On Dual Hyperbolic Generalized Pierre Numbers

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Abstract. In this paper, we introduce and develop the concept of generalized dual hyperbolic Pierre numbers, a novel class of number sequences that extends the structural framework of classical Pierre-type sequences through duality and hyperbolic transformations. This generalization offers a unified approach that encompasses both established and newly constructed numerical models.

As distinguished special cases, we examine the dual hyperbolic Pierre numbers and their Lucas-type counterparts, emphasizing their algebraic relationships and unique structural features. Our study presents a comprehensive set of mathematical results, including closed-form identities, matrix representations, and recurrence relations that define the behavior of these sequences.

We further derive Binet-type formulas for explicit term computation and construct generating functions that capture their combinatorial and analytical properties. Additionally, we explore Simson's formulas and establish various summation identities that reveal deeper interconnections among sequence elements.

This investigation contributes to the broader theory of Pierre-type sequences and offers new tools for research in discrete mathematics, algebraic structures, and computational number theory.

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1. Introduction

Hypercomplex number systems, as described in [14], extend the real numbers by introducing additional algebraic structures that generalize classical systems such as the complex and quaternionic numbers. Complex numbers are among the most well-known commutative examples of hypercomplex number systems,

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\},$$

hyperbolic (double, split-complex) numbers, [17],

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\},$$

and dual numbers, [11],

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\},$$

Some non-commutative examples of hypercomplex number systems are quaternions, [12],

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\},$$

octonions [2] and sedenions [18]. The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are real algebras constructed from the real numbers $\mathbb{R}$ via a recursive doubling procedure known as the Cayley–Dickson process. This process can be extended beyond the sedenions to generate higher-dimensional algebras referred to as2^n-ions (see, for example,[3], [13], [15].

Quaternions were introduced by the Irish mathematician W. R. Hamilton (1805–1865) [12] as a generalization of complex numbers. Hyperbolic numbers with complex coefficients were first proposed by J. Cockle in 1848 [5]. Later, H. H. Cheng and S. Thompson [5] introduced dual numbers with complex coefficients, termed complex dual numbers. More recently, Akar, Yüce, and Şahin [1] developed the concept of dual hyperbolic numbers, further enriching the landscape of hypercomplex number systems.

A dual hyperbolic number is a four-dimensional hypercomplex number defined as

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon j a_3,$$

where a_0, a_1, a_2 and a_3 are real numbers.

The set of all dual hyperbolic numbers are denoted by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon j a_3 : a_0, a_1, a_2, a_3 \in \mathbb{R},\ j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The base elements $\{1, j, \varepsilon, \varepsilon j\}$ of dual hyperbolic numbers satisfy the following properties (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j,\ \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0,\ j^2 = j.j = 1, \\ \varepsilon.j &= j.\varepsilon,\ \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0,\ j(\varepsilon j) = (\varepsilon j)j = \varepsilon, \end{aligned}$$

where ε denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j denotes the hyperbolic unit ($j^2 = 1$), and εj denotes the dual hyperbolic unit ($(j\varepsilon)^2 = 0$).

The product of two dual hyperbolic numbers $q = a_0 + ja_1 + \varepsilon a_2 + j\varepsilon a_3$ and $p = b_0 + jb_1 + \varepsilon b_2 + j\varepsilon b_3$ is

$$qp = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0) + \varepsilon(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + j\varepsilon(a_0b_3 + a_1b_2 + a_2b_1 + b_0a_3)$$

and addition of dual hyperbolic numbers is defined as componentwise.

The set of dual hyperbolic numbers constitutes a commutative ring, a real vector space, and a real algebra. However, the structure $\mathbb{H}_{\mathbb{D}}$ does not form a field, as not every dual hyperbolic number possesses a multiplicative inverse. For a detailed discussion, see[1].

To proceed, we briefly recall the definition of the generalized Pierre numbers.

A generalized Pierre sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations

$$W_n = 2W_{n-1} - W_{n-4}, \tag{1.1}$$

with the initial values W_0, W_1, W_2, W_3 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-3)} - W_{-(n-4)},$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

The first few generalized Pierre numbers, corresponding to positive and negative subscripts, are listed in Table 1.

Table 1. A few generalized Pierre numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$2W_2 - W_3$
2	W_2	$2W_1 - W_2$
3	W_3	$2W_0 - W_1$
4	$2W_3 - W_0$	$4W_2 - W_0 - 2W_3$
5	$4W_3 - W_1 - 2W_0$	$4W_1 - 4W_2 + W_3$
6	$8W_3 - 2W_1 - W_2 - 4W_0$	$4W_0 - 4W_1 + W_2$
7	$15W_3 - 4W_1 - 2W_2 - 8W_0$	$W_1 - 4W_0 + 8W_2 - 4W_3$
8	$28W_3 - 8W_1 - 4W_2 - 15W_0$	$W_0 + 8W_1 - 12W_2 + 4W_3$
9	$52W_3 - 15W_1 - 8W_2 - 28W_0$	$8W_0 - 12W_1 + 6W_2 - W_3$
10	$96W_3 - 28W_1 - 15W_2 - 52W_0$	$6W_1 - 12W_0 + 15W_2 - 8W_3$
11	$177W_3 - 52W_1 - 28W_2 - 96W_0$	$6W_0 + 15W_1 - 32W_2 + 12W_3$
12	$326W_3 - 96W_1 - 52W_2 - 177W_0$	$15W_0 - 32W_1 + 24W_2 - 6W_3$
13	$600W_3 - 177W_1 - 96W_2 - 326W_0$	$24W_1 - 32W_0 + 24W_2 - 15W_3$

If we set $W_0 = 0, W_1 = 1, W_2 = 2, W_3 = 4$ then $\{W_n\}$ is the well-known Pierre sequence and if we set $W_0 = 4, W_1 = 2, W_2 = 4, W_3 = 8$ then $\{W_n\}$ is the well-known Pierre -Lucas sequence. That is, both the Pierre sequence $\{P_n\}_{n\geq 0}$ and the Pierre–Lucas sequence$\{C_n\}_{n\geq 0}$ are governed by second-order recurrence relations.

$$P_n = 2P_{n-1} - P_{n-4}, \qquad P_0 = 0, P_1 = 1, P_2 = 2,\ P_3 = 4, \quad n \geq 4, \tag{1.2}$$

and

$$C_n = 2C_{n-1} - C_{n-4}, \qquad C_0 = 4, C_1 = 2, C_2 = 4, C_3 = 8, \qquad n \geq 4. \tag{1.3}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$P_{-n} = 2P_{-(n-3)} - P_{-(n-4)},$$

and

$$C_{-n} = 2C_{-(n-3)} - C_{-(n-4)},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.2) and (1.3) hold for all integer n.

The essential properties of the generalized Pierre numbers, required for the subsequent analysis, are listed below.

- The Binet formula for the generalized Pierre sequence can be derived using its characteristic equation, which is given by

$$z^4 - 2z^3 + 1 = (z^3 - z^2 - z - 1)(z - 1) = 0.$$

The roots of characteristic equation are

$$\begin{aligned}
\alpha &= \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3}, \\
\beta &= \frac{1 + \omega\sqrt[3]{19 + 3\sqrt{33}} + \omega^2\sqrt[3]{19 - 3\sqrt{33}}}{3}, \\
\gamma &= \frac{1 + \omega^2\sqrt[3]{19 + 3\sqrt{33}} + \omega\sqrt[3]{19 - 3\sqrt{33}}}{3}, \\
\delta &= 1,
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

The Binet formula, derived from the recurrence relation and its characteristic roots, is given by

$$\begin{aligned}
W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \qquad (1.4) \\
&= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-1)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-1)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-1)} + \frac{p_4}{(1-\alpha)(1-\beta)(1-\gamma)} \\
&= A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4
\end{aligned}$$

where p_1, p_2, p_3 and p_4 are given below

$$\begin{aligned} p_1 &= W_3-(\beta+\gamma+\delta)W_2+(\beta\gamma+\beta\delta+\gamma\delta)W_1-\beta\gamma\delta W_0, \\ p_2 &= W_3-(\alpha+\gamma+\delta)W_2+(\alpha\gamma+\alpha\delta+\gamma\delta)W_1-\alpha\gamma\delta W_0, \\ p_3 &= W_3-(\alpha+\beta+\delta)W_2+(\alpha\beta+\alpha\delta+\beta\delta)W_1-\alpha\beta\delta W_0, \\ p_4 &= W_3-W_2-W_1-W_0 \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} A_1 &= \frac{W_3-(\beta+\gamma+\delta)W_2+(\beta\gamma+\beta\delta+\gamma\delta)W_1-\beta\gamma\delta W_0}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}, \\ A_2 &= \frac{W_3-(\alpha+\gamma+\delta)W_2+(\alpha\gamma+\alpha\delta+\gamma\delta)W_1-\alpha\gamma\delta W_0}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\ A_3 &= \frac{W_3-(\alpha+\beta+\delta)W_2+(\alpha\beta+\alpha\delta+\beta\delta)W_1-\alpha\beta\delta W_0}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\ A_4 &= \frac{W_3-W_2-W_1-W_0}{-2}. \end{aligned} \tag{1.6}$$

Binet formula of Pierre and Pierre Lucas sequences are

$$P_n = \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} - \frac{1}{2},$$

and

$$C_n = \alpha^n + \beta^n + \gamma^n + 1,$$

respectively.

