A Study on Hyperbolic Generalized Adrien Numbers

Abstract. In this study, we define Hyperbolic Adrien numbers, focusing on two specific cases: Hyperbolic Adrien numbers and Hyperbolic Adrien-Lucas numbers. We then examine and present various properties of these sequences, including identities, matrix representations, recurrence relations, Binet's formulas, generating functions, exponential functions, Simson's formulas, and summation formulas.

Keywords. Hyperbolic Adrien numbers, Hyperbolic Adrien-Lucas numbers.

1. Introduction

In this section, we present the necessary background on the definition and fundamental properties of Adrien numbers.

1.1. Adrien Numbers. Numerous researchers have investigated the generalized (r, s, t, u) sequence, which encompasses various notable numerical constructs. Among these is the sequence of generalized Adrien numbers, formally introduced by Soykan [21]. Prior to presenting our original contributions, we briefly review several fundamental properties of the generalized Adrien numbers, including their recurrence relations, Binet-type formula, and generating function.

A generalized Adrien sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations

$$W_n = 3W_{n-1} - W_{n-2} - W_{n-4}, \ n \geq 4, \tag{1.1}$$

with the initial values W_0, W_1, W_2, W_3 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-2)} + 3W_{-(n-3)} - W_{-(n-4)},$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n. Soykan has investigated this specific numerical sequence in a recent study, for more details, see [21].

Characteristic equation of $\{W_n\}$ is

$$z^4 - 3z^3 + z^2 + 1 = (z^3 - 2z^2 - z - 1)(z - 1) = 0.$$

The roots of characteristic equation are

$$\begin{aligned}
\alpha &= \frac{2}{3} + \left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}, \\
\beta &= \frac{2}{3} + \omega\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega^2\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}, \\
\gamma &= \frac{2}{3} + \omega^2\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}, \\
\delta &= 1.
\end{aligned}$$

Where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned}
\alpha + \beta + \gamma + \delta &= 3, \\
\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= 1, \\
\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= 0, \\
\alpha\beta\gamma\delta &= 1.
\end{aligned}$$

We see that

$$\begin{aligned}
\alpha + \beta + \gamma &= 2, \\
\alpha\beta + \alpha\gamma + \beta\gamma &= -1, \\
\alpha\beta\gamma &= 1.
\end{aligned}$$

Using the roots and the recurrence relation of $\{W_n\}$ the Binet's formula for the generalized Adrien numbers can be expressed for all integers n as follows

$$\begin{aligned}
W_n &= \frac{p_1\alpha^n}{4\alpha^2 + 3\alpha - 1} + \frac{p_2\beta^n}{4\beta^2 + 3\beta - 1} + \frac{p_3\gamma^n}{4\gamma^2 + 3\gamma - 1} + \frac{p_4\delta^n}{3} \qquad (1.2) \\
&= S_1\alpha^n + S_2\beta^n + S_3\gamma^n + S_4\delta^n.
\end{aligned}$$

Where p_1, p_2, p_3 and p_4 are given below

$$\begin{aligned} p_1 &= (\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2 + (3-1)\alpha + 1)W_1 - W_0), \\ p_2 &= (\beta W_3 - \beta(3-\beta)W_2 + (-\beta^2 + (3-1)\beta + 1)W_1 - W_0), \\ p_3 &= (\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2 + (3-1)\gamma + 1)W_1 - W_0), \\ p_4 &= -(W_3 - 2W_2 - W_1 - W_0). \end{aligned}$$

And

$$\begin{aligned} S_1 &= \frac{p_1}{4\alpha^2 + 3\alpha - 1}, \\ S_2 &= \frac{p_2}{4\beta^2 + 3\beta - 1}, \\ S_3 &= \frac{p_3}{4\gamma^2 + 3\gamma - 1}, \\ S_4 &= -\frac{(W_3 - 2W_2 - W_1 - W_0)}{3}. \end{aligned} \tag{1.3}$$

