

# A GEOMETRIC PERSPECTIVE ON SOME SCHRÖDER SEQUENCE SPACES

ABSTRACT. Investigation of some structural properties of some sequence spaces constructed using Schröder numbers has recently become important. Some of the fundamental works on the investigation of the properties of these sequence spaces are given in [3, 4] and [5]. In this work we are going to investigate some geometric properties such as rotundity and uniformly smoothness of the sequence spaces  $\lambda(\mathcal{S})$  where  $\lambda$  is  $c_0, \ell_\infty$  or  $\ell_p$  and  $1 \leq p < \infty$ .

## 1. INTRODUCTION

The properties of rotundity and smoothness are important indicators that enable us to understand the geometric structure of Banach spaces. The classical Banach spaces known to possess these properties have been listed previously. For further information, see [1]. The investigation of these types of properties in recently defined quantum sequence spaces has been presented in [6]. The investigation of the properties of some sequence spaces constructed using Schröder numbers has recently become important. Some of the fundamental works on the investigation of the properties of these sequence spaces are [3, 4] and [5]. For further similar works one can see [7, 8].

Schröder numbers were named after the German mathematician Ernest Schröder and there are two kinds of Schröder numbers, the large Schröder numbers and the little Schröder numbers. First few numbers of large Schröder numbers are 1, 2, 6, 22, 90, 394, 1806, 8558, .... With the recurrence relation

$$S_{n+1} = S_n + \sum_{k=0}^n S_k S_{n-k}, \text{ for } n \geq 0 \text{ and } S_0 = 1,$$

we can write the large Schröder sequence as  $(S_n)$ .  $S_n$  also can be represented by the functional formula

$$S_n = 2 \cdot {}_2F_1(-n + 1, n + 2; 2; -1),$$

where  ${}_2F_1(-n + 1, n + 2; 2; -1)$  is known as the *hypergeometric function* and it is defined by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}.$$

Here we should specify the Pochhammer symbol  $(x)_n = x(x+1)\dots(x+n-1)$  for  $n \geq 1$ , and  $(x)_n = 1$  for  $n = 0$ . By using large Schröder sequence in [4]. MC. Dagh introduced a sub triangular matrix  $\mathcal{S} = (S_{nk})$  such that

---

2000 *Mathematics Subject Classification.* 46A45; 46B45; 47A10.

*Key words and phrases.* smoothness, rotundity .

$$\mathcal{S}_{nk} = \begin{cases} \frac{S_k S_{n-k}}{S_{n+1} - S_n}, & \text{if } 0 \leq k \leq n \\ 0, & \text{if } k > n \end{cases}.$$

More explicitly,

$$\mathcal{S} = \begin{bmatrix} \frac{S_0 S_0}{S_1 - S_0} & 0 & 0 & 0 & \cdots \\ \frac{S_0 S_1}{S_2 - S_1} & \frac{S_1 S_0}{S_2 - S_1} & 0 & 0 & \cdots \\ \frac{S_0 S_2}{S_3 - S_2} & \frac{S_1 S_1}{S_3 - S_2} & \frac{S_2 S_0}{S_3 - S_2} & 0 & \cdots \\ \frac{S_0 S_3}{S_4 - S_3} & \frac{S_1 S_2}{S_4 - S_3} & \frac{S_2 S_1}{S_4 - S_3} & \frac{S_3 S_0}{S_4 - S_3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{2}{4} & \frac{2}{4} & 0 & 0 & \cdots \\ \frac{6}{16} & \frac{6}{16} & \frac{6}{16} & 0 & \cdots \\ \frac{22}{68} & \frac{16}{68} & \frac{16}{68} & \frac{22}{68} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

By using  $\mathcal{S}$  in [4] he introduced the sequence spaces  $\ell_p(\mathcal{S})$ , where  $1 \leq p < \infty$ , and  $\ell_\infty(\mathcal{S})$  such that

$$\ell_p(\mathcal{S}) = \left\{ u = (u_n) \in w : \sum_{n=0}^{\infty} \left| \frac{1}{S_{n+1} - S_n} \sum_{k=0}^n S_k S_{n-k} u_n \right|^p < \infty \right\}$$

and

$$\ell_\infty(\mathcal{S}) = \left\{ u = (u_n) \in w : \sup_n \left| \frac{1}{S_{n+1} - S_n} \sum_{k=0}^n S_k S_{n-k} u_n \right| < \infty \right\}.$$