The generating function for generalized Pierre numbers is

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0+(W_1-2W_0)x+(W_2-2W_1)x^2+(W_3-2W_2)x^3}{1-2x+x^4}. \tag{1.7}$$

This paper defines the dual hyperbolic generalized Pierre numbers in the next section and outlines their key structural properties.

Next, we give the exponential generating function of $\sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence W_n.

Lemma 1. *[16, Lemma 1.4]. Suppose that $f_{GW_n}(x) = \sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is the exponential generating function of the generalized Pierre sequence $\{W_n\}$. Then*

$$\begin{aligned} \sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \frac{(\alpha W_3-\alpha(2-\alpha)W_2+(-\alpha^2+\alpha+1)W_1-W_0)}{2\alpha^2+2\alpha-2}e^{\alpha x} \\ &+ \frac{(\beta W_3-\beta(2-\beta)W_2+(-\beta^2+\beta+1)W_1-W_0)}{2\beta^2+2\beta-2}e^{\beta x} \\ &+ \frac{(\gamma W_3-\gamma(2-\gamma)W_2+(-\gamma^2+\gamma+1)W_1-W_0)}{2\gamma^2+2\gamma-2}e^{\gamma x} + (\frac{W_3-W_2+W_1-W_0}{-2})e^{x}. \end{aligned}$$

As direct consequences of the preceding lemma, we derive the following examples.

Corollary 2. *The Pierre and Pierre–Lucas numbers admit the following exponential generating functions:*

a): $\sum\limits_{n=0}^{\infty} P_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} (\frac{(\alpha^2+\alpha+1)\alpha^n}{2(\alpha^2+\alpha-1)} + \frac{(\beta^2+\beta+1)\beta^n}{2(\beta^2+\beta-1)} + \frac{(\gamma^2+\gamma+1)\gamma^n}{2(\gamma^2+\gamma-1)} - \frac{1}{2}) \frac{x^n}{n!}$
$= \frac{(\alpha^2+\alpha+1)}{2(\alpha^2+\alpha-1)} e^{\alpha x} + \frac{(\beta^2+\beta+1)}{2(\beta^2+\beta-1)} e^{\beta x} + \frac{(\gamma^2+\gamma+1)\gamma^n}{2(\gamma^2+\gamma-1)} e^{\gamma x} - \frac{1}{2} e^x.$

b): $\sum\limits_{n=0}^{\infty} C_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} (\alpha^n+\beta^n+\gamma^n+1) \frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^{\gamma x} + e^x.$

Next, we give some information on published papers related to hyperbolic and Dual hyperbolic numbers in literature.

- Cockle [5] presented the hyperbolic numbers with complex coefficients.
- Akar at al [1] introduced the dual hyperbolic numbers.
- Cheng and Thompson[4] studied dual numbers with complex coefficients.

Next, we give some information related to dual hyperbolic sequences presented in literature.

- Soykan at al [7] introduced dual hyperbolic generalized Pell numbers given by

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}$$

where generalized Pell numbers are given by $V_n = 2V_{n-1} + V_{n-2}$, $V_0 = a, V_1 = b$ $(n \geq 2)$ with the initial values V_0,V_1 not all being zero.

- Cihan at al [8] studied dual hyperbolic Fibonacci and Lucas numbers given by, respectively,

$$DHF_n = F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3},$$

$$DHL_n = L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3},$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1,$ $L_n = L_{n-1} + L_{n-2}, L_0 = 2, L_1 = 1.$

- Soykan at al [20] introduced dual hyperbolic generalized Jacopsthal numbers given by

$$\widehat{J}_n = J_n + jJ_{n+1} + \varepsilon J_{n+2} + j\varepsilon J_{n+3}$$

where $J_n = J_{n-1} + 2J_{n-2}, J_0 = a, J_1 = b.$

- Bród at al [6] studied dual hyperbolic generalized Balancing numbers are

$$DHB_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}$$

where $B_n = 6B_{n-1} - B_{n-2}, B_0 = 0, B_1 = 1.$

- Yılmaz and Soykan [10] introduced dual hyperbolic generalized Guglielmo numbers are

$$\widehat{T}_0 = T_0 + jT_1 + \varepsilon T_2 + j\varepsilon T_3$$

where $T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, T_0 = 0, T_1 = 1, T_2 = 3.$

- Dikmen [9] introduced dual hyperbolic generalised Leonardo numbers given by

$$\widehat{l}_0 = l_0 + jl_1 + \varepsilon l_2 + j\varepsilon l_3$$

where $l_n = 2l_{n-1} - l_{n-3}, l_0 = 1, l_1 = 1, l_2 = 3.$

We define the dual hyperbolic generalized Pierre numbers in the subsequent section and present selected properties.

2. Dual Hyperbolic Generalized Pierre Numbers and their Generating Functions and Binet's Formulas

In this section, we introduce the dual hyperbolic generalized Pierre numbers and derive their corresponding generating functions and Binet-type formulas. Specifically, we define these numbers over the algebra $\mathbb{H}_{\mathbb{D}}$dual hyperbolic numbers. The nth dual hyperbolic generalized Pierre number is

$$\widehat{W}_n = W_n + jW_{n+1} + \varepsilon W_{n+2} + j\varepsilon W_{n+3}. \tag{2.1}$$

To extend the sequence $\{\widehat{W}_n\}_{n\geq 0}$ to negative values of n, we define

$$\widehat{W}_{-n} = W_{-n} + jW_{-n+1} + \varepsilon W_{-n+2} + j\varepsilon W_{-n+3},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.2) holds for all integer n.

Note that

$$\begin{aligned}
\widehat{W}_0 &= W_0 + jW_1 + \varepsilon W_2 + j\varepsilon W_3 = W_0 + jW_1 + \varepsilon W_2 + j\varepsilon W_3, \\
\widehat{W}_1 &= W_1 + jW_2 + \varepsilon W_3 + j\varepsilon W_4 = W_1 + j\widehat{W}_2 + \varepsilon \widehat{W}_3 + j\varepsilon(2\widehat{W}_3 - \widehat{W}_0), \\
\widehat{W}_2 &= W_2 + jW_3 + \varepsilon W_4 + j\varepsilon W_5 = W_2 + j\widehat{W}_3 + \varepsilon(2\widehat{W}_3 - \widehat{W}_0) + j\varepsilon(4\widehat{W}_3 - \widehat{W}_1 - \widehat{2W}_0), \\
\widehat{W}_3 &= W_3 + jW_4 + \varepsilon W_5 + j\varepsilon W_6 = W_3 + j(2\widehat{W}_3 - \widehat{W}_0) + \varepsilon(4\widehat{W}_3 - \widehat{W}_1 - \widehat{2W}_0) \\
&\quad + j\varepsilon(8\widehat{W}_3 - \widehat{W}_2 - 2\widehat{W}_1 - 4\widehat{W}_0).
\end{aligned}$$

It can be easily shown that

$$\widehat{W}_n = 2\widehat{W}_{n-1} - \widehat{W}_{n-4} \tag{2.2}$$

and

$$\widehat{W}_{-n} = 2\widehat{W}_{-(n-3)} - \widehat{W}_{-(n-4)}.$$

The first few dual hyperbolic generalized Pierre numbers, corresponding to positive and negative subscripts, are listed in Table 2.