Binet's formula of Adrien and Adrien-Lucas sequences are

$$A_n = \frac{(2\alpha^2 + \alpha + 1)\alpha^n}{4\alpha^2 + 3\alpha - 1} + \frac{(2\beta^2 + \beta + 1)\beta^n}{4\beta^2 + 3\beta - 1} + \frac{(2\gamma^2 + \gamma + 1)\gamma^n}{4\gamma^2 + 3\gamma - 1} - \frac{1}{3},$$

and

$$B_n = \alpha^n + \beta^n + \gamma^n + 1,$$

respectively.

If we set $W_0 = 0, W_1 = 1, W_2 = 3, W_3 = 8$ then $\{W_n\}$ is the well-known Adrien sequence and if we set $W_0 = 4, W_1 = 3, W_2 = 7, W_3 = 18$ then $\{W_n\}$ is the well-known Lucas sequence. In other words, Adrien sequence $\{A_n\}_{n\geq 0}$ and Adrien-Lucas sequence $\{B_n\}_{n\geq 0}$ are defined by the fourth-order recurrence relations as;

$$A_n = 3A_{n-1} - A_{n-2} - A_{n-4}\ ,\ A_0 = 0, A_1 = 1, A_2 = 3, A_3 = 8,\ n \geq 4, \tag{1.4}$$

$$B_n = 3B_{n-1} - B_{n-2} - B_{n-4}\ ,\ B_0 = 4, B_1 = 3, B_2 = 7, B_3 = 18,\ n \geq 4. \tag{1.5}$$

The sequences $\{A_n\}_{n\geq 0}$, $\{B_n\}_{n\geq 0}$, can be extended to negative subscripts by defining,

$$\begin{aligned} A_{-n} &= -A_{-(n-2)} + 3A_{-(n-3)} - A_{-(n-4)}, \\ B_{-n} &= -B_{-(n-2)} + 3B_{-(n-3)} - B_{-(n-4)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. As a result, recurrences (1.4), (1.5) hold for all integer n. Binet's formulas as follows.

Table 1 presents the initial generalized Adrien numbers corresponding to both positive and negative subscripts.

Table 1. A few generalized Adrien numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$3W_2 - W_1 - W_3$
2	W_2	$3W_1 - W_0 - W_2$
3	W_3	$3W_0 - 3W_2 + W_3$
4	$3W_3 - W_2 - W_0$	$10W_2 - 6W_1 - 3W_3$
5	$8W_3 - W_1 - 3W_2 - 3W_0$	$10W_1 - 6W_0 - 3W_2$
6	$21W_3 - 3W_1 - 9W_2 - 8W_0$	$10W_0 + 3W_1 - 18W_2 + 6W_3$
7	$54W_3 - 8W_1 - 24W_2 - 21W_0$	$3W_0 - 28W_1 + 36W_2 - 10W_3$
8	$138W_3 - 21W_1 - 62W_2 - 54W_0$	$33W_1 - 28W_0 - W_2 - 3W_3$

We next present the generating function that characterizes the generalized Adrien numbers.

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - 3W_0)x + (W_2 - 3W_1 + W_0)x^2 + (W_3 - 3W_2 + W_1)x^3}{1 - 3x + x^2 + x^4}.$$

Next, we give the exponential generating function of $\sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence W_n.

Lemma 1. *[8]. Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is the exponential generating function of the generalized Adrien sequence $\{W_n\}$.*

Then $\sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is given by:

$$\begin{aligned}\sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \frac{(\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2 + (3-1)\alpha + 1)W_1 - W_0)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} e^{\alpha x} \\ &+ \frac{(\beta W_3 - \beta(3-\beta)W_2 + (-\beta^2 + (3-1)\beta + 1)W_1 - W_0)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} e^{\beta x} \\ &+ \frac{(\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2 + (3-1)\gamma + 1)W_1 - W_0)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} e^{\gamma x} \\ &+ (\frac{W_3 - 2W_2 - W_1 - W_0}{-3}) e^{x}.\end{aligned}$$

The previous Lemma 1 gives the following results as particular examples.