He also prove that  $\ell_p(\mathcal{S})$  and  $\ell_\infty(\mathcal{S})$  are BK- spaces with norms

$$\|u\|_{\ell_p(\mathcal{S})} = \left( \sum_{n=0}^{\infty} \left| \frac{1}{S_{n+1} - S_n} \sum_{k=0}^n S_k S_{n-k} u_n \right|^p \right)^{1/p}$$

and

$$\|u\|_{\ell_\infty(\mathcal{S})} = \sup_n \left| \frac{1}{S_{n+1} - S_n} \sum_{k=0}^n S_k S_{n-k} u_n \right|,$$

respectively. We can add the sequence space

$$c_0(\mathcal{S}) = \left\{ u = (u_n) \in w : \lim_n \left| \frac{1}{S_{n+1} - S_n} \sum_{k=0}^n S_k S_{n-k} u_n \right| = 0 \right\}$$

to these spaces which is a BK-space with the norm  $\|u\|_{\ell_\infty(\mathcal{S})}$ .  $\mathcal{S}$  is an invertible matrix, that is  $\mathcal{S}^{-1}$  exists and it is a bijection from  $\lambda$  to  $\lambda(\mathcal{S})$  where  $\lambda$  is  $c_0, \ell_\infty$  or  $\ell_p$ . In this work we are going to investigate some geometric properties of these new sequence spaces.

Previously let us present some preliminary definitions and results.

**Definition 1.** [2]. *A normed space  $X$  is rotund or strictly convex or strictly normed if  $\|tx_1 + (1-t)x_2\| < 1$  whenever  $x_1$  and  $x_2$  are different points of unit sphere  $S_X = \{x \in X : \|x\| = 1\}$  and  $0 < t < 1$ .*

An easier and more useful characterization of rotundity is the following theorem.

**Theorem 1.** [1] *1) Suppose that  $X$  is a normed space. Then  $X$  is rotund if and only if  $\|\frac{1}{2}(x_1 + x_2)\| < 1$  whenever  $x_1$  and  $x_2$  are different points of  $S_X$ .*

*2) A normed space is rotund if and only if each of its two-dimensional subspaces is rotund.*

**Definition 2.** [1] Suppose that  $x_0$  is an element of the unit sphere  $S_X$  of a normed space  $X$ . Then  $x_0$  is a point of smoothness of the unit ball  $B_X$  if there is no more than one support hyperplane for  $B_X$  that supports  $B_X$  at  $x_0$ . The space  $X$  is smooth if each point of  $S_X$  is a point of smoothness of  $B_X$ .

Suppose that  $X$  is a normed space, that  $x \in S_X$  and that  $y \in X$ . Let

$$G_-(x, y) = \lim_{t \rightarrow 0^-} \frac{\|x + ty\| - \|x\|}{t}$$

and

$$G_+(x, y) = \lim_{t \rightarrow 0^+} \frac{\|x + ty\| - \|x\|}{t}.$$

Then  $G_-(x, y)$  and  $G_+(x, y)$  are, respectively, the left-hand and right-hand Gateaux derivative of the norm at  $x$  in the direction  $y$ . The norm is Gateaux differentiable at  $x$  in the direction  $y$  if  $G_-(x, y) = G_+(x, y)$ , in which case the common value of  $G_-(x, y)$  and  $G_+(x, y)$  is denoted by  $G(x, y)$  and is called the Gateaux derivative of the norm at  $x$  in the direction  $y$ . If the norm is Gateaux differentiable at  $x$  in every direction  $y$ , then the norm is Gateaux differentiable at  $x$ . Finally, if the norm is Gateaux differentiable at every point of the unit sphere  $S_X$ , then it is simply said that the norm is Gateaux differentiable [1].

**Theorem 2.** 1) A normed space is smooth if and only if its norm is Gateaux differentiable [1].

2) A normed space is smooth if and only if each of its two-dimensional subspaces is smooth [1].

## 2. MAIN RESULTS

Now let us discuss some geometric properties of  $\ell_p(\mathcal{S})$ . Recall that the unit sphere of the  $n$ -dimensional Euclidean space is rotund. Also, we know that all  $\ell_p, 1 < p < \infty$ , spaces tells us that the unit spheres of them are rotund. Is it also true for  $\ell_p(\mathcal{S})$  spaces?

