Some results on pseudo-differential operators related coupled fractional Fourier Transform involving Semi-norms & Norms

Abhisekh Shekhar[1,*]

[1,*] Assistant Professor
Department of Mathematics
C.M.Science College, Darbhanga-846004, Bihar, India

October 14, 2025

1

Abstract

In this manuscript, some results on the Pseudo-differential operators (p.d.o.) $L(x,y,D'_{x,y})$ and $\mathscr{L}(x,y,D'_{x,y})$ are found by using semi-norms and norms in Herbitian Spaces, a set of compact operators and $L^2(\mathbb{R}\times\mathbb{R})$ with the symbol classes $\Lambda(\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R})$. In this paper, we also introduce the characterization of norms of the coupled fractional Fourier transform (CFrFT). Few mathematical properties on norms of pseudo-differential operators (P.D.O) connected with CFrFT are investigated. We conclude the manuscript by applying some of axioms to obtain the proofs of the Lemma and Theorems .

Keywords: Coupled fractional Fourier transform, Symbol Classes, pseudo-differential operators.
Subclass:35S05, 46E10, 46E35, 46F05, 47G30, 46F12.

---

[1] Email: abhi08.iitkgp@gmail.com (Corresponding author*)

1 Introduction

The well-known Fourier transform of a function $f \in L_1(\mathbb{R})$, represented by $\widehat{f}$, is described as

$$\widehat{f}(\eta) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\eta\zeta} f(\zeta) d\zeta$$

so that its inverse is given by

$$f(\zeta) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\eta\zeta} \widehat{f}(\eta) d\eta$$

provided the integrals exist.

The fractional Fourier transform , denoted by $\mathscr{F}_\theta$, which is a generalization of the Fourier transform [1]. In 2018, Ahmed I. Zayed introduced a novel generalization of the fractional Fourier transform to two dimensions, which is called the coupled fractional Fourier transform [2] and is denoted by $\mathscr{F}_{\theta_1,\theta_2}$.

More explicitly, the fractional Fourier transform [3, 4, 5, 6] can be viewed as a family of transformations, $\{\mathscr{F}_\theta\}$ indexed by a parameter θ, with $0 \leq \theta \leq 1$, such that $\mathscr{F}_0$ is the identity transformation and $\mathscr{F}_1$, is the standard Fourier transformation.
The fractional Fourier Transform or FrFT of a function $f \in L_1(\mathbb{R})$, is defined by [7, 8, 9, 10]

$$(\mathscr{F}_\theta f)(\eta) = \widehat{f}_\theta(\eta) = \int_{\mathbb{R}} K_\theta(\zeta, \eta) f(\zeta) d\zeta \tag{1}$$

where

$$K_\theta(\zeta,\eta) = \begin{cases} C_\theta e^{\frac{i(\zeta^2+\eta^2)\cot\theta}{2} - i\zeta\eta\csc\theta}, & \theta \neq n\pi, n \in \mathbb{Z} \\ \frac{1}{\sqrt{2\pi}} e^{-i\zeta\eta}, & \theta = \frac{\pi}{2} \\ \delta(\zeta - \eta), & \theta = 2n\pi \\ \delta(\zeta + \eta), & \theta = (2n+1)\pi \end{cases}$$

$C_\theta = \sqrt{\frac{1 - i\cot\theta}{2\pi}}$.
The corresponding inverse of $(\mathscr{F}_\theta \phi)(\eta)$ is

$$f(\zeta) = \int_{\mathbb{R}} \overline{K_\theta(\zeta,\eta)} (\mathscr{F}_\theta f)(\eta) d\eta \tag{2}$$

$$\overline{K_\theta}(\zeta,\eta) = \overline{C_\theta} e^{\frac{-i(\zeta^2+\eta^2)\cot\theta}{2} + i\zeta\eta\csc\theta}$$

and $\overline{C_\theta} = \sqrt{\frac{1+i\cot\theta}{2\pi}} = C_{-\theta}$.
Hence, $\overline{K_\theta}(\zeta,\eta) = K_{-\theta}(\zeta,\eta)$.

In two dimensions, we have

$$[\mathscr{F}_{\theta_1,\theta_2} f](\omega_1,\omega_2) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\theta_1}(t_1,\omega_1)K_{\theta_2}(t_2,\omega_2)f(t_1,t_2)dt_1dt_2. \tag{3}$$

This is the reason this transform is sometime called the coupled two-dimensional fractional Fourier transform (CFrFT) [11].

Definition 1. *We assume that* $\theta = (\theta_1,\theta_2)$, $\mathbf{x} = (x,\eta)$, $\mathbf{y} = (y,\zeta)$, $K_\theta(\mathbf{x},\mathbf{y}) = K_{\theta_1}(x,\eta).K_{\theta_2}(y,\zeta) = K_{\theta_1,\theta_2}(x,y,\eta,\zeta)$, *where* $K_{\theta_1}(x,\eta)$ *and* $\mathscr{K}_{\theta_2}(y,\zeta)$ *defined as above.*
The two-dimensional fractional Fourier transform [2, 12, 13] is introduced as follows:

$$\begin{aligned}[\mathscr{F}_\theta \phi](\eta,\zeta) = [\mathscr{F}_{\theta_1,\alpha_1}\phi](\eta,\zeta) &= \int_{\mathbb{R}}\int_{\mathbb{R}} K_\theta(\mathbf{x},\mathbf{y})\phi(x,y)dxdy \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\theta_1}(x,\eta)K_{\theta_2}(y,\zeta)\phi(x,y)dxdy \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\theta_1,\theta_2}(x,y,\eta,\zeta)\phi(x,y)dxdy. \end{aligned} \tag{4}$$

The inverse of (4) *is*

$$\phi(x,y) = \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{K_{\theta_1,\theta_2}(x,y,\eta,\zeta)}[\mathscr{F}_{\theta_1,\theta_1}\phi](\eta,\zeta)d\eta d\zeta. \tag{5}$$

For $\theta_1 = \theta_2 = \frac{\pi}{2}$, *the two-dimensional fractional Fourier transform* $\mathscr{F}_{\theta_1,\theta_2}$ *becomes a classical two-dimensional Fourier transform.*

Definition 2. *A tempered distribution* ϕ *belongs to the Sobolev type spce* $\mathscr{H}^s(\mathbb{R}\times\mathbb{R})$, *and* $s \in \mathbb{R}$ *if its coupled fractional Fourier transform* $\mathscr{F}_{\theta_1,\theta_2}\phi$ *corresponding to a locally integrable function* $(\mathscr{F}_{\theta_1,\theta_2}\phi)(\xi,\eta)$ *over* $\mathbb{R}\times\mathbb{R}$ *such that*

$$^{(\theta_1,\theta_2)}\|\phi\|_s = \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\{(1+|\xi|^2)(1+|\eta|^2)\}^{\frac{s}{2}}(\mathscr{F}_{\theta_1,\theta_2}\phi)(\xi,\eta)\right|^2 d\eta d\xi\right)^{\frac{1}{2}} < \infty \tag{6}$$

and

$$^{(0,0)}\|\phi\|_0 = \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\phi(\xi,\eta)\right|^2 d\eta d\xi\right)^{\frac{1}{2}} < \infty. \tag{7}$$

Definition 3. *The space* $\mathscr{S}(\mathbb{R}\times\mathbb{R})$ *is the collection of all complex valued infinitely differentiable functions* $\phi(\xi,\eta) \in \mathbb{R}\times\mathbb{R}$ *for every choice of* l_1, l_2, m_1, $m_2 \in \mathbb{N}_0$ *which for*

$$\Gamma^{l_1,\,l_2}_{m_1,\,m_2}(\phi) = \sup_{(x,y)\in\mathbb{R}\times\mathbb{R}} \left| x^{l_1}y^{l_2}\frac{\partial^{m_1}}{\partial x^{m_1}}\frac{\partial^{m_2}}{\partial y^{m_2}}\phi(x,y)\right| < \infty. \tag{8}$$

The dual of $\mathscr{S}(\mathbb{R}\times\mathbb{R})$ *is denoted by* $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$.
Then φ *generates a distribution in* $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$ *as follows:*

$$\langle \varphi,\ \phi\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(\xi,\eta)\phi(\xi,\eta)d\xi d\eta, \quad \forall\phi\in\mathscr{S}(\mathbb{R}\times\mathbb{R}). \tag{9}$$

The elements of $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$ *are known as tempered distributions.*

1.1 | Symbol Classes

Let $l(s,t,u,v)$ be a complex valued function defined in [11] for $s,t,u\neq 0, v\neq 0\in\mathbb{R}$. The function $l(s,t,u,v)\in\mathbb{C}^{\infty}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}-\{0\}\times\mathbb{R}-\{0\})$ is called to be an element of the class Λ if and only if $l(s,t,t_1u,t_2v) = l(s,t,u,v)$ for $t_1>0,\ t_2>0$, and assume also that

$$\lim_{(|s|,|t|)\to(\infty,\infty)} l(s,t,u,v) = l(\infty,\infty,u,v)$$

exists for $u\neq 0, v\neq 0\in\mathbb{R}$ and $l(\infty,\infty,u,v)$ is a mapping $\mathbb{C}^{\infty}$-function.

Now we define $l'(s,t,u,v) = l(s,t,u,v) - l(\infty,\infty,u,v)$, and assume the estimates

$$(1+s^2+t^2)^p\left|\frac{\partial^k}{\partial s^k}\frac{\partial^l}{\partial t^l}\frac{\partial^m}{\partial u^m}\frac{\partial^n}{\partial v^n}l'(s,t,u,v)\right| \leq \mathbb{C}_{p,k,l,m,n}, \quad \forall s,\ t,\ u\neq 0,\ v\neq 0\in\mathbb{R} \tag{10}$$

here p=1,2,3,.....,k, l, m, n are natural numbers.

Theorem 1. *(i) We get*
$|l(\infty,\infty,\xi,\zeta) - l(\infty,\infty,\delta,\eta)| \leq \mathbb{C}\big((|\xi-\delta|+|\zeta-\eta|)/\big(|\xi|+|\zeta|+|\delta|+|\eta|\big)\big)$,
$\forall\xi,\ \zeta,\ \delta,\ \eta$ *arbitray in* $\mathbb{R}-\{0\}$.
(ii) The estimates $(1+x^2csc^2\theta_1+y^2csc^2\theta_2)^p|\mathscr{F}_{\theta_1,\theta_2}(l')(x,y,\xi,\zeta)| \leq \mathbb{M}_p$
$\forall x,\ y,\ \xi\neq 0,\ \zeta\neq 0\in\mathbb{R},\ p=1,\ 2,\ 3,\ 4,\ 5...;$
(iii) $(1+x^2csc^2\theta_1+y^2csc^2\theta_2)^p|\mathscr{F}_{\theta_1,\theta_2}(l')(x,y,\xi,\zeta) - \mathscr{F}_{\theta_1,\theta_2}(l')(x,y,\delta,\eta)|$
$\leq \mathbb{M}_p(|\xi-\delta|+|\zeta-\eta|)(|\xi|+|\zeta|+|\delta|+|\eta|)^{-1},\ \ \forall\xi,\ \zeta,\ \delta,\ \eta\in\mathbb{R}-\{0\}$,
$\forall x,y\in\mathbb{R},\ p=1,2,...$*to* ∞ *being*
$\mathscr{F}_{\theta_1,\theta_2}(l')(x,y,\xi,\zeta) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\theta_1,\theta_2}(t,u,x,y)l'(t,u,\xi,\zeta)dtdu$,
$\forall x,y,\xi\neq 0,\zeta\neq 0\in\mathbb{R}$ *are verified.*

Proof. (i) Similar proof of Theorem 1 (a)[14].