Table 2. A few dual hyperbolic generalized Pierre numbers

n	$\widehat{W}_n$	$\widehat{W}_{-n}$
0	$\widehat{W}_0$	$\widehat{W}_0$
1	$\widehat{W}_1$	$2\widehat{W}_2-\widehat{W}_3$
2	$\widehat{W}_2$	$2\widehat{W}_1-\widehat{W}_3$
3	$\widehat{W}_3$	$2W_0-W_1$
4	$2\widehat{W}_3-\widehat{W}_0$	$4\widehat{W}_2-\widehat{W}_0-2\widehat{W}_3$
5	$4\widehat{W}_3-\widehat{W}_1-\widehat{2W}_0$	$4\widehat{W}_1-4\widehat{W}_2+\widehat{W}_3$
6	$8\widehat{W}_3-\widehat{W}_2-\widehat{2W}_1-4\widehat{W}_0$	$4\widehat{W}_0-4\widehat{W}_1+\widehat{W}_2$
7	$15\widehat{W}_3-2\widehat{W}_2-4\widehat{W}_1-8\widehat{W}_0$	$\widehat{W}_1-4\widehat{W}_0+8\widehat{W}_2-4\widehat{W}_3$
8	$28\widehat{W}_3-4\widehat{W}_2-8\widehat{W}_1-15\widehat{W}_0$	$\widehat{W}_0+8\widehat{W}_1-12\widehat{W}_2+4\widehat{W}_3$
9	$52\widehat{W}_3-8\widehat{W}_2-15\widehat{W}_1-28\widehat{W}_0$	$8\widehat{W}_0-12\widehat{W}_1+6\widehat{W}_2-\widehat{W}_3$
10	$96\widehat{W}_3-15\widehat{W}_2-28\widehat{W}_1-52\widehat{W}_0$	$6\widehat{W}_1-12\widehat{W}_0+15\widehat{W}_2-8\widehat{W}_3$
11	$177\widehat{W}_3-15\widehat{W}_2-28\widehat{W}_1-96\widehat{W}_0$	$6\widehat{W}_0+15\widehat{W}_1-32\widehat{W}_2+12\widehat{W}_3$
12	$326\widehat{W}_3-52\widehat{W}_2-96\widehat{W}_1-177\widehat{W}_0$	$15\widehat{W}_0-32\widehat{W}_1+24\widehat{W}_2-6\widehat{W}_3$
13	$600\widehat{W}_3-96\widehat{W}_2-177\widehat{W}_1-326\widehat{W}_0$	$24\widehat{W}_1-32\widehat{W}_0+24\widehat{W}_2-15\widehat{W}_3$

The following identities represent the nth dual hyperbolic Pierre and Pierre Lucas numbers as special cases

$$\widehat{P}_n = P_n + jP_{n+1} + \varepsilon P_{n+2} + j\varepsilon P_{n+3} \tag{2.3}$$

and

$$\widehat{C}_n = C_n + jC_{n+1} + \varepsilon C_{n+2} + j\varepsilon C_{n+3} \tag{2.4}$$

respectively. The sequences $\{\widehat{P}_n\}_{n\geq 0}$ and $\{\widehat{C}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\widehat{P}_{-n} = 2P_{-(n-3)} - P_{-(n-4)},$$

and

$$\widehat{C}_{-n} = 2C_{-(n-3)} - C_{-(n-4)},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.3) and (2.4) holds for all integer n.

For dual hyperbolic Pierre numbers (taking $W_n = P_n,\ \ P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 4,$) we get

$$\begin{aligned}
\widehat{P}_0 &= j + 2\varepsilon + 4j\varepsilon, \\
\widehat{P}_1 &= 2j + 4\varepsilon + 8j\varepsilon + 1, \\
\widehat{P}_2 &= 4j + 8\varepsilon + 15j\varepsilon + 2,
\end{aligned}$$

and for dual hyperbolic Pierre Lucas numbers (taking $W_n = C_n,\ \ C_0 = 4, C_1 = 2, C_2 = 4, C_3 = 8,$) we get

$$\begin{aligned}
\widehat{C}_0 &= 2j + 4\varepsilon + 8j\varepsilon + 4, \\
\widehat{C}_1 &= 4j + 8\varepsilon + 12j\varepsilon + 2, \\
\widehat{C}_2 &= 8j + 12\varepsilon + 22j\varepsilon + 4.
\end{aligned}$$

Tables 3 and 4 present selected values of the dual hyperbolic Pierre and dual hyperbolic Pierre–Lucas numbers for both positive and negative indices.

Table 3. Dual hyperbolic Pierre numbers

n	$\widehat{P}_n$	$\widehat{P}_{-n}$
0	$j + 2\varepsilon + 4j\varepsilon$	$j + 2\varepsilon + 4j\varepsilon$
1	$2j + 4\varepsilon + 8j\varepsilon + 1$	$\varepsilon + 2j\varepsilon$
2	$4j + 8\varepsilon + 15j\varepsilon + 2$	$j\varepsilon$
3	$8j + 15\varepsilon + 28j\varepsilon + 4$	-1
4	$15j + 28\varepsilon + 52j\varepsilon + 8$	$-j$
5	$28j + 52\varepsilon + 96j\varepsilon + 15$	$-\varepsilon$

Table 4. Dual hyperbolicPierre- Lucas numbers

n	$\widehat{C}_n$	$\widehat{C}_{-n}$
0	$2j + 4\varepsilon + 8j\varepsilon + 4$	$2j + 4\varepsilon + 8j\varepsilon + 4$
1	$4j + 8\varepsilon + 12j\varepsilon + 2$	$4j + 4j\varepsilon + 2\varepsilon$
2	$8j + 12\varepsilon + 22j\varepsilon + 4$	$4\varepsilon + 2j\varepsilon$
3	$12j + 22\varepsilon + 40j\varepsilon + 8$	$4j\varepsilon + 6$
4	$22j + 40\varepsilon + 72j\varepsilon + 12$	$-4 + 6j$
5	$40j + 72\varepsilon + 132j\varepsilon + 22$	$-4j + 6\varepsilon$

The Binet formula for the dual hyperbolic generalized Pierre numbers is introduced below, and the subsequent notations will be employed consistently throughout the rest of the paper.

$$\widehat{\alpha} = 1 + j\alpha + \varepsilon\alpha^2 + j\varepsilon\alpha^3, \tag{2.5}$$

$$\widehat{\beta} = 1 + j\beta + \varepsilon\beta^2 + j\varepsilon\beta^3. \tag{2.6}$$

$$\widehat{\gamma} = 1 + j\gamma + \varepsilon\gamma^2 + j\varepsilon\gamma^3 \tag{2.7}$$

$$\widehat{\delta} = \widehat{1} = 1 + j + \varepsilon + j\varepsilon, \tag{2.8}$$

We obtain the following identities:

$$
\begin{aligned}
\widehat{\alpha}^2 &= 1+\alpha^2+2\alpha j+2\alpha^2\left(\alpha^2+1\right)\varepsilon+4\alpha^3 j\varepsilon \\
\widehat{\beta}^2 &= 1+\beta^2+2j\beta+(2\beta^4+2\beta^2)\varepsilon+4j\varepsilon\beta^3, \\
\widehat{\alpha}\widehat{\beta} &= 1+\alpha\beta+(\alpha+\beta)\,j+\left(\alpha^2+\beta^2+2\alpha\beta^3+\alpha^3\beta\right)\varepsilon+(\alpha+\beta)\left(\alpha^2+\beta^2\right)j\varepsilon, \\
\widehat{\gamma}^2 &= 1+\gamma^2+2j\gamma+(2\gamma^4+2\gamma^4)\varepsilon+4j\varepsilon\gamma^3, \\
\widehat{\delta}^2 &= \widehat{1}^2=1+2\gamma+4\varepsilon+4j\varepsilon, \\
\widehat{\gamma}\widehat{\delta} &= 1+\gamma+(\gamma^3+\gamma)j+\left(\gamma^3+\gamma^2+\gamma+2\right)\varepsilon+\left(\gamma^2+\gamma+1\right)j\varepsilon
\end{aligned}
$$

THEOREM 3. *(Binet's Formula) For any integer n, the nth dual hyperbolic generalized Pierre number is*

$$\widehat{W}_n=\widehat{\alpha}A_1\alpha^n+\widehat{\beta}A_2\beta^n+\widehat{\gamma}A_3\gamma^n+\widehat{\delta}A_4 \tag{2.9}$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}$, $\widehat{\delta}$ are given as (2.5)-(2.8)

Proof. Using Binet's formula of the generalized Pierre numbers given below

$$W_n=A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4$$

where A_1,A_2,A_3,A_4 are given as in (1.6) we get

$$
\begin{aligned}
\widehat{W}_n &= W_n+jW_{n+1}+\varepsilon W_{n+2}+j\varepsilon W_{n+3}, \\
&= A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4+(A_1\alpha^{n+1}+A_2\beta^{n+1}+A_3\gamma^{n+1}+A_4)j \\
&\quad +(A_1\alpha^{n+2}+A_2\beta^{n+2}+A_3\gamma^{n+2}+A_4)\varepsilon+(A_1\alpha^{n+3}+A_2\beta^{n+3}+A_3\gamma^{n+3}+A_4)j\varepsilon. \\
&= \widehat{\alpha}A_1\alpha^n+\widehat{\beta}A_2\beta^n+\widehat{\gamma}A_3\gamma^n+\widehat{\delta}A_4.
\end{aligned}
$$

This proves (2.9).