**Theorem 3.** For  $1 < p < \infty$ , all  $\ell_p(\mathcal{S})$  spaces are rotund.

*Proof.* By the Theorem 1 it is sufficient to prove the rotundity of the space  $\text{span}\{e_1, e_2\} = Z$  in  $\ell_p(\mathcal{S})$  where  $e_1, e_2$  are elements of the unit vector basis of  $\ell_p$ . In other words, we will think about the two-dimensional subspace

$$Z = \{(x_1, x_2, 0, 0, \dots) : (x_1, x_2, 0, 0, \dots)\}$$

of  $\ell_p(\mathcal{S})$ . Let  $x$  and  $y$  be arbitrary elements of  $S_Z$  and then  $x+y = (x_1 + y_1, x_2 + y_2, 0, 0, \dots)$ .

$$\begin{aligned} & \left\| \frac{1}{2}(x+y) \right\|_{\ell_p(\mathcal{S})}^p \\ &= \left\| \left( \left( \mathcal{S} \left( \frac{1}{2}(x+y) \right) \right) \right)_{n=1}^\infty \right\|_{\ell_p}^p \\ &= \left\| \frac{1}{2} [((\mathcal{S}(x))_n)_{n=1}^\infty + ((\mathcal{S}(y))_n)_{n=1}^\infty] \right\|_{\ell_p}^p. \end{aligned}$$

Remember that

$$\|x\|_{\ell_p(\mathcal{S})}^p = \|((\mathcal{S}(x))_n)_{n=1}^\infty\|_{\ell_p}^p = 1 = \|y\|_{\ell_p(\mathcal{S})}^p = \|((\mathcal{S}(y))_n)_{n=1}^\infty\|_{\ell_p}^p$$

and  $\ell_p$  is rotund. Then

$$\left\| \frac{1}{2} [((\mathcal{S}(x))_n)_{n=1}^\infty + ((\mathcal{S}(y))_n)_{n=1}^\infty] \right\|_{\ell_p}^p < 1.$$

□

**Theorem 4.**  $c_0(\mathcal{S})$  and  $\ell_\infty(\mathcal{S})$  are not rotund.

*Proof.* Let us prove it only for  $c_0(\mathcal{S})$ . Because the proof for  $\ell_\infty(\mathcal{S})$  is done in the same way. Consider two special elements

$$\begin{aligned} x &= e_1 + e_2 = (1, 1, 0, 0, \dots) \\ \text{and } y &= e_1 - e_2 = (1, -1, 0, 0, \dots) \end{aligned}$$

in  $c_0(\mathcal{S})$  and let us see that  $x, y \in S_{c_0(\mathcal{S})}$ . Indeed,

$$\begin{aligned} \|y\|_{c_0(\mathcal{S})} &= \|(1, -1, 0, 0, \dots)\|_{c_0(\mathcal{S})} \\ &= \|\mathcal{S}y\|_\infty \\ &= \|y\|_\infty \\ &= 1 \end{aligned}$$

and similarly  $\|x\|_{c_0(\mathcal{S})} = 1$ . Because the triangular infinite matrix  $\mathcal{S}$  defines an isometry between  $c_0$  and  $c_0(\mathcal{S})$ . Now

$$\begin{aligned} \left\| \frac{1}{2}(x+y) \right\| &= \left\| \frac{1}{2}(2, 0, 0, \dots) \right\|_{c_0(\mathcal{S})} \\ &= \|(1, 0, 0, \dots)\|_{c_0(\mathcal{S})} \\ &= \left\| \left( \frac{S_0 S_0}{S_1 - S_0}, 0, 0, \dots \right) \right\|_\infty \\ &= \|(1, 0, 0, \dots)\|_\infty \\ &= 1. \end{aligned}$$

This means  $c_0(\mathcal{S})$  is not rotund. □

Uniform smoothness of a Banach spaces is an indication that the geometry of the balls in the space does not contain sharp edges or cliffs. First let us give the following definition.

**Definition 3.** [1] Suppose that  $X$  is a normed- space. Define a function  $\rho_X : (0, \infty) \rightarrow [0, \infty)$  by the formula

$$\rho_X(t) = \sup \left\{ \frac{1}{2} (\|x+ty\| + \|x-ty\|) - 1 : x, y \in S_X \right\}$$

if  $X \neq \{0\}$ , and by the formula

$$\rho_X(t) = \begin{cases} 0 & \text{if } 0 < t < 1 \\ t-1 & \text{if } t \geq 1 \end{cases}$$

if  $X = \{0\}$ . Then  $\rho_X$  is the modulus of smoothness of  $X$ . The space  $X$  is uniformly smooth if  $\lim_{t \rightarrow 0^+} \rho_X(t)/t = 0$ .