(ii) The proof is available in [11].

(iii) It can be easily proved from (ii), [11]. □

The term "pseudo-differential operators"[15, 16, 17, 18] has a fairly broad definition and covers such topics as harmonic analysis, partial differential equation, geometry, mathematical physics, microlocal analysis, time-frequency analysis, imaging, computations, and quantum mechanics. In mathematics, natural sciences, medicine, scientific computing, and engineering, current trends and novel applications are highlighted. The emphasis is on contemporary developments in different branches of engineering, mathematical sciences, the natural sciences, medicine, scientific computers.

In reality, Kohn-Nirenberg and Hörmander [19] were the ones who first introduced the pseudo-differential calculus.

Pseudo-differential operators on $\mathbb{R}_+$ are standard or conventional generalizations of partial differential operators or ordinary differential operators and singular integrals.

Many faculties, scientists, Ph.D students and researchers of other field developed the theory of pseudo-differential operators with the help of different types of integral operators like Fourier transforms (see [14, 20]), Hankel transform (see [21, 22, 23]), Fourier Bessel Transform on $\mathbb{R}_+$ (see [24, 25]), Weinstein transform (see [26]), Laguerre hypergroups (see [27]) and Jacobi differential operators (see [28]), two dimensional fractional Fourier transform [29], quadratic phase Fourier transform [30].

1.2 | Pseudo-Differential Operator $L(x,y,D'_{x,y})$ related to $\mathscr{F}_{\theta_1,\theta_2}$

Let $l(x,y,\xi,\zeta) = l'(x,y,\xi,\zeta) + l(\infty,\infty,\xi,\zeta)$ be a symbol, and, as previously,

$$\mathscr{F}_{\theta_1,\theta_2}(l')(x,y,\xi,\zeta) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\theta_1,\theta_2}(t,u,x,y)l'(t,u,\xi,\zeta)dtdu, \;\; \forall x,y,\xi \neq 0, \zeta \neq 0 \in \mathbb{R}.$$

Let us define, for any $\phi \in S(\mathbb{R}^2)$ and $x,y \in \mathbb{R}$, a function $\mu(x,y) = (L(x,y,D'_{x,y})\phi)(x,y)$, in [11] by

$$(L(x,y,D'_{x,y})\phi)(x,y) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\theta_1,\theta_2}(t,u,x,y)G_{\theta_1,\theta_2}(t,u)dtdu, \tag{11}$$

where the function $G_{\theta_1,\theta_2}(t,u)$ is given by

$$G_{\theta_1,\theta_2}(t,u) = l(\infty,\infty,t,u)\widehat{\phi}_{\theta_1,\theta_2}(t,u) + \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{l'}_{\theta_1,\theta_2}(t-\xi,u-\eta,t,u)\widehat{\phi}_{\theta_1,\theta_2}(\xi,\eta)d\xi d\eta. \tag{12}$$

1.3 | The pseudo-differential operator $\mathscr{L}(x,y,D'_{x,y})$

We consider a symbol $l(x,y,\xi,\zeta)$. We introduce an operator $\mathscr{L}(x,y,D'_{x,y})$ from

[11], of $\mathscr{S}(\mathbb{R}^2)$ in $\mathscr{S}'(\mathbb{R}^2)$ by means of the formula

$$[\mathscr{L}(x,y,D'_{x,y})\phi](x,y) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\theta_1,\theta_2}(t,u,x,y)\mathscr{H}_{\theta_1,\theta_2}(t,u)dtdu,$$

where, for $\phi \in \mathscr{S}$, the function $\mathscr{H}_{\theta_1,\theta_2}(t,u)$ is defined by the relation

$$\begin{aligned}\mathscr{H}_{\theta_1,\theta_2}(t,u) &= l(\infty,\infty,t,u)\widehat{\phi}_{\theta_1,\theta_2}(t,u)\\ &+\overline{C_{\theta_1}C_{\theta_2}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t\lambda_1-\lambda_1^2)cot\theta_1+i(u\lambda_2-\lambda_2^2)cot\theta_2}\\ &\times\widehat{l'}_{\alpha_1,\theta_2}(t-\lambda_1,u-\lambda_2,t,u)\widehat{\phi}_{\theta_1,\theta_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2,\end{aligned}$$

$\forall\phi \in \mathscr{S}$ and $t \neq 0,\ u \neq 0 \in \mathbb{R}$.

Theorem 2. *Let $l(s,t,u,v)$ be a symbol, $L(s,t,D'_{s,t})$ the associated pseudo-differential operator. Then, for every $\varepsilon > 0$, there ia a semi-norm ${}_{\varepsilon}\|.\|$ on $L^2(\mathbb{R}^2)$, dependent of ε, such that every L^2- bounded sequence contains a subsequence convergent in ${}_{\varepsilon}\|.\|$, and the inequality*

$$^{(0,0)}||L(s,t,D'_{s,t})||_0 \leq (\mathscr{W}+\varepsilon)^{(0,0)}||\phi||_0 + {}_{\varepsilon}\|\phi\|, \quad \forall\phi \in L^2(\mathbb{R}^2) \qquad (13)$$

is satisfied.

Proof. In fact, let us put $m_{\varepsilon}(s,t,u,v) = \sqrt{\mathscr{W}^2 + \overline{l(s,t,u,v)}l(s,t,u,v)+\varepsilon}$ which is still a homogeneous symbol as we can easily see, and besides is

$$m_{\varepsilon}(s,t,u,v) = \overline{m_{\varepsilon}(s,t,u,v)}, \quad \varepsilon > 0,\ s,\ t,\ u \neq 0,\ v \neq 0 \in \mathbb{R}.$$

Let us consider the operator $M_{\varepsilon}(s,t,D'_{s,t})$, $\mathscr{M}_{\varepsilon}(s,t,D'_{s,t})$ associated with $m_{\varepsilon}(s,t,u,v)$ and $\mathscr{L}(s,t,D'_{s,t})$ associated with $\overline{l(s,t,u,v)}$. We get □

Lemma 1. *The linear operator*

$$\mathscr{T}_{\varepsilon} = (\mathscr{W}^2+\varepsilon)I - \overline{\mathscr{L}}\cdot L - \mathscr{M}_{\varepsilon}\cdot M_{\varepsilon} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$$

is a compact operator.

Proof. In fact, we have first of all the relation

$$\begin{aligned}\mathscr{M}_{\varepsilon}\cdot M_{\varepsilon} &= (\mathscr{M}_{\varepsilon}-M_{\varepsilon})M_{\varepsilon}+M_{\varepsilon}^2\\ &= \mathscr{T}_1+M_{\varepsilon}^2,\end{aligned} \qquad (14)$$

where $\mathscr{T}_1 = (\mathscr{M}_{\varepsilon}-M_{\varepsilon})M_{\varepsilon}$ is a compact operator. So we arrive at the relation

$$\mathscr{T}_{\varepsilon} = (\mathscr{W}^2+\varepsilon)I - \overline{\mathscr{L}}\cdot L - \mathscr{T}_1 - M_{\varepsilon}^2.$$

On the other hand, we have

$$\begin{aligned}\overline{\mathscr{M}}\cdot M &= (\overline{\mathscr{M}}-\overline{M})M+\overline{M}\cdot M\\ &= \mathscr{T}_2+\overline{M}\cdot M\end{aligned}$$

where $\mathscr{T}_2 : L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)$ is compact, and hence we get

$$\mathscr{T}_\varepsilon = (\mathscr{W}^2+\varepsilon)I-\overline{L}\cdot L-M_\varepsilon^2-(\mathscr{T}_1+\mathscr{T}_2).$$

Finally, we have $\mathscr{M}_\varepsilon\cdot M_\varepsilon-((\mathscr{W}^2+\varepsilon)I-(\bar{l})l(s,t,D'_{s,t}))=\mathscr{T}_3 : L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)$ is compact and hence we derrive

$$M_\varepsilon(s,t,D'_{s,t})M_\varepsilon(s,t,D'_{s,t})=M_\varepsilon^2(s,t,D'_{s,t})=(\mathscr{W}^2+\varepsilon)I-(\bar{l}l)(s,t,D'_{s,t})+\mathscr{T}_3$$

and therefore

$$\begin{aligned}\mathscr{T}_\varepsilon &= (\mathscr{W}^2+\varepsilon)I-\overline{L}\cdot L-(\mathscr{W}^2+\varepsilon)I+(\bar{l}l)(s,t,D'_{s,t})-(\mathscr{T}_1+\mathscr{T}_2+\mathscr{T}_3)\\ &= (\bar{l}l)(s,t,D'_{s,t})-\overline{L}\cdot L-(\mathscr{T}_1+\mathscr{T}_2+\mathscr{T}_3)=\mathscr{T}_0\end{aligned}$$

where $\mathscr{T}_0 : L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)$ is compact.
Hence, Lemma 1 is proved. □

Lemma 2. *Given arbitrary $\varepsilon>0$, we have the relation*

$$Re(\mathscr{T}_\varepsilon\phi,\phi)_0+\varepsilon({}^{(0,0)}\|\phi\|_0)^2\geq-\frac{1}{4\varepsilon}({}^{(0,0)}\|\mathscr{T}_\varepsilon\phi\|_0)^2,\quad\forall\ \phi\in L^2(\mathbb{R}^2).$$

Proof. In fact, we have

$$\begin{aligned}\left|\mathrm{Re}(\mathscr{T}_\varepsilon\phi,\phi)_0\right| &\leq {}^{(0,0)}\|\mathscr{T}_\varepsilon\phi\|_0{}^{(0,0)}\|\phi\|_0\\ &= \frac{1}{2\sqrt{\varepsilon}}{}^{(0,0)}\|\mathscr{T}_\varepsilon\phi\|_0 2\sqrt{\varepsilon}{}^{(0,0)}\|\phi\|_0\\ &\leq \varepsilon({}^{(0,0)}\|\phi\|_0)^2+\frac{1}{4\varepsilon}({}^{(0,0)}\|\mathscr{T}_\varepsilon\phi\|_0)^2\end{aligned}$$

and consequently

$$\left|\mathrm{Re}(\mathscr{T}_\varepsilon\phi,\phi)_0\right|\geq-\varepsilon({}^{(0,0)}\|\phi\|_0)^2-\frac{1}{4\varepsilon}({}^{(0,0)}\|\mathscr{T}_\varepsilon\phi\|_0)^2,$$

follows. □

Lemma 3. *We have the relation, $\forall\varepsilon>0$*

$$({}^{(0,0)}||L(s,t,D'_{s,t})\phi||_0)^2\leq(\mathscr{W}^2+2\varepsilon)({}^{(0,0)}\|\phi\|_0)^2+\frac{1}{4\varepsilon}({}^{(0,0)}\|\mathscr{T}_\varepsilon\phi\|_0)^2,\quad\forall\phi\in L^2(\mathbb{R}^2).$$