Corollary 2. *Exponential generating function of Adrien and Adrien-Lucas numbers are given by:*

a): $\sum_{n=0}^{\infty} A_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} ((\frac{(2\alpha^2+\alpha+1)\alpha^n}{4\alpha^2+3\alpha-1} + \frac{(2\beta^2+\beta+1)\beta^n}{4\beta^2+3\beta-1} + \frac{(2\gamma^2+\gamma+1)\gamma^n}{4\gamma^2+3\gamma-1} - \frac{1}{3})\frac{x^n}{n!}$
$= (\frac{(2\alpha^2+\alpha+1)}{4\alpha^2+3\alpha-1} e^{\alpha x} + \frac{(2\beta^2+\beta+1)}{4\beta^2+3\beta-1} e^{\beta x} + \frac{(2\gamma^2+\gamma+1)\gamma^n}{4\gamma^2+3\gamma-1} e^{\gamma x} - \frac{1}{3} e^{x}).$

b): $\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (\alpha^n + \beta^n + \gamma^n + 1) \frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^{\gamma x} + e^x.$

For more details about generalized Adrien numbers, see [21].

In this section, we introduce several number systems relevant to our study, with particular emphasis on the hypercomplex framework, which includes complex numbers, hyperbolic numbers, and dual numbers. Among these, hyperbolic numbers are of particular significance and will play a central role in our analysis. Notably, hyperbolic functions and numbers have found applications across various branches of engineering, including electrical engineering (e.g., transmission line modeling), control theory (e.g., system dynamics), and signal processing (e.g., filter design). Furthermore, they are utilized in diverse areas of engineering physics such as special relativity, wave propagation, fluid dynamics, optics, and heat conduction. While hyperbolic numbers possess intriguing mathematical properties, their practical utility is context-dependent and hinges on whether they offer computational or conceptual advantages over alternative number systems for the problem under consideration.

We begin by exploring hypercomplex number systems, which generalize the real number line and provide a broader algebraic framework for mathematical analysis. For a more detailed treatment, the reader is referred to [18]. Notably, several commutative special cases of hypercomplex system such as complex numbers, hyperbolic numbers, and dual number are widely utilized across various domains of mathematics and physics due to their unique algebraic structures and diverse applications. In the following sections, we present these number systems in a systematic manner, highlighting their foundational properties and relevance to our study.

- Complex numbers represent the simplest and most fundamental extension within the broader class of hypercomplex number systems. A complex number is defined as $z = a + ib$, where a and b are real numbers, and i denotes the imaginary unit satisfying the relation $i^2 = -1$. The components a and b are referred to as the real and imaginary parts of z, respectively, and are denoted by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$. Accordingly, the formal definition of complex numbers is given by:

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\}.$$

- Hyperbolic (double, split-complex) numbers, for more detail see [29], Split complex numbers, commonly recognized as hyperbolic numbers, defined as $h = a + jb$ where a and b real numbers and j hyperbolic unit that satisfy $j^2 = 1$. In addition that a and b named, respectively, $\mathbb{R}$ and $\mathbb{H}$. Thus, the definition of hyperbolic numbers given by,

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\},$$

- Dual numbers, as discussed in [12], are defined in the form $d = a + \varepsilon b$, where a and b are real numbers, and ε is the dual unit satisfying $\varepsilon^2 = 0$. The components a and b are referred to as the real part and the dual part of d, respectively, and are denoted by $\mathbb{R}$ and and $\mathbb{D}$. Accordingly, the formal definition of dual numbers is given as follows:

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

- A dual hyperbolic number constitutes a distinct subclass of hypercomplex numbers, integrating the structural characteristics of both dual and hyperbolic number systems. A dual hyperbolic number is defined by,

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3$$

  where $a_0, a_1, a_2, a_3 \in \mathbb{R}$ and the set of all dual hyperbolic numbers are defined by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3 : a_0, a_1, a_2, a_3 \in \mathbb{R},\ j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The $\{1, j, \varepsilon, \varepsilon j\}$ is linear independent and $\mathbb{H}_{\mathbb{D}} = sp\{1, j, \varepsilon, \varepsilon j\}$ so that $\{1, j, \varepsilon, \varepsilon j\}$ is a basis of $\mathbb{H}_{\mathbb{D}}$. For more detail, see [1].