**Remark 1.** *The condition  $\lim_{t \rightarrow 0^+} \rho_X(t)/t = 0$  includes that the norm of the space is uniformly Gateaux differentiable, which includes the Frechet differentiability of the norm function at every point in every direction. Therefore, uniformly smooth spaces are smooth, but the reverse is not true.*

Let us now see that mostly  $\ell_p(\mathcal{S})$  spaces have this nice property.

**Theorem 5.** *For  $1 < p < \infty$ , the space  $\ell_p(\mathcal{S})$  is uniformly smooth.*

*Proof.* Let us first remember that

$$\|x + ty\|_{\ell_p(\mathcal{S})} = \|\mathcal{S}(x + ty)\|_{\ell_p}$$

and later we should calculate we  $\lim_{t \rightarrow 0^+} \rho_X(t)/t$ . But

$$\lim_{t \rightarrow 0^+} \rho_X(t)/t$$

gives the 0/0 uncertainty in the first stage. We can use the L'Hospital rule here and so

$$\lim_{t \rightarrow 0^+} \rho_X(t)/t = \lim_{t \rightarrow 0^+} \frac{d}{dt} (\rho_X(t)).$$

Let us now compute  $\frac{d}{dt} (\rho_X(t))$ . By the definition of modulus of smoothness we get

$$\begin{aligned} & \frac{d}{dt} (\rho_X(t)) \\ &= \sup \left\{ \frac{1}{2} \left( \frac{d}{dt} \|x + ty\| + \frac{d}{dt} \|x - ty\| \right) : x, y \in S_{\ell_p(\mathcal{S})} \right\}. \end{aligned}$$

Now

$$\begin{aligned} & \frac{d}{dt} (\|x + ty\|) = \frac{d}{dt} \left( \|\mathcal{S}(x + ty)\|_{\ell_p} \right) \\ &= \frac{d}{dt} \left( \sum_{n=1}^{\infty} |(\mathcal{S}(x + ty))_n|^p \right)^{1/p} \\ &= \frac{1}{p} \left( \sum_{n=1}^{\infty} |(\mathcal{S}(x + ty))_n|^p \right)^{1-1/p} \frac{d}{dt} \sum_{n=1}^{\infty} |(\mathcal{S}(x + ty))_n|^p \\ &= \frac{1}{p} \left( \sum_{n=1}^{\infty} |(\mathcal{S}(x + ty))_n|^p \right)^{1-1/p} \sum_{n=1}^{\infty} \frac{d}{dt} |(\mathcal{S}(x + ty))_n|^p. \end{aligned}$$

and similarly

$$\begin{aligned} & \frac{d}{dt} (\|x - ty\|) \\ &= \frac{1}{p} \left( \sum_{n=1}^{\infty} |(\mathcal{S}(x - ty))_n|^p \right)^{1-1/p} \sum_{n=1}^{\infty} \frac{d}{dt} |(\mathcal{S}(x - ty))_n|^p. \end{aligned}$$

In particular,

$$\begin{aligned} \frac{d}{dt} |(\mathcal{S}(x+ty))_n|^p &= p |(\mathcal{S}(x+ty))_n|^{p-1} \frac{d}{dt} |(\mathcal{S}(x+ty))_n| \\ \text{and now} \\ \frac{d}{dt} |(\mathcal{S}(x+ty))_n| &= \begin{cases} \frac{d}{dt} ((\mathcal{S}(x+ty))_n), & \text{if } (\mathcal{S}(x+ty))_n \geq 0 \\ -\frac{d}{dt} ((\mathcal{S}(x+ty))_n), & \text{if } (\mathcal{S}(x+ty))_n < 0 \end{cases} \\ &= \begin{cases} (\mathcal{S}y)_n, & \text{if } (\mathcal{S}(x+ty))_n \geq 0 \\ -(\mathcal{S}y)_n, & \text{if } (\mathcal{S}(x+ty))_n < 0 \end{cases} \end{aligned}$$

since the matrix  $\mathcal{S}$  is a linear operator.

Eventually

$$\frac{d}{dt} |(\mathcal{S}(x+ty))_n|^p = p |(\mathcal{S}(x+ty))_n|^{p-1} \begin{cases} (\mathcal{S}y)_n, & \text{if } (\mathcal{S}(x+ty))_n \geq 0 \\ -(\mathcal{S}y)_n, & \text{if } (\mathcal{S}(x+ty))_n < 0 \end{cases}.$$

Now similarly we get

$$\frac{d}{dt} |(\mathcal{S}(x-ty))_n|^p = p |(\mathcal{S}(x-ty))_n|^{p-1} \begin{cases} -(\mathcal{S}y)_n, & \text{if } (\mathcal{S}(x-ty))_n \geq 0 \\ (\mathcal{S}y)_n, & \text{if } (\mathcal{S}(x-ty))_n < 0 \end{cases}$$