Proof. From Lemma 2. We get

$$\begin{aligned}\langle \mathscr{T}_\varepsilon\phi,\phi\rangle &= (\mathscr{W}^2+\varepsilon)\left({}^{(0,0)}\|\phi\|_0\right)^2-\left({}^{(0,0)}\|L(s,t,D'_{s,t})\|_0\right)^2\\ &\quad -\left({}^{(0,0)}\|M_\varepsilon(s,t,D'_{s,t})\|\right)^2;\end{aligned}$$

$\langle \mathscr{T}_\varepsilon\phi,\phi\rangle$ is hence real-valued.
Applying Lemma 2, the estimate

$$\begin{aligned}\langle \mathscr{T}_\varepsilon\phi,\phi\rangle &= (\mathscr{W}^2+\varepsilon)\left({}^{(0,0)}\|\phi\|_0\right)^2-\left({}^{(0,0)}\|L(s,t,D'_{s,t})\phi\|_0\right)^2\\ &\quad -\left({}^{(0,0)}\|M_\varepsilon(s,t,D'_{s,t})\|\right)^2\geq-\frac{1}{4\varepsilon}\left({}^{(0,0)}\|\mathscr{T}_\varepsilon\phi\|_0\right)^2\end{aligned}$$

and therefore

$$\left({}^{(0,0)}\|L\phi\|_0\right)^2+\left({}^{(0,0)}\|M_\varepsilon\phi\|\right)^2\leq(\mathscr{W}^2+\varepsilon)\left({}^{(0,0)}\|\phi\|_0\right)^2+\frac{1}{4\varepsilon}\left({}^{(0,0)}\|\mathscr{T}_\varepsilon\phi\|_0\right)^2$$

and hence

$$\left({}^{(0,0)}\|L\phi\|_0\right)^2\leq(\mathscr{W}^2+\varepsilon)\left({}^{(0,0)}\|\phi\|_0\right)^2+\frac{1}{4\varepsilon}\left({}^{(0,0)}\|\mathscr{T}_\varepsilon\phi\|_0\right)^2$$

which proves Lemma3.

Extracting the square root and for $\sqrt{\kappa+\tau}\leq\sqrt{\kappa}+\sqrt{\tau},\ \kappa>0,\ \tau>0$, we have

$${}^{(0,0)}\|L\phi\|_0\leq(\mathscr{W}+\sqrt{2\varepsilon})^{(0,0)}\|\phi\|_0+\frac{1}{2\sqrt{\varepsilon}}{}^{(0,0)}\|\mathscr{T}_\varepsilon\phi\|_0.$$

Theorem 2 is proved if we put ${}_\varepsilon\|\phi\|=w_\varepsilon{}^{(0,0)}\|\mathscr{T}_\varepsilon\phi\|_0$ and if we observe that $\mathscr{T}_\varepsilon$ being compact in $L^2(\mathbb{R}^2)$ the semi-norm ${}_\varepsilon\|\phi\|=w_\varepsilon{}^{(0,0)}\|\mathscr{T}_\varepsilon\phi\|_0$ satisfies the required properties. □

Theorem 3. *Let $\mathscr{H}$ be a Hilbertian space, and $L\in\mathscr{L}(\mathscr{H}:\mathscr{H})$. Let us assume that $\forall\ \varepsilon>0$, There exists a seminorm ${}^{(\theta_1,\theta_2)\varepsilon}\|\cdot\|_s$ on $\mathscr{H}$ such that ${}^{(\theta_1,\theta_2)}\|\cdot\|_s$ is relatively compact with respect to ${}^{(\theta_1,\theta_2)\varepsilon}\|\cdot\|_s$ and such that ${}^{(\theta_1,\theta_2)\varepsilon}\|\phi\|_s\leq w^{(\theta_1,\theta_2)}\|\phi\|_s$, $\forall\ \phi\in\mathscr{H}$ and*

$${}^{(\theta_1,\theta_2)}\|L\phi\|_s\leq(W+\varepsilon)^{(\theta_1,\theta_2)}\|\phi\|_s+{}^{(\theta_1,\theta_2)\varepsilon}\|\phi\|_s,\quad\forall\phi\in\mathscr{H}.$$

Then

$$\inf\{{}^{(\theta_1,\theta_2)}\|L+\mathscr{T}\|_s:\mathscr{T}\in\mathscr{C}_c\}\leq\mathscr{W}.$$

Proof. For every $\varepsilon > 0$ we get a compact operator $\mathscr{T}_\varepsilon$ in $\mathscr{H}$, with

$$^{(\theta_1,\theta_2)}\|(L-\mathscr{T}_\varepsilon)\phi\|_s \leq (\mathscr{W}+\varepsilon)^{(\theta_1,\theta_2)}\|\phi\|_s, \quad \forall\ \phi\ \in \mathscr{H}.$$

Let be $\mathscr{H}_\varepsilon \subset \mathscr{H}$; for $\phi \in \mathscr{H}_\varepsilon$ we have, $^{(\theta_1,\theta_2)\varepsilon}\|\phi\|_s \leq \varepsilon^{(\theta_1,\theta_2)}\|\phi\|_s$ and $\mathscr{H}_\varepsilon^\perp$ of dimension $\mathscr{N}_\varepsilon-$finite.
Let us put $\mathscr{P}_\varepsilon$ the orthogonal Projection on $\mathscr{H}_\varepsilon$; hence, $\mathscr{I}-\mathscr{P}_\varepsilon$ projects on a space of finite dimension and is therefore compact $\mathscr{I}-\mathscr{P}_\varepsilon : \mathscr{H} \to \mathscr{H}$.
Hence, we put $\mathscr{T}_\varepsilon = L(\mathscr{I}-\mathscr{P}_\varepsilon)$; we get:

$$^{(\theta_1,\theta_2)}\|(L-\mathscr{T}_\varepsilon)\phi\|_s = {}^{(\theta_1,\theta_2)}\|(L\mathscr{P}_\varepsilon)\phi\|_s, \quad \forall\ \phi \in \mathscr{H}.$$

We obtain

$$^{(\theta_1,\theta_2)}\|(L-\mathscr{T}_\varepsilon)\phi\|_s \leq (\mathscr{W}+\varepsilon)^{(\theta_1,\theta_2)}\|\mathscr{P}_\varepsilon\phi\|_s + {}^{(\theta_1,\theta_2)\varepsilon}\|\mathscr{P}_\varepsilon\phi\|_s, \quad \forall\ \phi \in \mathscr{H}.$$

Being now $\mathscr{P}_\varepsilon\phi \in \mathscr{H}_\varepsilon$, we have

$$^{(\theta_1,\theta_2)\varepsilon}\|\mathscr{P}_\varepsilon\phi\|_s \leq \varepsilon \times {}^{(\theta_1,\theta_2)}\|\mathscr{P}_\varepsilon\phi\|_s \leq \varepsilon \times {}^{(\theta_1,\theta_2)}\|\phi\|_s \tag{15}$$

therefore we get,

$$^{(\theta_1,\theta_2)}\|(L-\mathscr{T}_\varepsilon)\phi\|_s \leq (\mathscr{W}+2\varepsilon)^{(\theta_1,\theta_2)}\|\phi\|_s \tag{16}$$

The proof is completed. □

Theorem 4. *Let $l(s,t,u,v)$ be a symbol and $\mathscr{W} = max\{|l(s,t,u,v)| : |u| = |v| = 1,\ s,\ t \in \mathbb{R}\}$; let $L(s,t,D'_{s,t})$ be the associated pseudo-differential operator; let $L(s,t,D'_{s,t})$ be the associated pseudo-differential operator. Let $\mathscr{C}_c$=$\{\mathscr{T} : \mathscr{T} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a linear compact operator$\}$. Then we have the upper estimates*

$$\inf\{\|L(s,t,D'_{s,t})+\mathscr{T}\| : \mathscr{T} \in \mathscr{C}_c\} \leq \mathscr{W}$$

$$\inf\{\|\mathscr{L}(s,t,D'_{s,t})+\mathscr{T}\| : \mathscr{T} \in \mathscr{C}_c\} \leq \mathscr{W}.$$

.

Proof. Applying Theorems 2 and 3, Theorem 4 is proved. □

2 A few further estimations

In this section, we consider the latter applications. We shall prove here the followings.

Theorem 5. *Let $l(s,t,u,v)$ be a symbol defined for $s,\ t,\ u \neq 0,\ v \neq 0 \in \mathbb{R}$, $\mho$ an open set, and $\mathscr{W}_{\mho} = \max\{|l(s,t,u,v)| : s,\ t \in \mho$ and $|u| = |v| = 1\}$. Then, for every $\varepsilon > 0$, constant $\Bbbk_\varepsilon$ and*

$$^{(0,0)}\|L(s,t,D'_{s,t})\phi\|_0 \leq (\mathscr{W}_{\mho} + \varepsilon)^{(0,0)}\|\phi\|_0 + \Bbbk_\varepsilon\ ^{(\theta_1,\theta_2)}\|\phi\|_{-\frac{1}{2}}, \quad \forall\ \phi \in C_0^\infty(\overline{\mho}) \tag{17}$$

be satisfied.

Proof. Firstly we have to prove Lemma4.

