psi)(\tau'+\delta',\tau''+\delta'')-(\widehat{\mathscr{L}_{\theta_1,\theta_2}}\psi)(\tau',\tau'')|^2 d\tau' d\tau'' = 0$$

uniformly for $\psi \in \mathscr{E}$*-bounded set in* $L^2(\mathbb{R}^2)$.

Proof. We have already shown this relation for $\psi \in \mathscr{E} \cap \mathscr{S}$. Let us remember that the space $\mathscr{S}$ is dense in $L^2(\mathbb{R}^2)$. Given $\varepsilon > 0$, and $\mathscr{E}$ a bounded set in $L^2(\mathbb{R}^2)$, there is $\forall \psi \in \mathscr{E}$, an element $\psi_\varepsilon \in \mathscr{S}$, such that ${}^{(0,0)}\|\psi-\psi_\varepsilon\|_0 < \varepsilon$. Hence, for $\psi \in \mathscr{E}$ we have ${}^{(0,0)}\|\psi\|_0 \leq M$, and

$${}^{(0,0)}\|\psi_\varepsilon\|_0 = {}^{(0,0)}\|-[\psi-\psi_\varepsilon]+\psi\|_0 \leq {}^{(0,0)}\|\psi-\psi_\varepsilon\|_0 + {}^{(0,0)}\|\psi\|_0 \leq \varepsilon + M < M+1$$

and therefore the set $\mathscr{E}_1 = \{\psi_\varepsilon : \psi \in \mathscr{E}\}$ is bounded in $L^2(\mathbb{R}^2)$ and included in $\mathscr{S}$. Here we have, for $|\delta'| \leq |\delta'_0(\varepsilon)|$ and $|\delta''| \leq |\delta''_0(\varepsilon)|$ such that in the case $l(\infty,\infty,\xi,\eta) \equiv 0$

$$\int_{-\rho_1}^{\rho_1}\int_{-\rho_2}^{\rho_2} \left|(\widehat{L}_{\theta_1,\theta_2}\psi_\varepsilon)(\tau'+\delta',\tau''+\delta'')-(\widehat{L}_{\theta_1,\theta_2}\psi_\varepsilon)(\tau',\tau'')\right|^2 d\tau' d\tau'' < \varepsilon,\ \forall \psi_\varepsilon \in \mathscr{E}_1$$

$$\int_{-\rho_1}^{\rho_1}\int_{-\rho_2}^{\rho_2} \left|(\widehat{\mathscr{L}_{\theta_1,\theta_2}}\psi_\varepsilon)(\tau'+\delta',\tau''+\delta'')-(\widehat{\mathscr{L}_{\theta_1,\theta_2}}\psi_\varepsilon)(\tau',\tau'')\right|^2 d\tau' d\tau'' < \varepsilon,\ \forall \psi_\varepsilon \in \mathscr{E}_1.$$

Hence, we deduce the inequalities

$$\begin{aligned}
& \int_{-\rho_1}^{\rho_1}\int_{-\rho_2}^{\rho_2}\left|(\widehat{L}_{\theta_1,\theta_2}\psi)(\tau'+\delta',\tau''+\delta'')-(\widehat{L}_{\theta_1,\theta_2}\psi)(\tau',\tau'')\right|^2 d\tau' d\tau'' \\
\leq\ & 3\int_{-\rho_1}^{\rho_1}\int_{-\rho_2}^{\rho_2}\left|(\widehat{L}_{\theta_1,\theta_2}\psi)(\tau'+\delta',\tau''+\delta'')-(\widehat{L}_{\theta_1,\theta_2}\psi_\varepsilon)(\tau'+\delta',\tau''+\delta'')\right|^2 d\tau' d\tau'' \\
& +3\int_{-\rho_1}^{\rho_1}\int_{-\rho_2}^{\rho_2}\left|(\widehat{L}_{\theta_1,\theta_2}\psi_\varepsilon)(\tau'+\delta',\tau''+\delta'')-(\widehat{L}_{\theta_1,\theta_2}\psi_\varepsilon)(\tau',\tau'')\right|^2 d\tau' d\tau'' \\
& +3\int_{-\rho_1}^{\rho_1}\int_{-\rho_2}^{\rho_2}\left|(\widehat{L}_{\theta_1,\theta_2}\psi_\varepsilon)(\tau',\tau'')-(\widehat{L}_{\theta_1,\theta_2}\psi)(\tau',\tau'')\right|^2 d\tau' d\tau'' \\
=\ & 6(^{(0,0)}\|L(\psi-\psi_\varepsilon)\|_0)^2 \\
& +3\int_{-\rho_1}^{\rho_1}\int_{-\rho_2}^{\rho_2}\left|(\widehat{L}_{\theta_1,\theta_2}\psi_\varepsilon)(\tau'+\delta',\tau''+\delta'')-(\widehat{L}_{\theta_1,\theta_2}\psi_\varepsilon)(\tau',\tau'')\right|^2 d\tau' d\tau'' \\
\leq\ & 6c(^{(0,0)}\|L(\psi-\psi_\varepsilon)\|_0)^2 \\
& +3\int_{-\rho_1}^{\rho_1}\int_{-\rho_2}^{\rho_2}\left|(\widehat{L}_{\theta_1,\theta_2}\psi_\varepsilon)(\tau'+\delta',\tau''+\delta'')-(\widehat{L}_{\theta_1,\theta_2}\psi_\varepsilon)(\tau',\tau'')\right|^2 d\tau' d\tau''.
\end{aligned} \tag{19}$$