For each integer n, the dual hyperbolic Pierre number admits the following Binet-type expression

$$\widehat{P}_n=\frac{(\alpha^2+\alpha+1)\alpha^n\widehat{\alpha}}{2(\alpha^2+\alpha-1)}+\frac{(\beta^2+\beta+1)\beta^n\widehat{\beta}}{2(\beta^2+\beta-1)}+\frac{(\gamma^2+\gamma+1)\gamma^n\widehat{\gamma}}{2(\gamma^2+\gamma-1)}-\frac{\widehat{1}}{2} \tag{2.10}$$

Furthermore, the nth dual hyperbolic Pierre Lucas number admits the following Binet-type representation

$$\widehat{C}_n=\widehat{\alpha}\alpha^n+\widehat{\beta}\beta^n+\widehat{\gamma}\gamma^n+\widehat{1}. \tag{2.11}$$

In the following, we derive the generating function.

THEOREM 4. *The generating function for the dual hyperbolic generalized Pierre numbers is*

$$f_{\widehat{W}_n}(x)=\sum_{n=0}^{\infty}\widehat{W}_nx^n=\frac{\widehat{W}_0+(\widehat{W}_1-2\widehat{W}_0)x+(\widehat{W}_2-2\widehat{W}_1)x^2+(\widehat{W}_3-2\widehat{W}_2)x^3}{1-2x+x^4}.$$

Proof. Suppose that $f_{\widehat{W}_n}(x)$ is the generating function of the dual hyperbolic generalized Pierre numbers and then we can write

$$f_{\widehat{W}_n}(x)=\sum_{n=0}^{\infty}\widehat{W}_nx^n.$$

Using the definition of the dual hyperbolic generalized Pierre numbers, and performing the subtraction $f(x)-xf(x)\ -x^2f(x)$, we arrive at the following result (observe the shift in the index n in the third line)

$$\begin{aligned}(1-2x+x^4)f_{\widehat{W}_n}(x) &= \sum_{n=0}^{\infty}\widehat{W}_nx^n-2x\sum_{n=0}^{\infty}\widehat{W}_nx^n+x^4\sum_{n=0}^{\infty}\widehat{W}_nx^n\\ &= \sum_{n=0}^{\infty}\widehat{W}_nx^n-2\sum_{n=0}^{\infty}\widehat{W}_nx^{n+1}+\sum_{n=0}^{\infty}\widehat{W}_nx^{n+4}\\ &= \sum_{n=0}^{\infty}\widehat{W}_nx^n-2\sum_{n=1}^{\infty}\widehat{W}_{(n-1)}x^n+\sum_{n=4}^{\infty}\widehat{W}_{(n-4)}x^n\\ &= (\widehat{W}_0+\widehat{W}_1x+\widehat{W}_2x^2+\widehat{W}_3x^3)-2(\widehat{W}_0x+\widehat{W}_1x^2+\widehat{W}_2x^3)\\ &\quad +\sum_{n=4}^{\infty}(\widehat{W}_n-2\widehat{W}_{n-1}+\widehat{W}_{n-4})x^n\\ &= \widehat{W}_0+(\widehat{W}_1-2\widehat{W}_0)x+(\widehat{W}_2-2\widehat{W}_1)x^2+(\widehat{W}_3-2\widehat{W}_2)x^3.\end{aligned}$$

The generating functions corresponding to the dual hyperbolic Pierre and dual hyperbolic Pierre Lucas numbers, in special cases, take the form

$$\sum_{n=0}^{\infty}\widehat{P}_nx^n=\frac{(j+2\varepsilon+4j\varepsilon)+(1)x+(-j\varepsilon)x^2+(-\varepsilon-2j\varepsilon)x^3}{1-2x+x^4}$$

and

$$\sum_{n=0}^{\infty}\widehat{C}_nx^n=\frac{(2j+4\varepsilon+8j\varepsilon+4)+(-6-4j\varepsilon)x+(-4\varepsilon-2j\varepsilon)x^2+(-4j-2\varepsilon-4j\varepsilon)x^3}{1-2x+x^4}$$

respectively.

Next, we give the exponential generating function of $\sum\limits_{n=0}^{\infty}\widehat{W}_n\frac{x^n}{n!}$ of the sequence $\widehat{W}_n$.

Lemma 5. *Suppose that $f_{\widehat{W}_n}(x)=\sum\limits_{n=0}^{\infty}\widehat{W}_n\frac{x^n}{n!}$ is the exponential dual hyperbolic generating function of the generalized Pierre sequence $\{\widehat{W}_n\}$.*

Then $\sum\limits_{n=0}^{\infty}\widehat{W}_n\frac{x^n}{n!}$ is given by

$$\sum_{n=0}^{\infty}\widehat{W}_n\frac{x^n}{n!}==A_1e^{\alpha x}\widehat{\alpha}+A_2e^{\beta x}\widehat{\beta}+A_3e^{\gamma x}\widehat{\gamma}+A_4e^{x}\widehat{1}.$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma},\widehat{\delta}$ are given as (2.5)-(2.8)

Proof. Using Binet's formula

$$W_n=A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4.$$

where A_1, A_2, A_3, A_4 are given as in (1.6) we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\widehat{W}_n\frac{x^n}{n!} &= \sum_{n=0}^{\infty}W_n\frac{x^n}{n!}+j\sum_{n=0}^{\infty}W_{n+1}\frac{x^n}{n!}+\varepsilon\sum_{n=0}^{\infty}W_{n+2}\frac{x^n}{n!}+j\varepsilon\sum_{n=0}^{\infty}W_{n+3}\frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty}(A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4)\frac{x^n}{n!}+j\sum_{n=0}^{\infty}(A_1\alpha^{n+1}+A_2\beta^{n+1}+A_3\gamma^{n+1}+A_4)\frac{x^n}{n!} \\
&\quad +\varepsilon\sum_{n=0}^{\infty}(A_1\alpha^{n+2}+A_2\beta^{n+2}+A_3\gamma^{n+2}+A_4)\frac{x^n}{n!} \\
&\quad +j\varepsilon\sum_{n=0}^{\infty}(A_1\alpha^{n+3}+A_2\beta^{n+3}+A_3\gamma^{n+3}+A_4)\frac{x^n}{n!} \\
&= (A_1e^{\alpha x}+A_2e^{\beta x}+A_3e^{\gamma x}+A_4e^{x})+j(A_1\alpha e^{\alpha x}+A_2\beta e^{\beta x}+A_3\gamma e^{\gamma x}+A_4e^{x}) \\
&\quad +\varepsilon(A_1\alpha^2e^{\alpha x}+A_2\beta^2e^{\beta x}+A_3\gamma^2e^{\gamma x}+A_4e^{x})+j\varepsilon(A_1\alpha^3e^{\alpha x}+A_2\beta^3e^{\beta x}+A_3\gamma^3e^{\gamma x}+A_4e^{x}) \\
&= A_1e^{\alpha x}(1+j\alpha++\varepsilon\alpha^2+j\varepsilon\alpha^3)+A_2e^{\beta x}(1+j\beta++\varepsilon\beta^2+j\varepsilon\beta^3) \\
&\quad +A_3e^{\gamma x}(1+j\gamma++\varepsilon\gamma^2+j\varepsilon\gamma^3)+A_4e^{x}(1+j++\varepsilon+j\varepsilon) \\
&= A_1e^{\alpha x}\widehat{\alpha}+A_2e^{\beta x}\widehat{\beta}+A_3e^{\gamma x}\widehat{\gamma}+A_4e^{x}\widehat{1}
\end{aligned}$$

This proves (5). □

From the previous lemma, we obtain the following illustrative cases.