When we apply  $t \rightarrow 0^+$  we get the following equations:

$$\begin{aligned} &\lim_{t \rightarrow 0^+} \frac{d}{dt} \|x+ty\| \\ &= \begin{cases} \frac{1}{p} \left( \sum_{n=1}^{\infty} |(\mathcal{S}(x))_n|^p \right)^{1-1/p} \sum_{n=1}^{\infty} p |(\mathcal{S}(x))_n|^{p-1} (\mathcal{S}y)_n, & \text{if } (\mathcal{S}x)_n \geq 0 \\ -\frac{1}{p} \left( \sum_{n=1}^{\infty} |(\mathcal{S}(x))_n|^p \right)^{1-1/p} \sum_{n=1}^{\infty} p |(\mathcal{S}(x))_n|^{p-1} (\mathcal{S}y)_n, & \text{if } (\mathcal{S}x)_n < 0 \end{cases} \end{aligned}$$

and

$$\begin{aligned} &\lim_{t \rightarrow 0^+} \frac{d}{dt} \|x-ty\| \\ &= \begin{cases} -\frac{1}{p} \left( \sum_{n=1}^{\infty} |(\mathcal{S}(x))_n|^p \right)^{1-1/p} \sum_{n=1}^{\infty} p |(\mathcal{S}(x))_n|^{p-1} (\mathcal{S}y)_n, & \text{if } (\mathcal{S}x)_n \geq 0 \\ \frac{1}{p} \left( \sum_{n=1}^{\infty} |(\mathcal{S}(x))_n|^p \right)^{1-1/p} \sum_{n=1}^{\infty} p |(\mathcal{S}(x))_n|^{p-1} (\mathcal{S}y)_n, & \text{if } (\mathcal{S}x)_n < 0 \end{cases} \end{aligned}$$

We just see that

$$\lim_{t \rightarrow 0^+} \frac{d}{dt} \|x+ty\| + \lim_{t \rightarrow 0^+} \frac{d}{dt} \|x-ty\| = 0.$$

Hence we get by this result that

$$\lim_{t \rightarrow 0^+} \frac{d}{dt} (\rho_X(t)) = 0$$

and the proof is now completed.  $\square$

**Conclusion 1.** *From this study, we understand that for  $1 < p < \infty$  the geometric structure of  $\ell_p(\mathcal{S})$  exhibits a certain degree of similarity to the geometric structure of  $n$ -dimensional Euclidean spaces. For example, the closed unit sphere of this space possesses a certain degree of roundness. The best roundness is given by  $p = 2$ . As  $p \rightarrow \infty$ , the roundness decreases and the geometry of the unit circle tends to form a corner point. This is just as it is in the  $\ell_p$  spaces for  $1 < p < \infty$ .*

## REFERENCES

- [1] R. E. Megginson, An Introduction to Banach Space Theory, Springer-Verlag, New York, 1998.
- [2] N. I. Akhiezer and M. G. Krein, Some questions in the theory of moments, Gosudarstv. Naucno-Tehn. Izdat. Ukrain., Kharkov, 1938.
- [3] Dağlı MC., Matrix mappings and compact operators for Schröder sequence spaces, *Turkish Journal of Mathematics*, (2022) 46: 2304 – 2320, doi:10.55730/1300-0098.3270.
- [4] Dağlı MC. A novel conservative matrix arising from Schröder numbers and its properties. *Linear and Multilinear Algebra* 2022; doi: 10.1080/03081087.2022.2061401
- [5] Y. Yılmaz, A. N. Tuncer, S. Yalçın, On Structural Properties of Some Banach Space-Valued Schröder Sequence Spaces, *Symmetry* 2025, 17(7), 977; <https://doi.org/10.3390/sym17070977>
- [6] Y. Yılmaz, A. O. Akdemir, On some topological and geometric properties of some q-Cesàro sequence spaces, *Symmetry*, 2023, 15(4), 791; <https://doi.org/10.3390/sym15040791>
- [7] Y. Yılmaz, H. Bozkurt, On Euler Sequence Spaces, *Open Journal of Mathematical Sciences*, Vol.8, 128-136, 2024.
- [8] Y. Yılmaz, S. Yalçın, On Some Properties of Banach Space-Valued Fibonacci Sequence Spaces, *Communications in Advanced Mathematical Sciences*, Volume: 7 Issue: 2, 80 - 87, 2024.