For $|\delta'| \leq |\delta_0'(\varepsilon)|$ and $|\delta''| \leq |\delta_0''(\varepsilon)|$, the 2^{nd} integral is $< \varepsilon$ and also $6c(^{(0,0)}\|L(\psi-\psi_\varepsilon)\|_0)^2 < 6c\varepsilon^2$; the result is so proven.
The proof for $\mathscr{L}(s,t,D_{s,t})$ is similar. □

Theorem 3 is herewith proven. □

Theorem 4. *If we consider three symbols $l(s,t,u,v)$, $m(s,t,u,v)$ and $n(s,t,u,v) = l(s,t,u,v)m(s,t,u,v)$ and $L(s,t,D_{s,t})$, $M(s,t,D_{s,t})$, $N(s,t,D_{s,t})$ the connected P.D.O. respectively. Then $L(s,t,D_{s,t})M(s,t,D_{s,t}) - N(s,t,D_{s,t}) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a compact operator.*

Proof. We consider the operator $P(s,t,D_{s,t}) = L(s,t,D_{s,t})M(s,t,D_{s,t}) - N(s,t,D_{s,t})$. If $\psi \in \mathscr{E}$, where $\mathscr{E}$ is a bounded set in $L^2(\mathbb{R}^2)$, then $\widehat{P}_{\theta_1,\theta_2}(\mathscr{E})$ is bounded in $L^2(\mathbb{R}^2)$ as easily seen. Therefore, we have to prove that, $\forall\ \rho_1 > 0$ and $\rho_2 > 0$

$$\lim_{(|\delta'|,|\delta''|)\to(0,0)}\int_{-\rho_1}^{\rho_1}\int_{-\rho_2}^{\rho_2}\left|(\widehat{T}_{\theta_1,\theta_2}\psi)(\tau'+\delta',\tau''+\delta'')-(\widehat{T}_{\theta_1,\theta_2}\psi)(\tau',\tau'')\right|^2 d\tau' d\tau'' = 0,$$

uniformly for $\psi \in \mathscr{E}$.
First of all, let us consider the case $l(\infty,\infty,\xi,\eta) \equiv m(\infty,\infty,\xi,\eta) \equiv n(\infty,\infty,\xi,\eta) \equiv 0$. If use Theorem 3, we get, $\forall\ \rho_1 > 0$ and $\rho_2 > 0$

$$\lim_{(|\delta'|,|\delta''|)\to(0,0)}\int_{-\rho_1}^{\rho_1}\int_{-\rho_2}^{\rho_2}\left|(\widehat{N}_{\theta_1,\theta_2}\psi)(\tau'+\delta',\tau''+\delta'')-(\widehat{N}_{\theta_1,\theta_2}\psi)(\tau',\tau'')\right|^2 d\tau' d\tau'' = 0,$$

uniformly for $\psi \in \mathscr{E}$. It is only left to consider

$$\lim_{(|\delta'|,|\delta''|)\to(0,0)} \int_{-\rho_1}^{\rho_1}\int_{-\rho_2}^{\rho_2} \left|(\widehat{LM}_{\theta_1,\theta_2}\psi)(\tau'+\delta',\tau''+\delta'') - (\widehat{LM}_{\theta_1,\theta_2}\psi)(\tau',\tau'')\right|^2 \times d\tau' d\tau''.$$