COROLLARY 6. *The exponential generating functions corresponding to the dual hyperbolic Pierre and dual hyperbolic Pierre–Lucas sequences are as follows.*

a):

$$\begin{aligned}
\sum_{n=0}^{\infty}\widehat{P}_n\frac{x^n}{n!} &= (\frac{(\alpha^2+\alpha+1)}{2(\alpha^2+\alpha-1)}e^{\alpha x}+\frac{(\beta^2+\beta+1)}{2(\beta^2+\beta-1)}e^{\beta x}+\frac{(\gamma^2+\gamma+1)}{2(\gamma^2+\gamma-1)}e^{\gamma x}-\frac{1}{2}e^{x}) \\
&\quad +j(\frac{\alpha(\alpha^2+\alpha+1)}{2(\alpha^2+\alpha-1)}e^{\alpha x}+\frac{\beta(\beta^2+\beta+1)}{2(\beta^2+\beta-1)}e^{\beta x}+\frac{\gamma(\gamma^2+\gamma+1)}{2(\gamma^2+\gamma-1)}e^{\gamma x}-\frac{1}{2}e^{x}) \\
&\quad +\varepsilon(\frac{\alpha^2(\alpha^2+\alpha+1)}{2(\alpha^2+\alpha-1)}e^{\alpha x}+\frac{\beta^2(\beta^2+\beta+1)}{2(\beta^2+\beta-1)}e^{\beta x}+\frac{\gamma^2(\gamma^2+\gamma+1)}{2(\gamma^2+\gamma-1)}e^{\gamma x}-\frac{1}{2}e^{x}) \\
&\quad +j\varepsilon(\frac{\alpha^3(\alpha^2+\alpha+1)}{2(\alpha^2+\alpha-1)}e^{\alpha x}+\frac{\beta^3(\beta^2+\beta+1)}{2(\beta^2+\beta-1)}e^{\beta x}+\frac{\gamma^3(\gamma^2+\gamma+1)}{2(\gamma^2+\gamma-1)}e^{\gamma x}-\frac{1}{2}e^{x}).
\end{aligned}$$

b):

$$\begin{aligned}
\sum_{n=0}^{\infty}\widehat{C}_n\frac{x^n}{n!} &= e^{\alpha x}+e^{\beta x}+e^{\gamma x}+e^{x}+j(\alpha e^{\alpha x}+\beta e^{\beta x}+\gamma e^{\gamma x}+e^{x}) \\
&\quad +\varepsilon(\alpha^2e^{\alpha x}+\beta^2e^{\beta x}+\gamma^2e^{\gamma x}+e^{x}) \\
&\quad +j\varepsilon(\alpha^3e^{\alpha x}+\beta^3e^{\beta x}+\gamma^3e^{\gamma x}+e^{x}).
\end{aligned}$$

3. Obtaining Binet Formula From Generating Function

Next, we derive the Binet formula for the generalized dual hyperbolic Pierre numbers $\{\widehat{W}_n\}$by employing the generating function associated with $\widehat{W}_n$.

THEOREM 7. *Binet's formula of generalized Gaussian Pierre numbers*

$$\widehat{W}_n = \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}, \tag{3.1}$$

where

$$\begin{aligned}
q_1 &= \widehat{W}_0\alpha^3 + \left(\widehat{W}_1 - 2\widehat{W}_0\right)\alpha^2 + \left(\widehat{W}_2 - 2\widehat{W}_1\right)\alpha + \widehat{W}_3 - 2\widehat{W}_2, \\
q_2 &= \widehat{W}_0\beta^3 + \left(\widehat{W}_1 - 2\widehat{W}_0\right)\beta^2 + \left(\widehat{W}_2 - 2\widehat{W}_1\right)\beta + \widehat{W}_3 - 2\widehat{W}_2, \\
q_3 &= \widehat{W}_0\gamma^3 + \left(\widehat{W}_1 - 2\widehat{W}_0\right)\gamma^2 + \left(\widehat{W}_2 - 2\widehat{W}_1\right)\gamma + \widehat{W}_3 - 2\widehat{W}_2, \\
q_4 &= \widehat{W}_0\delta^3 + \left(\widehat{W}_1 - 2\widehat{W}_0\right)\delta^2 + \left(\widehat{W}_2 - 2\widehat{W}_1\right)\delta + \widehat{W}_3 - 2\widehat{W}_2.
\end{aligned}$$

Proof. Let

$$h(x) = x^4 - 2x + 1.$$

Accordingly, for suitable parameters α, β, γ and δ we have

$$h(x) = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x).$$

i.e.,

$$x^4 - 2x + 1 = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x). \tag{3.2}$$

Therefore, since $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ and $\frac{1}{\delta}$ satisfy $h(x)$ the values α, β, γ and δ must satisfy

$$h(\frac{1}{x}) = h(\frac{1}{x}) = 1 - \frac{2}{x} + \frac{1}{x^4} = 0.$$

This implies $x^4 - 2x + 1 = 0$. Now, by it follows that

$$\sum_{n=0}^{\infty} \widehat{W}_n x^n = \frac{\left(\widehat{W}_3 - 2\widehat{W}_2\right)x^3 + \left(\widehat{W}_2 - 2\widehat{W}_1\right)x^2 + \left(\widehat{W}_1 - 2\widehat{W}_0\right)x + \widehat{W}_0}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}.$$

Consequently, we obtain

$$\frac{\left(\widehat{W}_3 - 2\widehat{W}_2\right)x^3 + \left(\widehat{W}_2 - 2\widehat{W}_1\right)x^2 + \left(\widehat{W}_1 - 2\widehat{W}_0\right)x + \widehat{W}_0}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)} = \frac{B_1}{(1-\alpha x)} + \frac{B_2}{(1-\beta x)} + \frac{B_3}{(1-\gamma x)} + \frac{B_4}{(1-\delta x)}. \tag{3.3}$$

So

$$\begin{aligned}
&\left(\widehat{W}_3-2\widehat{W}_2\right)x^3+\left(\widehat{W}_2-2\widehat{W}_1\right)x^2+\left(\widehat{W}_1-2\widehat{W}_0\right)x+\widehat{W}_0\\
=\ &B_1(1-\beta x)(1-\gamma x)(1-\delta x)+B_2(1-\alpha x)(1-\gamma x)(1-\delta x)\\
&+B_3(1-\alpha x)(1-\beta x)(1-\delta x)+B_3(1-\alpha x)(1-\beta x)(1-\gamma x).
\end{aligned}$$

Assume that we take $x=\frac{1}{\alpha}$, we get $\widehat{W}_0+\frac{1}{\alpha}\left(\widehat{W}_1-2\widehat{W}_0\right)+\frac{1}{\alpha^2}\left(\widehat{W}_2-2\widehat{W}_1\right)+\frac{1}{\alpha^3}\left(\widehat{W}_3-2\widehat{W}_2\right)$
$=-B_1\left(\frac{1}{\alpha}\beta-1\right)\left(\frac{1}{\alpha}\gamma-1\right)\left(\frac{1}{\alpha}\delta-1\right).$

This gives

$$\begin{aligned}
&\frac{B_1=\alpha^3\left(\widehat{W}_0+\frac{1}{\alpha^2}\left(\widehat{W}_2-2\widehat{W}_1\right)+\frac{1}{\alpha^3}\left(\widehat{W}_3-2\widehat{W}_2\right)+\frac{1}{\alpha}\left(\widehat{W}_1-2\widehat{W}_0\right)\right)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\\
=\ &\frac{\widehat{W}_0\alpha^3+\left(\widehat{W}_1-2\widehat{W}_0\right)\alpha^2+\left(\widehat{W}_2-2\widehat{W}_1\right)\alpha+\widehat{W}_3-2\widehat{W}_2}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
B_2&=\frac{\widehat{W}_0\beta^3+\left(\widehat{W}_1-2\widehat{W}_0\right)\beta^2+\left(\widehat{W}_2-2\widehat{W}_1\right)\beta+\widehat{W}_3-2\widehat{W}_2}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)},\\
B_3&=\frac{\widehat{W}_0\gamma^3+\left(\widehat{W}_1-2\widehat{W}_0\right)\gamma^2+\left(\widehat{W}_2-2\widehat{W}_1\right)\gamma+\widehat{W}_3-2\widehat{W}_2}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)},\\
B_4&=\frac{\widehat{W}_0\delta^3+\left(\widehat{W}_1-2\widehat{W}_0\right)\delta^2+\left(\widehat{W}_2-2\widehat{W}_1\right)\delta+\widehat{W}_3-2\widehat{W}_2}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}.
\end{aligned}$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty}\widehat{W}_nx^n=B_1(1-\alpha x)^{-1}+B_2(1-\beta x)^{-1}+B_3(1-\gamma x)^{-1}+B_4(1-\delta x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty}\widehat{W}_nx^n=B_1\sum_{n=0}^{\infty}\alpha^nx^n+B_2\sum_{n=0}^{\infty}\beta^nx^n+B_3\sum_{n=0}^{\infty}\gamma^nx^n+B_4\sum_{n=0}^{\infty}\delta^nx^n=\sum_{n=0}^{\infty}(B_1\alpha^n+B_2\beta^n+B_3\gamma^n+B_4\delta^n)x^n.$$