The next properties are true for the base elements $\{1, j, \varepsilon, \varepsilon j\}$ (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j,\ \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0,\ j^2 = j.j = 1 \\ \varepsilon.j &= j.\varepsilon,\ \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0,\ j(\varepsilon j) = (\varepsilon j)j = \varepsilon \end{aligned}$$

where ε satisfy the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j satisfy the hyperbolic unit ($j^2 = 1$), and εj satisfy the dual hyperbolic unit ($(j\varepsilon)^2 = 0$).

Furthermore, we introduce additional hypercomplex number system namely, quaternions, octonions, and sedenion each of which will be examined in detail in the subsequent sections.

- Quaternions, which represent a non-commutative subclass of hypercomplex number systems, constitute a four-dimensional extension of complex numbers.
- They are expressed as $a_0+ia_1+ja_2+ka_3$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$, and i, j, and k are the quaternion units that satisfy specific multiplication rules. For more detail,see [14]. Quaternion numbers are defined by

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\},$$

- The octonions, denoted by $\mathbb{O}$, form an algebraic set in which each element is expressed as a linear combination of the eight unit octonions $\{e_i : i = 0, 1.2, ..., 7\}$. Octonions are defined by,

$$\mathbb{O} = \{\sum_{i=0}^{7} a_i e_i : a_i \in \mathbb{R}, e_0 e_i = e_i e_0 = e_i,\ e_i e_j = -\delta_{ij} e_0 + \varepsilon_{ijk} e_k\ \}$$

where $e_e = 1$, δ_{ij} is Kroneker delta (equal to 1 if and only if $i = j$), ε_{ijk} is anti-symetric tensor. For more detail see [16, 27]

- Sedenions is a set, every element of the set linear combinations of unit sedenions $\{e_i : i = 0, 1.2, ..., 15\}$, denoted by $\mathbb{S}$. It can be seen from here that ever sedenion can be written as

$$\sum_{i=0}^{15} a_i e_i,$$

where a_i is real number. For more detail, see[20, 27].

Next we give some proporties on two hyperbolic numbers, $h_1 = a + jb$ and $h_2 = c + jd$, as

$$\begin{aligned}
h_1 + h_2 &= (a+b) + j(c+d), \\
h_1.h_2 &= (ac+bd) + j(ad+bc), \\
\overline{h_1} &= a - jb \\
\frac{h_1}{h_2} &= \frac{(ac-bd)+j(cb-ad)}{c^2-d^2}, \\
h_1 &= h_2 \text{ if only if } a = c \text{ and } b = d, \\
\langle h_1, h_2 \rangle &= (ac+bd) + j(bc+ad), \\
\|h_1\| &= \sqrt{|a^2-b^2|}, \text{ called norm of } h_1, \\
\text{if } \left|a^2-b^2\right| &> 0, \ h_1 \text{ is named spacelike vector}, \\
\text{if } \left|a^2-b^2\right| &< 0, \ h_1 \text{ is named timelike vector}, \\
\text{if } \left|a^2-b^2\right| &= 0, \ h_1 \text{ is named null(light-like) vector.}
\end{aligned}$$

Note that$\{\mathbb{R}^2, H, \langle , \rangle\}$ is called Lorentz plane and denoted as $\mathbb{R}^2_1$. There is an isomorphism relationship between the Lorentz plane and hyperbolic numbers. For more detail, see [27].Hence, the algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are all real algebras derived from the field of real numbers $\mathbb{R}$ through successive applications of a doubling procedure known as the Cayley–Dickson process.This doubling process can be extended beyond the sedenions to generate higher-dimensional algebras collectively referred to as the 2^n-ions, where n denotes the number of doublings applied starting from the real numbers (see for example [3, 14, 15, 19, 13].