Lemma 4. *Let $l(s,t,u,v)$ be a symbol, $\mho$ an open subset of $\mathbb{R}$, $\mathscr{W}_{\mho} = \max\{|l(s,t,u,v)| : s,\ t \in \mho$ and $|u| = |v| = 1\}$. Then, $\forall\ \varepsilon > 0$ there is an open set $\overline{\mho} \subset \mho_\varepsilon$ such that the relation $\mathscr{W}_{\mho_\varepsilon} \leq \mathscr{W}_{\mho} + \varepsilon$ is satisfied.*

Proof. Actually, we have, for every $s_0,\ t_0 \in \mathbb{R}$, $|l(s,t,u,v) - l(s_0,t_0,u,v)| \leq \varepsilon$ if $|s - s_0| < \delta'_\varepsilon,\ |t - t_0| < \delta''_\varepsilon$ and $u \neq 0,\ v \neq 0 \in \mathbb{R}$; here δ'_ε and δ''_ε are independent of s_0 and t_0 respectively. Let us consider here if $\partial\mho$ is the boundary of $\mho$, for every $s_0,\ t_0 \in \partial\mho$ the square $\{(s,t) : |s - s_0| < \delta'_\varepsilon$ and $|t - t_0| < \delta''_\varepsilon\}$.
Let us take

$$\mho_\varepsilon = \mho \cup \{\mathbb{S}(s_0,t_0,\delta'_\varepsilon,\delta''_\varepsilon) : s_0,t_0 \in \partial\mho\};$$

where $\mathbb{S}(s_0,t_0,\delta'_\varepsilon,\delta''_\varepsilon) = \{(s_0,t_0) : |s - s_0| \leq \delta'_\varepsilon$ and $|t - t_0| \leq \delta''_\varepsilon\}$.
Therefore, if $\upsilon,\ \vartheta \in \mho_\varepsilon$, we have $\upsilon,\ \vartheta \in \mho$ or $\upsilon,\ \vartheta \in \mathbb{S}(s^*,t^*,\delta'_\varepsilon,\delta''_\varepsilon)$ for a certain $s^*,\ t^* \in \partial\mho$. In the first case, we have

$$|l(\upsilon,\vartheta,u,v)| \leq \max\{|l(s,t,u,v)| : |u| = |v| = 1, s,t \in \overline{\mho}\} = \mathscr{W}_{\mho}.$$

In the 2^{nd} case, we get

$$|l(\upsilon,\vartheta,u,v)| \leq |l(\upsilon,\vartheta,u,v) - l(\upsilon^*,\vartheta^*,u,v)| + |l(\upsilon^*,\vartheta^*,u,v)| \leq \varepsilon + \mathscr{W}_{\mho}.$$

Hence, for every $\upsilon,\vartheta \in \mho_\varepsilon,\ u \neq 0,\ v \neq 0 \in \mathbb{R}$ we get $|l(\upsilon,\vartheta,u,v)| \leq \varepsilon + \mathscr{W}_{\mho}$.
Thus, $\mathscr{W}_{\mho_\varepsilon} \leq \mathscr{W}_{\mho} + \varepsilon$. □

Proof of the Theorem 5: Given $\varepsilon > 0$, and $\phi \in C_0^\infty(\overline{\mho})$ we build $\mho_\varepsilon$ given in the Lemma 4. There exists also, a function $\Psi_\varepsilon(\upsilon,\vartheta) \in C_0^\infty(\mathbb{R}^2)$, such that

$$\Psi_\varepsilon(\upsilon,\vartheta) = \begin{cases} 1, & \forall\ (\upsilon,\vartheta) \in supp\phi, \\ 0, & \forall\ (\upsilon,\vartheta) \notin \mho_\varepsilon. \end{cases}$$

Obviously $\Psi_\varepsilon(s,t)$ is a symbol, and $\Phi_\varepsilon(s,t,u,v)=\Psi_\varepsilon(s,t)ll(s,t,u,v)$ is another symbol. Furthermore $\Phi_\varepsilon(s,t,u,v)=0$ if $s,\ t\in\mho_\varepsilon^c$; hence we have

$$\begin{aligned}&\max\{|\Phi_\varepsilon(s,t,u,v)|:\ s,\ t\in\mathbb{R}\ \mathit{and}\ |u|=|v|=1\}\\&\le\max\{|l(s,t,u,v)|:\ s,\ t\in\mho_\varepsilon\ \mathit{and}\ |u|=|v|=1\}=\mathscr{W}_\mho\le\mathscr{W}_{\mho_\varepsilon}+\varepsilon.\end{aligned}$$

We define $\Upsilon_\varepsilon(s,t,D'_{s,t})$ the pseudo-differential operator associated with $\Phi_\varepsilon(s,t,u,v)$. We have

$$\Upsilon_\varepsilon(s,t,D'_{s,t})=L(s,t,D'_{s,t})(\Psi_\varepsilon(s,t)).$$

Actually,

$$\begin{aligned}&\mathscr{F}_{\theta_1,\theta_1}[\Upsilon_\varepsilon(s,t,D'_{s,t})\phi](u,v)=\int_{\mathbb{R}}\int_{\mathbb{R}}K_{\theta_1,\theta_2}(s,t,u,v)[l(s,t,u,v)\Psi_\varepsilon(s,t)]\phi(s,t)dsdt\\=\ \ &\mathscr{F}_{\theta_1,\theta_1}[L(s,t,D'_{s,t})(\Psi_\varepsilon\phi)](u,v),\ \ \forall\ \phi\in\mathscr{S},\ \forall\ u\neq0,\ v\neq0\in\mathbb{R}.\end{aligned}$$

Hence we get

$$\Upsilon_\varepsilon(s,t,D'_{s,t})\phi=L(s,t,D'_{s,t})(\Psi_\varepsilon(s,t)\phi(s,t)),\quad\forall\ \phi\in\mathscr{S}(\mathbb{R}^2).$$

Now we have decomposition

$$\phi(s,t)=\Psi_\varepsilon(s,t)\phi(s,t)+\{1-\Psi_\varepsilon(s,t)\}\phi(s,t)$$

and

$$\begin{aligned}L(s,t,D'_{s,t})\phi&=L(s,t,D'_{s,t})\Psi_\varepsilon\phi+L(s,t,D'_{s,t})\{1-\Psi_\varepsilon(s,t)\}\phi\\&=\Upsilon_\varepsilon(s,t,D'_{s,t})\phi+L(s,t,D'_{s,t})\{1-\Psi_\varepsilon(s,t)\}\phi,\end{aligned}$$

as it is $1-\Psi_\varepsilon(s,t)=0$ *on* $supp\phi$, then it is $(1-\Psi_\varepsilon(s,t))\phi(s,t)=0$ *on* $\mathbb{R}^2$, and therefore

$$L(s,t,D'_{s,t})\phi=\Upsilon_\varepsilon(s,t,D'_{s,t})\phi.$$

Hence, we obtain

$$\begin{aligned}&{}^{(0,0)}\|L(s,t,D'_{s,t})\phi\|_0={}^{(0,0)}\|\Upsilon_\varepsilon(s,t,D'_{s,t})\phi\|_0\\&\le(\max\{|\Phi_\varepsilon(s,t,u,v)|:s,\ t\in\mathbb{R}\ \mathit{and}\ |u|=|v|=1\}+\varepsilon)^{(0,0)}\|\phi\|_0+\Bbbk_\varepsilon{}^{(\theta_1,\theta_2)}\|\phi\|_{-\frac{1}{2}}\\&\le(\mathscr{W}_\varepsilon+2\varepsilon)^{(0,0)}\|\phi\|_0+\Bbbk_\varepsilon{}^{(\theta_1,\theta_2)}\|\phi\|_{-\frac{1}{2}}.\end{aligned}$$

This completes the proof of the Throrem 5. □

Theorem 6. *Let* $l(s,t,u,v)$ *be a symbol, and* $l(s_0,t_0,u_0,v_0) = \Bbbk_0$ *for a certain* $s_0,\ t_0 \in \mathbb{R},\ |u_0| = |v_0| = 1$. *Then* $\forall\ \varepsilon > 0,\ \exists\ \phi_\varepsilon(s,t) \in C_0^\infty(\mathbb{R}^2)$, *such that* ${}^{(0,0)}\|\phi\|_0 \neq 0$ *and estimates*

$$\left|{}^{(0,0)}\|L(s,t,D'_{s,t})\phi_\varepsilon\|_0 - \Bbbk_0{}^{(0,0)}\|\phi_\varepsilon\|_0\right| \leq \varepsilon\ {}^{(0,0)}\|\phi_\varepsilon\|_0 \tag{18}$$

$${}^{(\theta_1,\theta_2)}\|\phi_\varepsilon\|_{-1} \leq \varepsilon\ {}^{(0,0)}\|\phi_\varepsilon\|_0 \tag{19}$$

are satisfied.

Corollary to the above Theorem 6**:** Let $l(s,t,u,v)$ be a symbol such that the estimate

$${}^{(0,0)}\|\phi\|_0 \leq \Bbbk'\left({}^{(0,0)}\|L(s,t,D'_{s,t})\phi\|_0 + {}^{(\theta_1,\theta_2)}\|\phi\|_{-1}\right),\ \ \forall\ \phi \in \mathscr{S}(\mathbb{R}^2), \tag{20}$$

is satisfied.

Proof. $\exists$ a real number $\delta > 0$, and

$$|l(s,t,u,v)| > \delta > 0,\ \forall\ s,\ t \in \mathbb{R},\ u,\ v \in \mathbb{R} - \{0\}.$$

In fact, otherwise we could find the sequences $\{s_n\},\ \{t_n\} \in \mathbb{R}$ and $\{u_n,\ v_n\}$ on the unit interval, such that $|l(s_n,t_n,u_n,v_n)| \leq \frac{1}{n},\ n = 1,2,3,\dots$. Then, take $\phi_n(s,t) \in C_0^\infty(\mathbb{R}^2)$ corresponding to $\varepsilon_n = \frac{1}{n}$. We get

$${}^{(0,0)}\|\phi_n\|_0 \leq \Bbbk'\left({}^{(0,0)}\|L(s,t,D'_{s,t})\phi_n\|_0 + {}^{(\theta_1,\theta_2)}\|\phi_n\|_{-1}\right),\ \ \forall\ \phi \in \mathscr{S}(\mathbb{R}^2),$$

and applying (18) we deduce

$${}^{(0,0)}\|\phi_n\|_0 \leq \Bbbk'\left((|l(s_n,t_n,u_n,v_n)|^{(0,0)}\|\phi_n\|_0 + \frac{1}{n}{}^{(0,0)}\|\phi_n\|_0 + \frac{1}{n}{}^{(0,0)}\|\phi_n\|_0)\right)$$

(when (19) is also used): it follows $1 \leq 3\Bbbk'/n,\ n = 1,2,3,\dots$, which is impossible.