Hence, comparing the coefficients of corresponding terms, we derive

$$\widehat{W}_n=B_1\alpha^n+B_2\beta^n+B_3\gamma^n+B_4\delta^n.$$

THEOREM 8. *The following identities are valid for every pair of integers m and n:*

$$\widehat{W}_{m+n}=P_{m-2}\widehat{W}_{n+3}-P_{m-5}\widehat{W}_{n+2}-P_{m-4}\widehat{W}_{n+1}-P_{m-3}\widehat{W}_n.$$

Proof.Assuming $m,n\geq 0$, we prove Theorem 8 by induction on m. In the base case $m=0$, we have

$$\widehat{W}_n=P_{-2}\widehat{W}_{n+3}-P_{-5}\widehat{W}_{n+2}-P_{-4}\widehat{W}_{n+1}-P_{-3}\widehat{W}_n.$$

which is true since $P_{-2}=0, P_{-3}=-1, P_{-4}=0, P_{-5}=0$. Suppose the identity is valid for $m\leq k$. For $m=k+1$, it follows that

$$\begin{aligned}
\widehat{W}_{k+1+n} &= 2\widehat{W}_{n+k}+-\widehat{W}_{n+k-3} \\
&= 2\left(P_{k-2}\widehat{W}_{n+3}-P_{k-5}\widehat{W}_{n+2}-P_{k-4}\widehat{W}_{n+1}-P_{k-3}\widehat{W}_{n}\right) \\
&\quad -\left(2P_{k-5}\widehat{W}_{n+3}-P_{k-8}\widehat{W}_{n+2}-P_{k-6}\widehat{W}_{n+1}-P_{k-6}\widehat{W}_{n}\right)
\end{aligned}$$

By applying mathematical induction on m, we complete the proof of Theorem 8.

The remaining cases of m and n can be proved similarly for all integers m,n. □

Taking $\widehat{W}_n=\widehat{P}_n$ or $\widehat{W}_n=\widehat{C}_n$ in above Theorem, respectively, we obtain:

Corollary 9.

$$\begin{aligned}
\widehat{P}_{m+n} &= P_{m-2}\widehat{P}_{n+3}-P_{m-5}\widehat{P}_{n+2}-P_{m-4}\widehat{P}_{n+1}-P_{m-3}\widehat{P}_{n} \\
\widehat{C}_{m+n} &= P_{m-2}\widehat{C}_{n+3}-P_{m-5}\widehat{C}_{n+2}-P_{m-4}\widehat{C}_{n+1}-P_{m-3}\widehat{C}_{n}.
\end{aligned}$$

4. SIMSON'S FORMULA

This section introduces Simpson's formula for the dual hyperbolic generalized Pierre numbers, derived as a particular instance of [22,Theorem 4.1].

Theorem 10. *(Simpson's formula for dual hyperbolic generalized Pierre numbers) For all integers n we have,*

$$\begin{vmatrix}
\widehat{W}_{n+3} & \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_{n} \\
\widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_{n} & \widehat{W}_{n-1} \\
\widehat{W}_{n+1} & \widehat{W}_{n} & \widehat{W}_{n-1} & \widehat{W}_{n-2} \\
\widehat{W}_{n} & \widehat{W}_{n-1} & \widehat{W}_{n-2} & \widehat{W}_{n-3}
\end{vmatrix} = \begin{vmatrix}
\widehat{W}_{3} & \widehat{W}_{2} & \widehat{W}_{1} & \widehat{W}_{0} \\
\widehat{W}_{2} & \widehat{W}_{1} & \widehat{W}_{0} & \widehat{W}_{-1} \\
\widehat{W}_{1} & \widehat{W}_{0} & \widehat{W}_{-1} & \widehat{W}_{-2} \\
\widehat{W}_{0} & \widehat{W}_{-1} & \widehat{W}_{-2} & \widehat{W}_{-3}
\end{vmatrix}$$

$$=(\widehat{\widehat{W}}_3-\widehat{W}_2-\widehat{W}_1-\widehat{W}_0)(\widehat{W}_3^3-\widehat{W}_2^3-\widehat{W}_1^3-\widehat{W}_0^3+(-5\widehat{W}_2+\widehat{W}_1+\widehat{W}_0)\widehat{W}_3^2+(7\widehat{W}_3-3\widehat{W}_0-\widehat{W}_1)\widehat{W}_2^2$$
$$+(3\widehat{W}_3+\widehat{W}_2-\widehat{W}_0)\widehat{W}_1^2+(\widehat{W}_3+\widehat{W}_2+\widehat{W}_1)\widehat{W}_0^2+4(-\widehat{W}_2\widehat{W}_3-\widehat{W}_0\widehat{W}_3+\widehat{W}_0\widehat{W}_2)\widehat{W}_1).$$

Proof. Using Theorem 10 it can be proved by using induction or use

[22,Theorem 4.1]

From Theorem 10 we get the following corollary.

Corollary 11. *The Simson formulas for the dual hyperbolic Pierre and Pierre–Lucas numbers hold for all integers n and are stated as follows.*

a):
$$\begin{vmatrix}
\widehat{P}_{n+3} & \widehat{P}_{n+2} & \widehat{P}_{n+1} & \widehat{P}_{n} \\
\widehat{P}_{n+2} & \widehat{P}_{n+1} & \widehat{P}_{n} & \widehat{P}_{n-1} \\
\widehat{P}_{n+1} & \widehat{P}_{n} & \widehat{P}_{n-1} & \widehat{P}_{n-2} \\
\widehat{P}_{n} & \widehat{P}_{n-1} & \widehat{P}_{n-2} & \widehat{P}_{n-3}
\end{vmatrix} = 2+2j+8\varepsilon+8j\varepsilon,$$

b): $\begin{vmatrix} \widehat{C}_{n+3} & \widehat{C}_{n+2} & \widehat{C}_{n+1} & \widehat{C}_{n} \\ \widehat{C}_{n+2} & \widehat{C}_{n+1} & \widehat{C}_{n} & \widehat{C}_{n-1} \\ \widehat{C}_{n+1} & \widehat{C}_{n} & \widehat{C}_{n-1} & \widehat{C}_{n-2} \\ \widehat{C}_{n} & \widehat{C}_{n-1} & \widehat{C}_{n-2} & \widehat{C}_{n-3} \end{vmatrix} = -352-352j-1408\varepsilon-1408j\varepsilon,$

5. Linear Sums

This section provides the summation identities for the dual hyperbolic generalized Pierre numbers, encompassing cases with positive and negative indices.

We proceed to present the summation identities pertaining to the generalized Pierre numbers.

Theorem 12. *For the dual hyperbolic Pierre numbers, we have the following formulas:*

(a): $\sum_{k=0}^{n} W_k = \frac{1}{2}(-(n+3))W_{n+3}+(n+4)W_{n+2}+(n+3)W_{n+1}+(n+4)W_n+3W_3-4W_2-3W_1-2W_0).$

(b): $\sum_{k=0}^{n} W_{2k} = \frac{1}{2}(-(n+2)W_{2n+2}+(n+3)W_{2n+1}+(n+3)W_{2n}+(n+2)W_{2n-1}+2W_3-2W_2-3W_1-W_0).$

(c): $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{2}(-(n+1)W_{2n+2}+(n+3)W_{2n+1}+(n+2)W_{2n}+(n+2)W_{2n-1}+2W_3-3W_2-W_1-2W_0).$

Proof. For the proof, see Soykan [19, Theorem 3.10]. □

Theorem 13. *The formulas governing the dual hyperbolic Pierre numbers are presented below.*

(a): $\sum_{k=0}^{n} \widehat{W}_k = \frac{1}{2}(-(n+3))\widehat{W}_{n+3}+(n+4)\widehat{W}_{n+2}+(n+3)\widehat{W}_{n+1}+(n+4)\widehat{W}_n+3\widehat{W}_3-4\widehat{W}_2-3\widehat{W}_1-2\widehat{W}_0).$

(b): $\sum_{k=0}^{n} \widehat{W}_{2k} = \frac{1}{2}(-(n+2)\widehat{W}_{2n+2}+(n+3)\widehat{W}_{2n+1}+(n+3)\widehat{W}_{2n}+(n+2)\widehat{W}_{2n-1}+2\widehat{W}_3-2\widehat{W}_2-3\widehat{W}_1-\widehat{W}_0).$

(c): $\sum_{k=0}^{n} \widehat{W}_{2k+1} = \frac{1}{2}(-(n+1)\widehat{W}_{2n+2}+(n+3)\widehat{W}_{2n+1}+(n+2)\widehat{W}_{2n}+(n+2)\widehat{W}_{2n-1}+2\widehat{W}_3-3\widehat{W}_2-\widehat{W}_1-2\widehat{W}_0).$

Proof. Use Theorem 12 and the definition of $\widehat{W}n$. □

As a special case of Theorem 13, we state the following corollary.