Several authors have conducted research on dual numbers, hyperbolic numbers, dual hyperbolic numbers, and other specialized numerical systems, exploring their algebraic structures, geometric interpretations, and applications across various fields.We now present a selection of information from published papers in the literature.

- Cockle [7] explored hyperbolic numbers with complex coefficients, contributing to the early development of hypercomplex algebra.
- Eren and Soykan [11] studied the generalized Generalized Woodall Numbers.
- Cheng and Thompson [5] introduced dual numbers with complex coefficients, expanding the algebraic versatility of dual number systems for applications in polynomial equations and transformation theory.

- Akar at al [1] introduced the concept of dual hyperbolic numbers, combining characteristics of dual and hyperbolic systems into a unified algebraic structure.

Next, we present a selection of information from the literature concerning hyperbolic numbers, including their algebraic properties, historical development, and applications.

- Aydın [2] ntroduced the concept of hyperbolic Fibonacci numbers, defined by the following expression:

$$\widetilde{F}_n = F_n + hF_{n+1},$$

where Fibonacci numbers are given by $F_{n+2} = F_{n+1} + F_n$, with the initial conditation $F_0 = 0$, $F_1 = 1$.

- Soykan and Taşdemir [22] studied hyperbolic generalized Jacobsthal numbers given by

$$\widetilde{V}_n = V_n + hV_{n+1},$$

where generalized Jacobsthal numbers are $V_{n+2} = V_{n+1} + 2V_n$ with the initial conditation $V_0 = a$, $V_1 = b$.

- Taş [28] introduced hyperbolic Jacobsthal-Lucas sequence written by

$$HJ_n = J_n + hJ_{n+1},$$

where Jacobsthal-Lucas numbers given by $J_{n+2} = J_{n+1} + 2J_n$ with the inintial conditation $J_0 = 2$, $J_1 = 1$.

- Dikmen and Altınsoy, [10] introduced On Third Order Hyperbolic Jacobsthal Numbers are

$$\begin{aligned} \widehat{J}_n^{(3)} &= J_n^{(3)} + hJ_{n+1}^{(3)}, \\ \widehat{j}_n^{(3)} &= j_n^{(3)} + hj_{n+1}^{(3)}, \end{aligned}$$

where Jacobsthal numbers, respectively, given by $J_n^{(3)} = J_{n-1}^{(3)} + J_{n-2}^{(3)} + 2J_{n-3}^{(3)}$, $J_0^{(3)} = 0$, $J_1^{(3)} = 1$, $J_2^{(3)} = 1$, $j_n^{(3)} = j_{n-1}^{(3)} + j_{n-2}^{(3)} + 2j_{n-3}^{(3)}$, $j_0^{(3)} = 2$, $j_1^{(3)} = 1$, $j_2^{(3)} = 5$.

Following this, we provide details on dual hyperbolic sequences as they are presented in literature.

- Soykan at al [23] introduced dual hyperbolic generalized Pell numbers are

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3},$$

where generalized Pell numbers, with the initial values V_0, V_1 not all being zero, are given by $V_n = 2V_{n-1} + V_{n-2}$, $V_0 = a$, $V_1 = b$ $(n \geq 2)$.

- Cihan at al [6] introduced dual hyperbolic Fibonacci and Lucas numbers are,

$$\begin{aligned} DHF_n &= F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3}, \\ DHL_n &= L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3}, \end{aligned}$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$, $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$.