We have: $\inf\{{}^{(\theta_1,\theta_2)}\|L + \mathscr{T}\|_s : \mathscr{C}_{-1}\} = w^\star < W$, there could be taken w such that $w^\star < w < W$ at least one $\mathscr{T}_n \in \mathscr{C}_{-1}$ so that

$$w^\star \leq {}^{(\theta_1,\theta_2)}\|L + \mathscr{T}_n\|_s \leq w < W$$

and therefore

$$w^\star \leq \sup\left\{\frac{1}{{}^{(0,0)}\|\phi\|_0}{}^{(0,0)}\|(L + \mathscr{T}_n)\phi\|_0 : \phi \in L^2(\mathbb{R}^2)\right\} \leq w < W$$

whence ${}^{(0,0)}\|(L+\mathscr{T}_n)\phi\|_0 \leq w^{(0,0)}\|\phi\|_0,\ \forall\ \phi \in L^2(\mathbb{R}^2)$.
Being $w < W = \max\{|l(s,t,u,v)| : s,\ t \in \mathbb{R},\ \&\ |u| = |v| = 1\}$ we find at least two $s_0,\ t_0 \in \mathbb{R}$ and $u_0,\ v_0,\ |u_0| = |v_0| = 1$ such that $w < |l(s_0,t_0,u_0,v_0)| = \Bbbk_0' < W$.
We find $\phi_\varepsilon(s,t) \in C_0^\infty(\mathbb{R}^2)$ such that

$$-\varepsilon^{(0,0)}\|\phi_\varepsilon\|_0 \leq {}^{(0,0)}\|L\phi_\varepsilon\|_0 - \Bbbk_0'{}^{(0,0)}\|\phi_\varepsilon\|_0$$

or

$$\begin{aligned}
(\Bbbk_0' - \varepsilon)^{(0,0)}\|\phi_\varepsilon\|_0 &\leq {}^{(0,0)}\|L\phi_\varepsilon\|_0 \\
&= {}^{(0,0)}\|(L+\mathscr{T}_n)\phi_\varepsilon - \mathscr{T}_n\phi_\varepsilon\|_0 \\
&\leq {}^{(0,0)}\|(L+\mathscr{T}_n)\phi_\varepsilon\|_0 + {}^{(0,0)}\|\mathscr{T}_n\phi_\varepsilon\|_0 \\
&\leq w^{(0,0)}\|\phi_\varepsilon\|_0 + \Bbbk'^{(\theta_1,\theta_2)}\|\phi\|_{-1} \\
&\leq w^{(0,0)}\|\phi_\varepsilon\|_0 + \Bbbk' \cdot \varepsilon^{(0,0)}\|\phi_\varepsilon\|_0 \\
&= (w + \Bbbk_\varepsilon')^{(0,0)}\|\phi_\varepsilon\|_0
\end{aligned}$$

and being ${}^{(0,0)}\|\phi_\varepsilon\|_0 \neq 0$ we get, $\forall\ \varepsilon$

$$\Bbbk_0' - \varepsilon < w + \Bbbk' \cdot \varepsilon$$

and

$$w < \Bbbk_0'$$

which is impossible.

Proof of Theorem 6: Let us take ε'; we have $|l(s,t,u,v) - l(s_0,t_0,u,v)| \leq \varepsilon'$ if $|s-s_0| < \delta_\varepsilon'$ *and* $|t-t_0| < \delta_\varepsilon'$, $u \neq 0,\ v \neq 0 \in \mathbb{R}$. We consider a function $\psi_{\varepsilon'} \in C_0^\infty$ with support contained in the square $\{(s,t) : |s-s_0| \leq \delta_{\varepsilon'}\ \textit{and}\ |t-t_0| \leq \delta_{\varepsilon'}\}$, and the sequence

$$\phi_{n,\varepsilon'}(s,t) = e^{\frac{i}{2}[-2n(uu_0 cot\theta_1 + vv_0\cot\theta_2)] + in(su_0 csc\theta_1 + tv_0\csc\theta_2)}\psi_{\varepsilon'}(s,t) \tag{21}$$

where by hpothesis is

$$|l(s_0,t_0,u_0,v_0)| = \Bbbk_0' \text{ and } |u_0| = |v_0| = 1.$$

Let be $\Psi(\lambda,\mu) \in C_0^\infty(\mathbb{R}^2)$ such that

$$\Psi(\lambda,\mu) = \begin{cases} 1, & |\lambda| \leq 1 \text{ and } |\mu| \leq 1, \\ 0, & \text{for } |\lambda| > 2 \text{ and } |\mu| > 2. \end{cases}$$

Hence we write

$$\Phi_n(u,v)=\Psi\left(\frac{u-nu_0}{\sqrt{n}},\frac{v-nv_0}{\sqrt{n}}\right).$$

The following estimate is valid:

$$|grad\Phi_n(u,v)|\leq\frac{\Bbbk'}{\sqrt{n}}.$$

We obtain

$$\begin{aligned}L(s,t,D'_{s,t})\phi_{n,\varepsilon'} &= l(s_0,t_0,u_0,v_0)\phi_{n,\varepsilon'}+\Phi_n(D'_{s,t})\Big(L(s,t,D'_{s,t})-l(s_0,t_0,u_0,v_0)I\Big)\phi_{n,\varepsilon'}\\ &\quad+\Big(I-\Phi_n(D'_{s,t})\Big)\Big(L(s,t,D'_{s,t})-l(s_0,t_0,u_0,v_0)I\Big)\phi_{n,\varepsilon'}\\ &= l(s_0,t_0,u_0,v_0)\phi_{n,\varepsilon'}+\mathscr{I}_1+\mathscr{I}_2\quad(\text{ say }),\end{aligned}$$

where I being the identity mapping and therefore we get

$$^{(0,0)}\|L(s,t,D'_{s,t})\phi_{n,\varepsilon'}\|_0={}^{(0,0)}\|l(s_0,t_0,u_0,v_0)\phi_{n,\varepsilon'}+\mathscr{I}_1+\mathscr{I}_2\|_0$$

ans hence

$$\begin{aligned}&\Big|{}^{(0,0)}\|L(s,t,D'_{s,t})\phi_{n,\varepsilon'}\|_0-\Bbbk_0{}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0\Big|\\ &=\Big|{}^{(0,0)}\|l(s_0,t_0,u_0,v_0)\phi_{n,\varepsilon'}+\mathscr{I}_1+\mathscr{I}_2\|_0-{}^{(0,0)}\|l(s_0,t_0,u_0,v_0)\phi_{n,\varepsilon'}\|_0\Big|\\ \leq\ &\Big|{}^{(0,0)}\|l(s_0,t_0,u_0,v_0)\phi_{n,\varepsilon'}\|_0+{}^{(0,0)}\|\mathscr{I}_1+\mathscr{I}_2\|_0-{}^{(0,0)}\|l(s_0,t_0,u_0,v_0)\phi_{n,\varepsilon'}\|_0\Big|\\ \leq\ &{}^{(0,0)}\|\mathscr{I}_1+\mathscr{I}_2\|_0\leq{}^{(0,0)}\|\mathscr{I}_1\|_0+{}^{(0,0)}\|\mathscr{I}_2\|_0.\end{aligned}$$

We consider hence the expression

$$^{(0,0)}\|\mathscr{I}_1\|_0={}^{(0,0)}\|\Phi_n(D'_{s,t})\Big(L(s,t,D'_{s,t})-l(s_0,t_0,u_0,v_0)I\Big)\phi_{n,\varepsilon'}\|_0$$

which is estimated by

$$\begin{aligned}&{}^{(0,0)}\|\Phi_n(D'_{s,t})\Big(L(s,t,D'_{s,t})-L(s_0,t_0,D'_{s_0,t_0})\Big)\phi_{n,\varepsilon'}\|_0\\ &+{}^{(0,0)}\|\Phi_n(D'_{s,t})\Big(L(s,t,D'_{s,t})-l(s_0,t_0,u_0,v_0)I\Big)\phi_{n,\varepsilon'}\|_0\end{aligned}$$

where

$$\mathscr{F}_{\theta_1,\theta_2}[L(s_0,t_0,D'_{s_0,t_0})\phi](u,v) = l(s_0,t_0,u,v)[\mathscr{F}_{\theta_1,\theta_2}\phi](u,v), \quad \forall \phi \in \mathscr{S}(\mathbb{R}^2).$$

Hence, we have

$$\begin{aligned}
& {}^{(0,0)}\|\Phi_n(D'_{s,t})\Big(L(s,t,D'_{s,t}) - l(s_0,t_0,u_0,v_0)I\Big)\phi_{n,\varepsilon'}\|_0 \\
= {} & {}^{(0,0)}\|\Phi_n(D'_{s,t})\Big(L(s,t,D'_{s,t}) - l(s_0,t_0,nu_0,nv_0)I\Big)\phi_{n,\varepsilon'}\|_0 \\
= {} & \Big(\int_{\mathbb{R}^2} |\Phi_n(u,v)|^2 |l(s_0,t_0,u,v) - l(s_0,t_0,nu_0,nv_0)|^2 \\
& \times |[\mathscr{F}_{\theta_1,\theta_{\theta_2}}]\phi_{n,\varepsilon'}(u,v)|^2 dudv\Big)^{\frac{1}{2}}.
\end{aligned}$$

By the inequality, we have

$$\begin{aligned}
|l(s_0,t_0,u,v) - l(s_0,t_0,nu_0,nv_0)| & \leq \mathbb{k}\frac{|u-nu_0| + |v-nv_0|}{|u| + |nu_0| + |v| + |nv_{0|}} \\
& \leq \mathbb{k}\frac{|u-nu_0| + |v-nv_0|}{2n},
\end{aligned}$$

$p = 1,2,3,\dots$ *to* ∞, $u \neq 0,\ v \neq 0 \in \mathbb{R},\ |u_0| = |v_0| = 1,\ s,\ t \in \mathbb{R}$.
Therefore, considering too that

$$\Phi_n(u,v) = 0$$

for $|u - nu_0| > 2\sqrt{n}$ *and* $|v - nv_0| > 2\sqrt{n}$, we have

$$\begin{aligned}
& {}^{(0,0)}\|\Phi_n(D'_{s,t})\Big(L(s_0,t_0,D'_{s_0,t_0}) - l(s_0,t_0,u_0,v_0)I\Big)\phi_{n,\varepsilon'}\|_0 \\
\leq {} & \mathbb{k}\Big(\int_{-2\sqrt{n}<u-nu_0}^{u-nu_0<2\sqrt{n}} \int_{-2\sqrt{n}<v-nv_0}^{v-nv_0<2\sqrt{n}} \frac{1}{4n^2}(|u-nu_0|^2 + |u-nu_0|^2) \\
& \times |[\mathscr{F}_{\theta_1,\theta_{\theta_2}}]\phi_{n,\varepsilon'}(u,v)|^2 dudv\Big)^{\frac{1}{2}} \\
\leq {} & \frac{\mathbb{k}_1}{2\sqrt{n}}\,{}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0. \qquad (22)
\end{aligned}$$

Besides, we observe that we have also estimate

$$\begin{aligned}
& {}^{(0,0)}\|\Phi_n(D'_{s,t})(L(s,t,D'_{s,t}) - L(s_0,t_0,D'_{s_0,t_0}))\phi_{n,\varepsilon'}\|_0 \\
& \leq {}^{(0,0)}\|(L(s,t,D'_{s,t}) - L(s_0,t_0,D'_{s_0,t_0}))\phi_{n,\varepsilon'}\|_0.
\end{aligned}$$