Let us remember Lemma 3. Then, $\forall\ \varepsilon > 0,\ \exists\ \delta_Q(\varepsilon)$, where Q is a real number, such that

$$\lim_{(|\delta'|,|\delta''|)\to(0,0)} \int_{-\rho_1}^{\rho_1}\int_{-\rho_2}^{\rho_2} \left|(\widehat{L}_{\theta_1,\theta_2}\phi)(\tau'+\delta',\tau''+\delta'') - (\widehat{L}_{\theta_1,\theta_2}\phi)(\tau',\tau'')\right|^2 \times d\tau' d\tau'' \leq \varepsilon, \quad \text{if } |\delta'| < \delta_Q(\varepsilon),\ |\delta''| < \delta_Q(\varepsilon) \text{ and } {}^{(0,0)}||\psi||_0 \leq Q.$$

Remark that if ψ is arbitrary in $L^2(\mathbb{R})$, $\frac{\psi}{{}^{(0,0)}\|\psi\|_0} = \psi'$ is of norm 1, therefore

$$\lim_{(|\delta'|,|\delta''|)\to(0,0)} \int_{-\rho_1}^{\rho_1}\int_{-\rho_2}^{\rho_2} \left|(\widehat{L}_{\theta_1,\theta_2}\psi')(\tau'+\delta',\tau''+\delta'') - (\widehat{L}_{\theta_1,\theta_2}\psi')(\tau',\tau'')\right|^2 \times d\tau' d\tau'' \leq \varepsilon, \quad \text{if } |\delta'| < \delta_1(\varepsilon),\ |\delta''| < \delta_2(\varepsilon).$$

that is

$$\lim_{(|\delta'|,|\delta''|)\to(0,0)} \int_{-\rho_1}^{\rho_1}\int_{-\rho_2}^{\rho_2} \left|(\widehat{L}_{\theta_1,\theta_2}\psi)(\tau'+\delta',\tau''+\delta'') - (\widehat{L}_{\theta_1,\theta_2}\psi)(\tau',\tau'')\right|^2 \times d\tau' d\tau'' \leq \varepsilon^{(0,0)}\|\psi\|_0^2, \quad \text{if } |\delta'| < \delta_1(\varepsilon),\ |\delta''| < \delta_2(\varepsilon),\ \forall\ \psi \in L^2(\mathbb{R}^2).$$

We apply this relation to $LM\psi,\ \psi \in L^2(\mathbb{R}^2)$; we have then

$$\lim_{(|\delta'|,|\delta''|)\to(0,0)} \int_{-\rho_1}^{\rho_1}\int_{-\rho_2}^{\rho_2} \left|(\widehat{LM}_{\theta_1,\theta_2}\psi)(\tau'+\delta',\tau''+\delta'') - (\widehat{LM}_{\theta_1,\theta_2}\psi)(\tau',\tau'')\right|^2 \times d\tau' d\tau'' \leq \varepsilon^{(0,0)}\|M\psi\|_0^2, \quad \text{if } |\delta'| < \delta_1(\varepsilon),\ |\delta''| < \delta_2(\varepsilon),\ \forall\ \psi \in L^2(\mathbb{R}^2).$$

But ${}^{(0,0)}\|M\psi\|_0^2 \leq {}^{(0,0)}\|\psi\|_0^2$; the relation is proven then, as easily seen. □

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6 Conflicts of Interest

There is no potential conflict of interest of author.

7 Date Availability Statement

Not applicable.

8 Conclusion

The main aim of this paper is to be shown that the operator $L(s,t,D_{s,t}) - \mathscr{L}(s,t,D_{s,t})$ is a compact operator in $L^2(\mathbb{R}^2)$ by using three lemmas. $L(s,t,D_{s,t})M(s,t,D_{s,t}) - N(s,t,D_{s,t}) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is also a compact operator.

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