Corollary 14. *For $n \geq 0$, dual hyperbolic Pierre numbers have the following properties:*

(a): $\sum_{k=0}^{n} \widehat{P}_k = \frac{1}{2}(-(n+3))\widehat{P}_{n+3}+(n+4)\widehat{P}_{n+2}+(n+3)\widehat{P}_{n+1}+(n+4)\widehat{P}_n+1-8j\varepsilon-3\varepsilon).$

(b): $\sum_{k=0}^{n} \widehat{P}_{2k} = \frac{1}{2}(-(n+2))\widehat{P}_{2n+2}+(n+3)\widehat{P}_{2n+1}+(n+3)\widehat{P}_{2n}+(n+2)\widehat{P}_{2n-1}+j-2j\varepsilon+1).$

(c): $\sum_{k=0}^{n} \widehat{P}_{2k+1} = \frac{1}{2}(-(n+1)\widehat{P}_{2n+2}+(n+3)\widehat{P}_{2n+1}+(n+2)\widehat{P}_{2n}+(n+2)\widehat{P}_{2n-1}+1-5j\varepsilon-2\varepsilon).$

As a second illustrative case of the preceding theorem, the dual hyperbolic Pierre–Lucas numbers satisfy the following summation identities:

Corollary 15. *For $n \geq 0$, dual hyperbolic Pierre Lucas numbers have the following properties.*

(a): $\sum_{k=0}^{n} \widehat{C}_k = \frac{1}{2}(-(n+3)\widehat{C}_{n+3}+(n+4)\widehat{C}_{n+2}+(n+3)\widehat{C}_{n+1}+(n+4)\widehat{C}_n-12j-14\varepsilon-20j\varepsilon-6).$

(b): $\sum\limits_{k=0}^{n}\widehat{C}_{2k}=\frac{1}{2}(-(n+2)\widehat{C}_{2n+2}+(n+3)\widehat{C}_{2n+1}+(n+3)\widehat{C}_{2n}+(n+2)\widehat{C}_{2n-1}+-6j-8\varepsilon-8j\varepsilon-2).$

(c): $\sum\limits_{k=0}^{n}\widehat{C}_{2k+1}=\frac{1}{2}(-(n+1)\widehat{C}_{2n+2}+(n+3)\widehat{C}_{2n+1}+(n+2)\widehat{C}_{2n}+(n+2)\widehat{C}_{2n-1}+-8j-8\varepsilon-14j\varepsilon-6).$

The ordinary generating functions for representative special cases of the dual hyperbolic generalized Pierre numbers are given below.

Theorem 16. *The ordinary generating functions of the sequences $\widehat{W}_{2n}$, $\widehat{W}_{2n+1}$ are given as follows:*

(a): $\sum_{n=0}^{\infty}\widehat{W}_{2n}x^n=\frac{\widehat{W}_3(2x^2)+\widehat{W}_2\left(x^3-4x^2+x\right)-\widehat{W}_1\left(2x^3\right)+\widehat{W}_0\left(x^2-4x+1\right)}{x^4+2x^2-4x+1}.$

(b): $\sum_{n=0}^{\infty}\widehat{W}_{2n+1}x^n=\frac{\widehat{W}_3\left(x^3+x\right)-\widehat{W}_2\left(2x^3\right)-\widehat{W}_1\left(x^2-4x+1\right)-\widehat{W}_0(2x^2)}{x^4+2x^2-4x+1}.$

Proof.Similarly,the proof can be constructed as in[4].

From the last Theorem, we have the following Corollary which gives sum formula of dual hyperbolic Pierre numbers (Take $\widehat{W}_n=\widehat{P}_n$ with $\widehat{P}_0=j+2\varepsilon+4j\varepsilon$,$\widehat{P}_1=2j+4\varepsilon+8j\varepsilon+1$, $\widehat{P}_2=4j+8\varepsilon+15j\varepsilon+2$, $\widehat{P}_3=8j+15\varepsilon+28j\varepsilon+4$)

Corollary 17. *For $n\geq 0$ the dual hyperbolic Pierre numbers satisfy the following properties.*

(a): $\sum_{n=0}^{\infty}\widehat{P}_{2n}x^n=\frac{(j+2\varepsilon+4j\varepsilon)+(2-j\varepsilon)x+jx^2-j\varepsilon x^3}{x^4+2x^2-4x+1}.$

(b): $\sum_{n=0}^{\infty}\widehat{P}_{2n+1}x^n=\frac{(-2j-4\varepsilon-8j\varepsilon-1)+(8+31\varepsilon+16j+60j\varepsilon)x+(-4j-8\varepsilon-1-16j\varepsilon)x^2+(-\varepsilon-2j\varepsilon)x^3}{x^4+2x^2-4x+1}.$

6. Matrices related with Dual Hyperbolic Generalized Pierre Numbers

The matrix A, a square matrix of order 4, is defined as

$$A=\begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where $detA=1$. Note that

$$A^n=\begin{pmatrix} P_{n+1} & -P_{n-2} & -P_{n-1} & -P_n \\ P_n & -P_{n-3} & -P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-4} & -P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-5} & -P_{n-4} & -P_{n-3} \end{pmatrix}$$

for the proof see [21].

We now state the following lemma.

Lemma 18. *For $n\geq 0$ the following identity holds:*

$$\begin{pmatrix} \widehat{W}_{n+3} \\ \widehat{W}_{n+2} \\ \widehat{W}_{n+1} \\ \widehat{W}_n \end{pmatrix}=\begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n\begin{pmatrix} \widehat{W}_3 \\ \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix}.$$

Proof. Identity(18) admits a proof via mathematical induction on n. If $n = 0$ we obtain

$$\begin{pmatrix} \widehat{W}_3 \\ \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} \widehat{W}_3 \\ \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix}$$

which is true. Assume that the identity holds for $n = k$. Then, the following identity is valid.

$$\begin{pmatrix} \widehat{W}_{k+3} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \\ \widehat{W}_k \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \widehat{W}_3 \\ \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix}.$$

For $n = k + 1$, we get

$$\begin{aligned} \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} \widehat{W}_3 \\ \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix} &= \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \widehat{W}_3 \\ \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{W}_{k+3} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \\ \widehat{W}_k \end{pmatrix} \\ &= \begin{pmatrix} \widehat{W}_{k+4} \\ \widehat{W}_{k+3} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \end{pmatrix}. \end{aligned}$$

Hence,the proof completed.□

We define

$$N_{\widehat{W}} = \begin{pmatrix} \widehat{W}_3 & \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} & \widehat{W}_{-3} \end{pmatrix}, \tag{6.1}$$

$$E_{\widehat{W}} = \begin{pmatrix} \widehat{W}_{n+3} & \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n \\ \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} \\ \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} \\ \widehat{W}_n & \widehat{W}_{n-1-} & \widehat{W}_{n-2} & \widehat{W}_{n-3} \end{pmatrix}. \tag{6.2}$$