If $m(s,t,u,v) = l(s,t,u,v) - l(s_0,t_0,u,v)$ is related to the operator $L(s,t,D'_{s,t}) - L(s_0,t_0,D'_{s_0,t_0})$, we have $|m(s,t,u,v)| < \varepsilon'$ for $|s-s_0| < \delta_{\varepsilon'}$, $|t-t_0| < \delta_{\varepsilon'}$, $|u_0| = 1$ & $|v_0| = 1$. On the other hand, the functions $\phi_{n,\varepsilon'}$ in (21) belong to $C_0^\infty(\{(s,t) : |s-s_0| < \delta_{\varepsilon'} \ \& \ |t-t_0| < \delta_{\varepsilon'}\})$ and hence (*by Theorem* 5), we have, given $\varepsilon' > 0$, a constant $\Bbbk_{\varepsilon'}$, such that

$$ {}^{(0,0)}\|(L(s,t,D'_{s,t}) - L(s_0,t_0,D'_{s_0,t_0}))\phi_{n,\varepsilon'}\|_0 \le (2\varepsilon')\ {}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 + \Bbbk_{\varepsilon'}\,{}^{(\theta_1,\theta_2)}\|\phi_{n,\varepsilon'}\|_{-1}, $$

$n = 1,2,3,4,5\ldots$ to ∞.
Up to now, we have arrived at estimate

$$ {}^{(0,0)}\|\mathscr{I}_1\|_0 \le \frac{\Bbbk}{\sqrt{n}}\ {}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 + 2\varepsilon'\,{}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 + \Bbbk_{\varepsilon'}\,{}^{(\theta_1,\theta_2)}\|\phi_{n,\varepsilon'}\|_{-1}, $$

Obviously, we get

$$ \begin{aligned} \mathscr{I}_2 &= \Big(L(s,t,D'_{s,t}) - l(s_0,t_0,nu_0,nv_0)(I - \Phi_n(D'_{s,t}))\Big)\phi_{n,\varepsilon'} \\ &\quad - \Big[L(s,t,D'_{s,t}) - l(s_0,t_0,nu_0,nv_0)I,\ I - \Phi_n(D'_{s,t})\Big]\phi_{n,\varepsilon'}. \end{aligned} $$

The commutator $[L(s,t,D'_{s,t}),\ \Phi_n(D'_{s,t})]$, and therefore

$$ \begin{aligned} \mathscr{I}_2 &= \Big(L(s,t,D'_{s,t}) - l(s_0,t_0,nu_0,nv_0)(I - \Phi_n(D'_{s,t}))\Big)\phi_{n,\varepsilon'} \\ &\quad - \Big[L(s,t,D'_{s,t}),\ \Phi_n(D'_{s,t})\Big]\phi_{n,\varepsilon'} = \mathscr{I}_3 + \mathscr{I}_4 \quad \text{(say)}. \end{aligned} $$

Hence, first of all we have (being $|l(s_0,t_0,nu_0,nv_0)| \le \Bbbk'$) that

$$ \begin{aligned} {}^{(0,0)}\|\mathscr{I}_3\|_0 &\le \Bbbk\ {}^{(0,0)}\|(\mathscr{I}_3 - \Phi_n(D'_{s,t}))\phi_{n,\varepsilon'}\|_0 \\ &\le \Bbbk\left(\int_{\mathbb{R}}\int_{\mathbb{R}}(1 - \Phi_n(u,v))|[\mathscr{F}_{\theta_1,\ \theta_2}\phi_{n,\varepsilon'}](u,v)|^2 dudv\right)^{\frac{1}{2}} \end{aligned} $$

Now we observe that we have $\Phi_n(u,v) = 1$ for $|u - nu_0| < \sqrt{n}$ & $|v - nv_0| < \sqrt{n}$; hence $1 - \Phi_n(u,v) = 0$ for $|u - nu_0| < \sqrt{n}$ & $|v - nv_0| < \sqrt{n}$ and besides it is

$$ \begin{aligned} [\mathscr{F}_{\theta_1,\ \theta_2}\phi_{n,\varepsilon'}](u,v) &= \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\theta_1,\ \theta_2}(s,t,u,v)e^{\frac{i}{2}[-2n(uu_0cot\theta_1 + vv_0\cot\theta_2)]} \\ &\quad \times e^{in(su_0csc\theta_1 + tv_0\csc\theta_2)}\psi_{\varepsilon'}(s,t)dsdt \\ &= [\mathscr{F}_{\theta_1,\ \theta_2}\psi_{\varepsilon'}](u - nu_0, v - nv_0) \end{aligned} $$

and therefore

$$\begin{aligned}
{}^{(0,0)}\|\mathscr{I}_3\|_0 &\leq \Bbbk\left(\int_{|u-nu_0|\geq\sqrt{n}}\int_{|v-nv_0|\geq\sqrt{n}}|[\mathscr{F}_{\theta_1,\ \theta_2}\psi_{\varepsilon'}](u-nu_0,v-nv_0)|^2 dudv\right)^{\frac{1}{2}} \\
&= \Bbbk\left(\int_{|U|\geq\sqrt{n}}\int_{|V|\geq\sqrt{n}}|[\mathscr{F}_{\theta_1,\ \theta_2}\psi_{\varepsilon'}](U,V)|^2 dUdV\right)^{\frac{1}{2}},
\end{aligned}$$

where $u-nu_0=U$ and $v-nv_0=V$, we have:

$$\begin{aligned}
&\left(\int_{|U|\geq\sqrt{n}}\int_{|V|\geq\sqrt{n}}|[\mathscr{F}_{\theta_1,\ \theta_2}\psi_{\varepsilon'}](U,V)|^2 dUdV\right)^{\frac{1}{2}} \\
&\leq \varepsilon'\left(\int_{\mathbb{R}}\int_{\mathbb{R}}|[\mathscr{F}_{\theta_1,\ \theta_2}\psi_{\varepsilon'}](U,V)|^2 dUdV\right)^{\frac{1}{2}} \\
&= \varepsilon'\ {}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 \quad \text{if } n\geq N_0(\varepsilon',\ \mathscr{F}_{\theta_1,\ \theta_2}\psi_{\varepsilon'}).
\end{aligned}$$

Then we have

$$\begin{aligned}
{}^{(0,0)}\|\mathscr{I}_4\|_0 &= {}^{(0,0)}\|\int_{\mathbb{R}}\int_{\mathbb{R}}[\mathscr{F}_{\theta_1,\ \theta_2}l'](u-\eta,v-\zeta,u,v)(\Phi_n(u,v)-\Phi_n(\eta,\zeta)) \\
&\quad\times[\mathscr{F}_{\theta_1,\ \theta_2}\phi_{n,\varepsilon'}](\eta,\zeta)d\eta d\zeta\|_0.
\end{aligned}$$

We see that

$$\begin{aligned}
&|\Phi_n(u,v)-\Phi(\eta,\zeta)|\leq(|u-\eta|+|v-\zeta|)|grad\Phi_n(U,V)| \\
&\leq\Bbbk\frac{1}{\sqrt{n}}\left(1+|u-\eta|^2+|v-\zeta|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\int_{\mathbb{R}}[\mathscr{F}_{\theta_1,\ \theta_2}l'](u-\eta,v-\zeta,u,v)(\Phi_n(u,v)-\Phi_n(\eta,\zeta))[\mathscr{F}_{\theta_1,\ \theta_2}\phi_{n,\varepsilon'}](\eta,\zeta)d\eta d\zeta\right| \\
&\leq\frac{\Bbbk_q}{\sqrt{n}}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(1+|u-\eta|^2+|v-\zeta|^2\right)^{-q+\frac{1}{2}}|[\mathscr{F}_{\theta_1,\ \theta_2}\phi_{n,\varepsilon'}](\eta,\zeta)|d\eta d\zeta,
\end{aligned}$$

$\forall\ q=1,2,3,\ldots$
we obtain

$$ {}^{(0,0)}\|\mathscr{I}_4\|_0\leq\frac{\Bbbk}{\sqrt{n}}{}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0,\quad n=1,2,3,\ldots$$

Adding, we get

$$\begin{aligned}
&\left|{}^{(0,0)}\|L(s,t,D'_{s,t})\phi_{n,\varepsilon'}\|_0-\Bbbk_0\ {}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0\right|\leq\frac{\Bbbk}{\sqrt{n}}{}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0+2\varepsilon'^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 \\
&+\Bbbk_{\varepsilon'}{}^{(\theta_1,\theta_2)}\|\phi_{n,\varepsilon'}\|_{-1}+\varepsilon'^{(0,0)}\|\phi_{n,\varepsilon'}\|_0+\frac{\Bbbk}{\sqrt{n}}{}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0,\quad\text{for } n\geq N_0(\varepsilon').
\end{aligned}$$

For every $\varepsilon'' > 0$ there is $[\mathscr{F}_{\theta_1,\ \theta_2} n](\varepsilon'', \varepsilon')$ such that we have

$$^{(\theta_1,\theta_2)}\|\phi_{n,\varepsilon'}\|_{-1} \leq \mathbb{k}\varepsilon''^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 \quad \text{for } n \geq [\mathscr{F}_{\theta_1,\theta_2} n](\varepsilon'', \varepsilon').$$

In fact, we have

$$\begin{aligned}
&\left(^{(\theta_1,\theta_2)}\|\phi_{n,\varepsilon'}\|_{-1}\right)^2 \\
= &\int\int (1+|u|^2+|v|^2)^{-1}\left|[\mathscr{F}_{\theta_1,\theta_2}\psi_{\varepsilon'}](u-nu_0, v-nv_0)\right|^2 dudv \\
= &\int_{|u-nu_0|>\rho_1}\int_{|v-nv_0|>\rho_2}\left|[\mathscr{F}_{\theta_1,\theta_2}\psi_{\varepsilon'}](u-nu_0, v-nv_0)\right|^2 dudv \\
+ &\int_{|u-nu_0|<\rho_1}\int_{|v-nv_0|<\rho_2}(1+|u|^2+|v|^2)^{-1}\left|[\mathscr{F}_{\theta_1,\theta_2}\psi_{\varepsilon'}](u-nu_0, v-nv_0)\right|^2 dudv,
\end{aligned}$$

for every $\rho_1 > 0$ & $\rho_2 > 0$.
Given now $\varepsilon'' > 0$ there is $\rho_1^*(\varepsilon', \varepsilon'')$ & $\rho_2^*(\varepsilon', \varepsilon'')$ such that

$$\int_{|U|>\rho_1^*}\int_{|V|>\rho_2^*}[\mathscr{F}_{\theta_1,\ \theta_2}\psi_{\varepsilon'}](U,V)dUdV \leq \varepsilon''\left(^{(0,0)}\|\phi_{n,\varepsilon'}\|_0\right)^2.$$