- Soykan at al [22] introduced dual hyperbolic generalized Jacopsthal numbers are

$$\widehat{J}_n = J_n + jJ_{n+1} + \varepsilon J_{n+2} + j\varepsilon J_{n+3},$$

where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = a$, $J_1 = b$.

- Bród et al [4] investigated dual hyperbolic generalized balancing numbers, examining their algebraic formulation, recurrence relations, and potential applications within number theory and symbolic computation

$$DHB_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3},$$

where $B_n = 6B_{n-1} - B_{n-2}$, $B_0 = 0$, $B_1 = 1$.

- Yılmaz and Soykan [30] introduced dual hyperbolic generalized Guglielmo numbers are

$$\widehat{T}_0 = T_0 + jT_1 + \varepsilon T_2 + j\varepsilon T_3,$$

where $T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, T_0 = 0, T_1 = 1, T_2 = 3.$

- Kalça and Soykan [17] introduced dual hyperbolic generalized Pandita numbers are

$$\widehat{P}_0 = P_0 + jP_1 + \varepsilon P_2 + j\varepsilon P_3,$$

where $P_n = 2P_{n-1} - P_{n-2} + P_{n-3} - P_{n-4}, P_0 = 0P_1 = 1, P_2 = 2, P_3 = 3.$

- Demirci and Soykan [9] introduced dual hyperbolic generalized Adrien numbers are

$$\widehat{A}_0 = A_0 + jA_1 + \varepsilon A_2 + j\varepsilon A_3,$$

where $A_n = 3A_{n-1} - A_{n-2} - A_{n-4}$, $A_0 = 0, A_1 = 1, A_2 = 3, A_3 = 8.$

Next section, we define the hyperbolic generalized Adrien numbers and some properties, generating function and Binet's formula, of these numbers.

2. Hyperbolic Generalized Adrien Numbers and their Generating Functions and Binet's Formulas

In this section, we introduce the concept of hyperbolic generalized Adrien numbers, formulated within the framework of the hyperbolic algebra $\mathbb{H}$. Based on this definition, we proceed to derive their corresponding generating function and Binet type formula. We now examine the structure of these numbers in the algebra $\mathbb{H}$, where the nth hyperbolic generalized Adrien number is defined as follows:

$$HW_n = W_n + jW_{n+1} \tag{2.1}$$

with the initial values HW_0, HW_1, HW_2, HW_3. (2.1). The hyperbolic Adrien numbers, as defined above, can be extended to negative subscripts by introducing the following definition,

$$HW_{-n} = W_{-n} + jW_{-n+1} \tag{2.2}$$

so identity (2.1) holds for all integers n.

We now define several special cases of the hyperbolic generalized Adrien numbers, highlighting particular parameter choices that yield notable variations or simplifications. The nth hyperbolic Adrien numbers, the nth hyperbolic Adrien-Lucas numbers, respectively, are given as the nth hyperbolic Adrien numbers is given $HA_n = A_n + jA_{n+1}$, with the initial values

$$\begin{aligned} HA_0 &= A_0 + jA_1, \\ HA_1 &= A_1 + jA_2, \\ HA_2 &= A_2 + jA_3, \end{aligned}$$

the nth hyperbolic Adrien-Lucas numbers is given $HB_n = B_n + jB_{n+1}$ with the initial values

$$\begin{aligned} HB_0 &= B_0 + jB_1, \\ HB_1 &= B_1 + jB_2, \\ HB_2 &= B_2 + jB_3, \end{aligned}$$

Note that, hyperbolic Adrien numbers (by using $W_n = A_n$, $A_0 = 0$, $A_1 = 1$, $A_2 = 3$) we get

$$\begin{aligned} HA_0 &= j, \\ HA_1 &= 1 + 3j, \\ HA_2 &= 3 + 8j, \end{aligned}$$

for hyperbolic Adrien-Lucas numbers (bu using $W_n = B_n$, $B_0 = 4$, $B_1 = 3$, $B_2 = 7$) we obtain

$$\begin{aligned} HB_0 &= 4 + 3j, \\ HB_1 &= 3 + 7j, \\ HB_2 &= 7 + 18j. \end{aligned}$$