Now, we have the following theorem with $N_{\widehat{W}}$ and $E_{\widehat{W}}$,

THEOREM 19. *Using $N_{\widehat{W}}$ and $E_{\widehat{W}}$, we get*

$$A^n N_{\widehat{W}} = E_{\widehat{W}}.$$

Proof. Note that we get

$$\begin{aligned} A^n N_{\widehat{W}} &= \begin{pmatrix} P_{n+1} & -P_{n-2} & -P_{n-1} & -P_n \\ P_n & -P_{n-3} & -P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-4} & -P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-5} & -P_{n-4} & -P_{n-3} \end{pmatrix} \begin{pmatrix} \widehat{W}_3 & \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} & \widehat{W}_{-3} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned}
a_{11} &= P_{n+1}\widehat{W}_3 - P_{n-2}\widehat{W}_2 - P_{n-1}\widehat{W}_1 - P_n\widehat{W}_0 = \widehat{W}_{n+3}, \\
a_{12} &= P_{n+1}\widehat{W}_2 - P_{n-2}\widehat{W}_1 - P_{n-1}\widehat{W}_0 - P_n\widehat{W}_{-1} = \widehat{W}_{n+2}, \\
a_{13} &= P_{n+1}\widehat{W}_1 - P_{n-2}\widehat{W}_0 - P_{n-1}\widehat{W}_{-1} - P_n\widehat{W}_{-2} = \widehat{W}_{n+1}, \\
a_{14} &= P_{n+1}\widehat{W}_0 - P_{n-2}\widehat{W}_{-1} - P_{n-1}\widehat{W}_{-2} - P_n\widehat{W}_{-3} = \widehat{W}_n, \\
a_{21} &= P_n\widehat{W}_3 - P_{n-3}\widehat{W}_2 - P_{n-2}\widehat{W}_1 - P_{n-1}\widehat{W}_0 = \widehat{W}_{n+2}, \\
a_{22} &= P_n\widehat{W}_2 - P_{n-3}\widehat{W}_1 - P_{n-2}\widehat{W}_0 - P_{n-1}\widehat{W}_{-1} = \widehat{W}_{n+1}, \\
a_{23} &= P_n\widehat{W}_1 - P_{n-3}\widehat{W}_0 - P_{n-2}\widehat{W}_{-1} - P_{n-1}\widehat{W}_{-2} = \widehat{W}_n, \\
a_{24} &= P_n\widehat{W}_0 - P_{n-3}\widehat{W}_{-1} - P_{n-2}\widehat{W}_{-2} - P_{n-1}\widehat{W}_{-3} = \widehat{W}_{n-1}, \\
a_{31} &= P_{n-1}\widehat{W}_3 - P_{n-4}\widehat{W}_2 - P_{n-3}\widehat{W}_1 - P_{n-2}\widehat{W}_0 = \widehat{W}_{n+1}, \\
a_{32} &= P_{n-1}\widehat{W}_2 - P_{n-4}\widehat{W}_1 - P_{n-3}\widehat{W}_0 - P_{n-2}\widehat{W}_{-1} = \widehat{W}_n, \\
a_{33} &= P_{n-1}\widehat{W}_1 - P_{n-4}\widehat{W}_0 - P_{n-3}\widehat{W}_{-1} - P_{n-2}\widehat{W}_{-2} = \widehat{W}_{n-1}, \\
a_{34} &= P_{n-1}\widehat{W}_0 - P_{n-4}\widehat{W}_{-1} - P_{n-3}\widehat{W}_{-2} - P_{n-2}\widehat{W}_{-3} = \widehat{W}_{n-2}, \\
a_{41} &= P_{n-2}\widehat{W}_3 - P_{n-5}\widehat{W}_2 - P_{n-4}\widehat{W}_1 - P_{n-3}\widehat{W}_0 = \widehat{W}_n, \\
a_{42} &= P_{n-2}\widehat{W}_2 - P_{n-5}\widehat{W}_1 - P_{n-4}\widehat{W}_0 - P_{n-3}\widehat{W}_{-1} = \widehat{W}_{n-1}, \\
a_{43} &= P_{n-2}\widehat{W}_1 - P_{n-5}\widehat{W}_0 - P_{n-4}\widehat{W}_{-1} - P_{n-3}\widehat{W}_{-2} = \widehat{W}_{n-2}, \\
a_{44} &= P_{n-2}\widehat{W}_0 - P_{n-5}\widehat{W}_{-1} - P_{n-4}\widehat{W}_{-2} - P_{n-3}\widehat{W}_{-3} = \widehat{W}_{n-3}.
\end{aligned}$$

Using the theorem (8) the proof is done. □

By setting $\widehat{W}_n = \widehat{P}_n$ with $\widehat{P}_0, \widehat{P}_1, \widehat{P}_2, \widehat{P}_3$ in (6.1) and (6.2)

$\widehat{W}_n = C_n$ with $\widehat{C}_0, \widehat{C}_1, \widehat{C}_2, \widehat{C}_3$ in (6.1) and (6.2)

respectively, we get:

$$N_{\widehat{P}} = \begin{pmatrix} 8j+15\varepsilon+28j\varepsilon+4 & 4j+8\varepsilon+15j\varepsilon+2 & 2j+4\varepsilon+8j\varepsilon+1 & j+2\varepsilon+4j\varepsilon \\ 4j+8\varepsilon+15j\varepsilon+2 & 2j+4\varepsilon+8j\varepsilon+1 & j+2\varepsilon+4j\varepsilon & \varepsilon+2j\varepsilon \\ 2j+4\varepsilon+8j\varepsilon+1 & j+2\varepsilon+4j\varepsilon & \varepsilon+2j\varepsilon & j\varepsilon \\ j+2\varepsilon+4j\varepsilon & \varepsilon+2j\varepsilon & j\varepsilon & -1 \end{pmatrix},$$

$$E_{\widehat{P}} = \begin{pmatrix} \widehat{P}_{n+3} & \widehat{P}_{n+2} & \widehat{P}_{n+1} & \widehat{P}_n \\ \widehat{P}_{n+2} & \widehat{P}_{n+1} & \widehat{P}_n & \widehat{P}_{n-1} \\ \widehat{P}_{n+1} & \widehat{P}_n & \widehat{P}_{n-1} & \widehat{P}_{n-2} \\ \widehat{P}_n & \widehat{P}_{n-1} & \widehat{P}_{n-2} & \widehat{P}_{n-3} \end{pmatrix},$$

$$N_{\widehat{C}} = \begin{pmatrix} 12j+22\varepsilon+40j\varepsilon+8 & 8j+12\varepsilon+22j\varepsilon+4 & 4j+8\varepsilon+12j\varepsilon+2 & 2j+4\varepsilon+8j\varepsilon+4 \\ 8j+12\varepsilon+22j\varepsilon+4 & 4j+8\varepsilon+12j\varepsilon+2 & 2j+4\varepsilon+8j\varepsilon+4 & 4j+4j\varepsilon+2\varepsilon \\ 4j+8\varepsilon+12j\varepsilon+2 & 2j+4\varepsilon+8j\varepsilon+4 & 4j+4j\varepsilon+2\varepsilon & 4\varepsilon+2j\varepsilon \\ 2j+4\varepsilon+8j\varepsilon+4 & 4j+4j\varepsilon+2\varepsilon & 4\varepsilon+2j\varepsilon & 4j\varepsilon+6 \end{pmatrix},$$

$$E_{\widehat{C}} = \begin{pmatrix} \widehat{C}_{n+3} & \widehat{C}_{n+2} & \widehat{C}_{n+1} & \widehat{C}_n \\ \widehat{C}_{n+2} & \widehat{C}_{n+1} & \widehat{C}_n & \widehat{C}_{n-1} \\ \widehat{C}_{n+1} & C_n & \widehat{C}_{n-1} & \widehat{C}_{n-2} \\ \widehat{C}_n & \widehat{C}_{n-1} & \widehat{C}_{n-2} & \widehat{C}_{n-3} \end{pmatrix}.$$

From Theorem [19], we can write the following corollary.

Corollary 20. *The following identities hold:*

- **a):** $A^n N_{\widehat{P}} = E_{\widehat{P}}$.
- **b):** $A^n N_{\widehat{C}} = E_{\widehat{C}}$.

7. Conclusions

Sequences governed by recurrence relations have long stood as a cornerstone of mathematical inquiry, valued for their inherent structure and wide-ranging applicability across disciplines such as physics, engineering, architecture, biology, and the arts. Classical second-order integer sequences—such as the Fibonacci, Lucas, Pell, and Jacobsthal numbers—exemplify this versatility. The Fibonacci sequence, introduced by Leonardo of Pisa in his 1202 treatise Liber Abaci, was originally formulated through a rabbit population model and has since become a foundational tool for analyzing recursive behavior and mathematical identities.

In this study, we extend the classical framework to fourth-order recurrence systems by introducing the dual hyperbolic Pierre numbers, along with two distinguished subclasses. For these newly defined sequences, we establish Binet-type formulas, derive ordinary and exponential generating functions, and formulate generalized Simson-type identities. Our investigation further includes closed-form summation expressions, algebraic characterizations, recurrence dynamics, and matrix-based representations. We present our fourth-order generalizations as a natural evolution within this broader mathematical landscape—offering novel insights and robust tools for modeling, analysis, and optimization in both pure and applied domains.

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