We observe that if $|u-nu_0| < \rho_1^*$ and $|v-nv_0| < \rho_2^*$, it results $|u| \geq n-\rho_1^*$ and $|v| \geq n-\rho_2^*$ and therefore, for $n > \rho_1^*+\rho_2^*+1$, we get

$$\begin{aligned}
&\int_{|u-nu_o|\leq\rho_1^*}\int_{|v-nv_o|\leq\rho_2^*}(1+|u|^2+|v|^2)^{-1}|[\mathscr{F}_{\theta_1,\theta_2}\psi_{\varepsilon'}](u-nu_o, v-nv_o)|^2 dudv \\
&\leq \left(1+(n-\rho_1^*-\rho_2^*)^2\right)^{-1}\left(\int\int|[\mathscr{F}_{\theta_1,\theta_2}\psi_{\varepsilon'}](u-nu_o, v-nv_o)|^2 dudv\right) \\
&= \left(1+(n-\rho_1^*-\rho_2^*)^2\right)^{-1}\left(^{(0,0)}\|\phi_{n,\varepsilon'}\|_0\right)^2 \\
&\leq (\varepsilon'')^2\left(^{(0,0)}\|\phi_{n,\varepsilon'}\|_0\right)^2, \quad \text{if } n > \max\{\rho_1^*+\rho_2^*+1,\ N_{\varepsilon'}\}
\end{aligned}$$

and therefore, for $n \geq N_1(\varepsilon', \varepsilon'')$, we get

$$^{(\theta_1,\theta_2)}\|\phi_{n,\varepsilon'}\|_{-1} \leq 2\varepsilon''^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 .$$

Hence we arrive at inequalities

$$\left|^{(0,0)}\|L(s,t,D'_{s,t})\phi_{n,\varepsilon'}\|_0 - \mathbb{k}_o{}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0\right| \leq \frac{\mathbb{k}}{\sqrt{n}}{}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 + 2\varepsilon'^{(0,0)}\|\phi_{n,\varepsilon'}\|_0$$

for $n > N(\varepsilon', \varepsilon'')$ and ${}^{(\theta_1,\theta_2)}\|\phi_{n,\varepsilon'}\|_{-1} \le \Bbbk^{(0,0)}\|\phi_{n,\varepsilon'}\|_0$ for $n \ge N_1(\varepsilon', \varepsilon'')$.
Let us take $\varepsilon''(\varepsilon')$ small enough to have $\Bbbk\varepsilon'' < \varepsilon'$ and $2\Bbbk_{\varepsilon'}.\varepsilon'' < \varepsilon'$; hence, for $n \ge \mathscr{N}'(\varepsilon')$.
We have

$$
{}^{(\theta_1,\theta_2)}\|\phi_{n,\varepsilon'}\|_{-1} \le \varepsilon'^{(0,0)}\|\phi_{n,\varepsilon'}\|_0
$$

and

$$
\begin{aligned}
&\left| {}^{(0,0)}\|L(s,t,D'_{s,t})\phi_{n,\varepsilon'}\|_0 - \Bbbk_o{}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 \right| \le \frac{\Bbbk}{\sqrt{n}}{}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 + 3\varepsilon'^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 \\
&\le 4\varepsilon'^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 \quad \text{if } n \ge \mathscr{N}_1^{''}(\varepsilon').
\end{aligned}
$$

Let us take $\varepsilon' < \frac{\varepsilon}{4}$ and the result achieves. □

Theorem 7. *If $l(s,t,u,v)$ is a symbol, $L(s,t,D'_{s,t})$ the associated pseudo-differential operatot, $\mathscr{C}_c = \{\mathscr{T} | \mathscr{T} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a compact operator$\}$,*
$\mathscr{M} = \max\{|l(s,t,u,v)| :\ s,\ t \in \mathbb{R}$ and $|u| = |v| = 1\}$, we get

$$
\mathscr{M} \le inf\{\|L(s,t,D'_{s,t})\| : \mathscr{T} \in \mathscr{C}_c\}. \tag{23}
$$

Remark 1. *As a simple corollary of* (23) *we obtain also the estimate*

$$
\mathscr{M} \le inf\{\|\mathscr{L}(s,t,D'_{s,t}) + \mathscr{T}\| : \mathscr{T} \in \mathscr{C}_c\}. \tag{24}
$$

In fact, if we take an arbitrary $\mathscr{T}_o \in \mathscr{C}_c$, we obtain

$$
\begin{aligned}
\mathscr{L}(s,t,D'_{s,t}) + \mathscr{T}_o &= \mathscr{L}(s,t,D'_{s,t}) - L(s,t,D'_{s,t}) + L(s,t,D'_{s,t}) + \mathscr{T}_o \\
&= L(s,t,D'_{s,t}) + \mathscr{T}_1
\end{aligned}
$$

where $\mathscr{T}_1 \in \mathscr{C}_c$. Consequently, applying (23)*, we obtain $\|\mathscr{L}(s,t,D'_{s,t}) + \mathscr{T}_o\| = \|L(s,t,D'_{s,t}) + \mathscr{T}_1\| \ge \mathscr{M}$. As $\mathscr{T}_o$ is arbitrary in $\mathscr{C}_c$, we get the equality*

$$
inf\{\|\mathscr{L}(s,t,D'_{s,t}) + \mathscr{T}\| : \mathscr{T} \in \mathscr{C}_c\} = \mathscr{M}. \tag{25}
$$

Corollary 2.1. *Combining with Theorem* 4 *we get the interesting the result*

$$
\inf\{\|L(s,t,D'_{s,t}) + \mathscr{T}\| : \mathscr{T} \in \mathscr{C}_c\} = \mathscr{M}. \tag{26}
$$

Proof. First of all, we have the following. □

Lemma 5. *Let $l(s,t,u,v)$ be a symbol and $\Bbbk_o = |l(s_o,t_o,u_o,v_o)|$ for a certain $s_o,\ t_o \in \mathbb{R}$ and $|u_o| = |v_o| = 1$. There is then, for every $\varepsilon > 0$ a sequence $\phi_n(s,t) \in C_o^\infty(\mho_n \times \mho'_n)$; $\mho_n \times \mho'_n = \{(s,t) : |s - s_o| \le \frac{1}{n}$ and $|t - t_o| \le \frac{1}{n}\}$ with ${}^{(0,0)}\|\phi_n\|_0 = 1$ and $\Bbbk_o - \varepsilon \le {}^{(0,0)}\|L\phi_n\|_0$.*

Proof. The function ϕ_ε is got $= e^{in(s.u_o+t.v_o)}\psi_\varepsilon(s,t)$, from Theorem 6, where $\psi_\varepsilon \in C_o^\infty\{(s,t) : |s-s_o| < \delta_\varepsilon \ \ \& \ |t-t_o| < \delta'_\varepsilon\}$. Hence, for $n \geq n_o$ we get $\frac{1}{n} \leq \delta_\varepsilon$, and all the functions

$$\phi_{n,\varepsilon}(s,t) = e^{iq_n(su_o+tv_o)}\psi_n(s,t)$$

(with q_n big enough, fixed, dependent from $\varepsilon > 0$ and from ψ_n), verify estimate

$$(\Bbbk_o - \varepsilon)^{(0,0)}\|\phi_{n,\varepsilon}\|_0 \leq {}^{(0,0)}\|L(s,t,D'_{s,t})\phi_{n,\varepsilon}\|_0.$$

Dividing by ${}^{(0,0)}\|\phi_{n,\varepsilon}\|_0$, we can have the sequence of norm 1. We have

$$\Bbbk_o - \varepsilon \leq {}^{(0,0)}\|L(s,t,D'_{s,t})\phi_n\|_0.$$

□

Lemma 6. *We have*

$$\lim_{n\to\infty}\iint \phi_{n,\varepsilon}\psi(s,t)dsdt = 0, \quad \forall\psi \in L^2(\mathbb{R}\times\mathbb{R}).$$

Proof. In fact, we have

$$\begin{aligned}\iint \phi_{n,\varepsilon}(s,t)\psi(s,t)dsdt &= \int_{|s-s_o|>\rho}\int_{|t-t_o|>\rho}\phi_{n,\varepsilon}(s,t)\psi(s,t)dsdt \\ &\quad + \int_{|s-s_o|<\rho}\int_{|t-t_o|<\rho}\phi_{n,\varepsilon}(s,t)\psi(s,t)dsdt.\end{aligned}$$

For n big enough, $\phi_{n,\varepsilon} = 0$ when $|s-s_o| > \rho,\ |t-t_o| > \rho$ and therefore

$$\begin{aligned}\iint \phi_{n,\varepsilon}(s,t)\psi(s,t)dsdt &= \int_{|s-s_o|<\rho}\int_{|t-t_o|<\rho}\phi_{n,\varepsilon}(s,t)\psi(s,t)dsdt \\ &\leq {}^{(0,0)}\|\phi_{n,\varepsilon}\|_0\left(\int_{|s-s_o|<\rho}\int_{|t-t_o|<\rho}|\psi(s,t)|^2dsdt\right)^{\frac{1}{2}} \\ &= \left(\int_{|s-s_o|<\rho}\int_{|t-t_o|<\rho}|\psi(s,t)|^2dsdt\right)^{\frac{1}{2}}.\end{aligned}$$

Hence, given $\upsilon > 0$, we take $\rho(\upsilon)$ such that

$$\left(\int_{|s-s_o|<\rho}\int_{|t-t_o|<\rho}|\psi(s,t)|^2dsdt\right)^{\frac{1}{2}} < \upsilon.$$

At last, we take n big enough to have $\phi_{n,\varepsilon}(s,t) = 0$ when $|s-s_o| > \frac{1}{n}$
& $|t-t_o| > \frac{1}{n}$. □

Proof of the Theorem: We assume, that

$$\inf\{\|L(s,t,D'_{s,t})+\mathscr{T}\| : \mathscr{T}\in\mathscr{C}_c\} = m < \mathscr{M}. \tag{27}$$

Hence, take m' such that $m < m' < M$ there is at least a $\mathscr{T}\in\mathscr{C}_c$ such that $\|L(s,t,D'_{s,t})+\mathscr{T}\| < m'$. Hence we get

$${}^{(0,0)}\|(L+\mathscr{T})\phi\|_0 \le m' \; {}^{(0,0)}\|\phi\|_0, \quad \phi\in L^2(\mathbb{R}\times\mathbb{R}).$$

Being $m' < M$, we find at least two $s_o,\ t_o\in\mathbb{R},\ u_o\neq 0,\ v_o\neq 0\in\mathbb{R}$ and $|u_o| = |v_o| = 1$ such that $m' < |l(s_o,t_o,u_o,v_o)| = \Bbbk_o < \mathscr{M}$.
Hence, we get, for $\phi - \phi_{n,\varepsilon}$ (applying Lemma 5), that

$$\begin{aligned}(\Bbbk_o-\varepsilon) \le {}^{(0,0)}\|[L(s,t,D'_{s,t})]\phi_{n,\varepsilon}\|_0 &\le {}^{(0,0)}\|[L+\mathscr{T}]\phi_{n,\varepsilon}\|_0 + {}^{(0,0)}\|\mathscr{T}\phi_{n,\varepsilon}\|_0 \\ &\le m' + {}^{(0,0)}\|\mathscr{T}\phi_{n,\varepsilon}\|_0.\end{aligned}$$

If $n\to\infty$, $\mathscr{T}\phi_{n,\varepsilon}\to 0$ strongly in $L^2(\mathbb{R}\times\mathbb{R})$; hence $\Bbbk_o - \varepsilon \le m'$, absurd for ε small enough.
Taken then $|s_o| \le \mathscr{N}_o$, $|u_o| = 1$, such that $|l(s_o,t_o,u_o,v_o)| = \mathscr{M}_{\mathscr{N}_o}$; then $l(s,t,u,v)\in C_o^\infty(|s|\le\mathscr{N}_o\times|t|\le\mathscr{N}_o)$ and the function $\phi(s,t)\not\equiv 0$ and the sequence

$$\phi_\vartheta(s,t) = \vartheta^{\frac{n}{4}}\phi((s-s_o)\sqrt{\vartheta},(t-t_o)\sqrt{\vartheta})e^{i(s.u_o+t.v_o)\vartheta}, \quad \vartheta = 1,\ 2,\ 3,\ldots \text{to}\ \infty.$$

It follows $\|\phi_\vartheta\|_{L^2(\mathbb{R}^2)} = \|\phi\|_{L^2(\mathbb{R}^2)}$ and

$$\lim_{\vartheta\to\infty}\phi_\vartheta(s,t) = 0 \text{ in } L^2(\mathbb{R}^2)$$

By direct computation one gets

$$[\mathscr{L}\phi_\vartheta](s,t) = \vartheta^{\frac{n}{4}}\chi_\vartheta((s-s_o)\sqrt{\vartheta},(t-t_o)\sqrt{\vartheta})e^{i(s.u_o+t.v_o)\sqrt{\vartheta}}$$

where

$$\begin{aligned}\chi_\vartheta(s,t) &= \iint l(s_o+\frac{1}{\sqrt{\vartheta}}s,\ t_o+\frac{1}{\sqrt{\vartheta}}t,\ \vartheta u_o+\eta\sqrt{\vartheta},\ \vartheta v_o+\xi\sqrt{\vartheta})[\mathscr{F}_{\theta_1,\theta_2}\phi](\eta,\xi) \\ &\quad\times e^{i(s.\eta+t.\xi)}d\eta d\xi;\end{aligned} \tag{28}$$

it follows $\|\mathscr{L}\phi_\vartheta\|_{L^2(\mathbb{R}^2)} = \|\chi_\vartheta\|_{L^2(\mathbb{R}^2)}$; some simple estimates give also that

$$\lim_{\vartheta\to\infty}|\chi_\vartheta(s,t)|^2 = |l(s_o,\ t_o,u_o,v_o)|^2|\phi(s,t)|^2,$$

uniformly on bounded sets in $\mathbb{R}^2$.
Using FATOU's lemma to sequence $|\chi_\vartheta(s,t)|^2$. We get

$$\begin{aligned}\iint |l(s_o,t_o,u_o,v_o)|^2|\phi(s,t)|^2 dsdt &= |l(s_o,t_o,u_o,v_o)|^2\|\phi\|^2_{L^2(\mathbb{R}^2)} \\ &\leq \lim_{\vartheta\to\infty} \inf\|\mathscr{L}\phi_\vartheta\|^2_{L^2(\mathbb{R}^2)}.\end{aligned}$$

Consider $\mathscr{T} \in \mathscr{C}_c$. Then the estimate

$$\|\mathscr{L}\phi_\vartheta\|^2_{L^2(\mathbb{R}^2)} \leq \left(\|\mathscr{L}+\mathscr{T}\|_{L^2(\mathbb{R}^2)}\|\phi\| + \|\mathscr{T}\phi_\vartheta\|_{L^2(\mathbb{R}^2)}\right)^2$$

and consequently

$$\lim_{\vartheta\to\infty} \inf\|\mathscr{L}\phi_\vartheta\|^2_{L^2(\mathbb{R}^2)} \leq \|\mathscr{L}+\mathscr{T}\|^2_{L^2(\mathbb{R}^2)}\|\phi\|^2.$$

We obtained this way the inequality

$$|l(s_0,t_o,u_o,v_o)|^2\|\phi\|^2_{L^2(\mathbb{R}^2)} \leq \|\mathscr{L}+\mathscr{T}\|^2_{L^2(\mathbb{R}^2)}\|\phi\|^2_{L^2(\mathbb{R}^2)};$$

hence $M_{N_o} \leq \|\mathscr{L}+\mathscr{T}\|$, which gives the desired result.
The proof is completed.

3 Conclusion

In 1972, S. Zaidman discussed norms of pseudo-differential operator involving Fourier Transform in [14]. In this paper, i introduced norms of pseudo-differential operator involving coupled fractional Fourier transform. Paper studies the norms of pseudo-differential operators (PDOs) associated with the coupled fractional Fourier transform (CFrFT). Establish several norm estimates, compactness results, and upper/lower bounds for PDOs under the CFrFT setting.The work contributes to functional analysis and operator theory by generalizing Fourier-based operator results to a coupled fractional domain.

4 Scope for future work

My manuscript addresses the study of norms of pseudo-differential operators related to the coupled fractional Fourier transform, which is a relevant area in mathematical analysis with potential applications in signal processing and applied mathematics. The development of theoretical results in this direction may provide deeper insights into operator theory and transform analysis. Such work can also serve as a foundation for future research in fractional calculus, wave propagation, and time–frequency analysis.

5 Acknowledgments

For useful advices and instructions of the referees, we would be grateful.

References

[1] Ahmed I Zayed. On the relationship between the fourier and fractional fourier transforms. *IEEE signal processing letters*, 3(12):310–311, 1996.

[2] Ahmed Zayed. Two-dimensional fractional fourier transform and some of its properties. *Integral Transforms and Special Functions*, 29(7):553–570, 2018.

[3] T Alieva, Vicente Lopez, Fernando Agulló-López, and LB Almeida. The fractional fourier transform in optical propagation problems. *Journal of modern optics*, 41(5):1037–1044, 1994.

[4] RS Pathak, Akhilesh Prasad, and Manish Kumar. Fractional fourier transform of tempered distributions and generalized pseudo-differential operator. *Journal of Pseudo-Differential Operators and Applications*, 3:239–254, 2012.

[5] Haldun M Ozaktas and M Alper Kutay. The fractional fourier transform. In *2001 European Control Conference (ECC)*, pages 1477–1483. IEEE, 2001.

[6] Ahmed I Zayed. A class of fractional integral transforms: a generalization of the fractional fourier transform. *IEEE transactions on signal processing*, 50(3):619–627, 2002.

[7] Abhisekh Shekhar and Nawin Kumar Agrawal. Fractional fourier transform on sobolev spaces connected to negative definite functions. *Asian Journal of Pure and Applied Mathematics*, pages 112–122, 2023.

[8] Abhisekh Shekhar. Fractional fourier-bessel type transform. In *AIP Conference Proceedings*, volume 3180. AIP Publishing, 2024.

[9] Abhisekh Shekhar and Nawin Kumar Agrawal. Inequality and estimate of generalized pseudo-differential operators involving fractional fourier transform. In *American Institute of Physics Conference Series*, volume 3087, page 090001, 2024.

[10] Abhisekh Shekhar and Nawin Agrawal. Generalized pseudo-differential operators associated with symbol classes involving fractional fourier transform. *Serdica Mathematical Journal*, 48(4):247–270, 2022.

[11] Abhisekh Shekhar. The estimation of pseudo-differential operators utilising the coupled fractional fourier transform and a certain inequality. *Asian Research Journal of Mathematics*, 21(9):99–112, Sep. 2025.

[12] Shraban Das, Kanailal Mahato, and Ahmed I Zayed. Characterization of pseudo-differential operators associated with the coupled fractional fourier transform. *Axioms*, 13(5):296, 2024.

[13] R Kamalakkannan, R Roopkumar, and A Zayed. On the extension of the coupled fractional fourier transform and its properties. *Integral Transforms and Special Functions*, 33(1):65–80, 2022.

[14] S Zaidman. Pseudo-differential operators. *Annali di Matematica Pura ed Applicata*, 92(1):345–399, 1972.

[15] Jean-Marc Bouclet. An introduction to pseudo-differential operators. *Lecture Notes, Available at http://www. math. univ-toulouse. fr/˜ bouclet*, 2012.

[16] Man-Wah Wong. *Introduction To Pseudo-differential Operators, An*, volume 6. World Scientific Publishing Company, 2014.

[17] Akhilesh Prasad and Manish Kumar. Product of two generalized pseudo-differential operators involving fractional fourier transform. *Journal of Pseudo-Differential Operators and Applications*, 2(3):355–365, 2011.

[18] RS Pathak, Akhilesh Prasad, and Manish Kumar. An n-dimensional pseudo-differential operator involving the hankel transformation. *Proceedings-Mathematical Sciences*, 122(1):99–120, 2012.

[19] Joseph J Kohn and Louis Nirenberg. An algebra of pseudo-differential operators. *Communications on Pure and Applied Mathematics*, 18(1-2):269–305, 1965.

[20] Samuel Zaidman. *Distributions and pseudo-differential operators*, volume 248. Harlow, England: Longman Scientific & Technical, 1991.

[21] Ram S Pathak and Pradip K Pandey. A class of pseudo-differential operators associated with bessel operators. *Journal of mathematical analysis and applications*, 196(2):736–747, 1995.

[22] RS Pathak and S Pathak. Certain pseudo-differential operator associated with the bessel operator. *INDIAN JOURNAL OF PURE & APPLIED MATHEMATICS*, 31(3):309–317, 2000.

[23] Ram Shankar Pathak and S Pathak. The pseudodifferential operator a (x, d). *International Journal of Mathematics and Mathematical Sciences*, 2004(8):407–419, 2004.

[24] Akhilesh Prasad and Vishal Kumar Singh. On pseudo-differential operator associated with bessel operator. *Int. J. Contemp. Math. Sciences*, 6(25):1237–1243, 2011.

[25] Akhilesh Prasad and Kanailal Mahato. On the sobolev boundedness results of the product of pseudo-differential operators involving a couple of fractional hankel transforms. *Acta Mathematica Sinica, English Series*, 34(2):221–232, 2018.

[26] Abdessalem Gasmi and Anis El Garna. Properties of the linear multiplier operator for the weinstein transform and applications. *Electronic Journal of Differential Equations*, 124:1–18, 2017.

[27] Miloud Assal. Pseudo-differential operators associated with laguerre hypergroups. *Journal of computational and applied mathematics*, 233(3):617–620, 2009.

[28] N Ben Salem and A Dachraoui. Pseudo-differential operators associated with the jacobi differential operator. *Journal of mathematical analysis and applications*, 220(1):365–381, 1998.

[29] Shraban Das, Kanailal Mahato, and Sourav Das. L p-sobolev spaces and coupled potential operators associated with coupled fractional fourier transform. *Journal of Pseudo-Differential Operators and Applications*, 15(4):72, 2024.

[30] Manish Kumar and Bhawna. An n- dimensional pseudo-differential operator involving quadratic phase fourier transform and applications in quantum mechanics. *Journal of Pseudo-Differential Operators and Applications*, 16(2):1–24, 2025.