So, using (2.1), we can write the following identity for non negative integers n,

$$HW_n = 3HW_{n-1} - HW_{n-2} - HW_{n-4}, \tag{2.3}$$

and the sequence $\{HW_n\}_{n\geq 0}$ can be given as

$$HW_{-n} = -HW_{-(n-2)} + 3HW_{-(n-3)} - HW_{-(n-4)},$$

for $n = 1, 2, 3, ...$ by using (2.2). As a result, recurrence (2.3) holds for all integer n.

Table 2 displays the initial values of the hyperbolic generalized Adrien numbers HW_n, incorporating both positive and negative indices to provide a comprehensive representation of the sequence's symmetric behavior.

Table 2. A few hyperbolic generalized Adrien numbers

n	HW_n	HW_{-n}
0	HW_0	HW_0
1	HW_1	$3HW_2 - HW_1 - HW_3$
2	HW_2	$3HW_1 - HW_0 - HW_2$
3	HW_3	$3HW_0 - 3HW_2 + HW_3$
4	$3HW_3 - HW_2 - HW_0$	$10HW_2 - 6HW_1 - 3HW_3$
5	$8HW_3 - HW_1 - 3HW_2 - 3HW_0$	$10HW_1 - 6HW_0 - 3HW_2$
6	$21HW_3 - 3HW_1 - 9HW_2 - 8HW_0$	$10HW_0 + 3HW_1 - 18HW_2 + 6HW_3$
7	$54HW_3 - 8HW_1 - 24HW_2 - 21HW_0$	$3HW_0 - 28HW_1 + 36HW_2 - 10HW_3$
8	$138HW_3 - 21HW_1 - 62HW_2 - 54HW_0$	$33HW_1 - 28HW_0 - HW_2 - 3HW_3$

Note that

$$\begin{aligned} HW_0 &= W_0 + jW_1, \\ HW_1 &= W_1 + jW_2, \\ HW_2 &= W_2 + jW_3. \end{aligned}$$

A selection of hyperbolic Adrien numbers and hyperbolic Adrien–Lucas numbers with both positive and negative subscripts are presented in Table 3 and Table 4, respectively.

Table 3.Some hyperbolic Adrien numbers

n	HA_n	HA_{-n}
0	j	j
1	$1 + 3j$	0
2	$3 + 8j$	0
3	$8 + 21j$	-1
4	$21 + 54j$	$-j$
5	$54 + 138j$	1
6	$138 + 352j$	$-3 + j$
7	$352 + 897j$	$-3j$
8	$897 + 2285j$	6

Table 4. Some hyperbolic Adrien-Lucas numbers

n	HB_n	HB_{-n}
0	$4 + 3j$	$4 + 3j$
1	$3 + 7j$	$4j$
2	$7 + 18j$	-2
3	$18 + 43j$	$9 - 2j$
4	$43 + 108j$	$-2 + 9j$
5	$108 + 274j$	$-15 - 2j$
6	$274 + 696j$	$31 - 15j$
7	$696 + 1771j$	$31j$
8	$1771 + 4509j$	-74

Now, we will give some expressions that we will use in the rest of the paper and then we define Binet's formula for the hyperbolic generalized Adrien numbers. First, we define

$$\widetilde{\alpha} = 1 + j\alpha, \tag{2.4}$$

$$\widetilde{\beta} = 1 + j\beta, \tag{2.5}$$

$$\widetilde{\gamma} = 1 + j\gamma, \tag{2.6}$$

$$\widetilde{\lambda} = 1 + j, \tag{2.7}$$

note that using above equalities we can write the following identities:

$$\begin{aligned}
\widetilde{\alpha}^2 &= 1 + 2j\alpha + \alpha^2, \\
\widetilde{\beta}^2 &= 1 + 2j\beta + \beta^2, \\
\widetilde{\gamma}^2 &= 1 + 2j\gamma + \gamma^2, \\
\widetilde{\lambda}^2 &= 2 + 2j, \\
\widetilde{\alpha}\widetilde{\beta} &= 1 + j(\alpha + \beta) + \alpha\beta, \\
\widetilde{\alpha}\widetilde{\gamma} &= 1 + j(\alpha + \gamma) + \alpha\gamma, \\
\widetilde{\gamma}\widetilde{\beta} &= 1 + j(\alpha + \beta) + \alpha\beta, \\
\widetilde{\alpha}\widetilde{\lambda} &= 1 + j(1 + \alpha) + \alpha.
\end{aligned}$$

Theorem 3. *(Binet's Formula) For any integer n, the nth hyperbolic generalized Adrien number is*

$$\widehat{W}_n = \widetilde{\alpha} S_1 \alpha^n + \widetilde{\beta} S_2 \beta^n + \widetilde{\gamma} S_3 \gamma^n + \widetilde{\delta} S_4, \tag{2.8}$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}$, $\widehat{\delta}$ are given as (2.4), (2.5), (2.6), (2.7).

Proof. Using Binet's formula of the generalized Adrien numbers given below

$$W_n = S_1 \alpha^n + S_2 \beta^n + S_3 \gamma^n + S_4,$$

where S_1, S_2, S_2, S_4 are given (1.3) we get

$$\begin{aligned}
HW_n &= W_n + jW_{n+1}, \\
&= S_1 \alpha^n + S_2 \beta^n + S_3 \gamma^n + S_4 \\
&\quad + j(S_1 \alpha^{n+1} + S_2 \beta^{n+1} + S_3 \gamma^{n+1} + S_4) \\
&= \widetilde{\alpha} S_1 \alpha^n + \widetilde{\beta} S_2 \beta^n + \widetilde{\gamma} S_3 \gamma^n + \widetilde{\delta} S_4,
\end{aligned}$$

This proves (2.8). □

As special cases, for any integer n, the Binet's Formula of nth hyperbolic Adrien number is

$$\widetilde{A}_n = \frac{(2\alpha^2+\alpha+1)\alpha^n\widetilde{\alpha}}{4\alpha^2+3\alpha-1} + \frac{(2\beta^2+\beta+1)\beta^n\widetilde{\beta}}{4\beta^2+3\beta-1} + \frac{(2\gamma^2+\gamma+1)\gamma^n\widetilde{\gamma}}{4\gamma^2+3\gamma-1} - \frac{\widetilde{1}}{3}, \tag{2.9}$$

and the Binet's Formula of nth hyperbolic Adrien-Lucas number is

$$\widetilde{B}_n = \widetilde{\alpha}\alpha^n + \widetilde{\beta}\beta^n + \widetilde{\gamma}\gamma^n + 1. \tag{2.10}$$

In the following section, we present the generating function associated with the hyperbolic generalized Adrien numbers.

THEOREM 4. *The generating function for the hyperbolic generalized Adrien numbers is*

$$\sum_{n=0}^{\infty} HW_n x^n = \frac{HW_0 + (HW_1 - 3HW_0)x + (HW_2 - 3HW_1 + HW_0)x^2 + (HW_3 - 3HW_2 + HW_1)x^3}{1-3x+x^2+x^4}. \tag{2.11}$$

Proof. Let

$$f_{HW_n}(x) = \sum_{n=0}^{\infty} HW_n x^n$$

be generating function of the hyperbolic generalized Adrien numbers. Then, using the definition of the hyperbolic generalized Adrien numbers, and substracting $xf_{HW_n}(x)$ and $x^2 f_{HW_n}(x)$ from $f_{HW_n}(x)$, we obtain $(1-3x+x^2+x^4)f_{HW_n}(x)$

$$\begin{aligned}
(1-3x+x^2+x^4)f_{HW_n}(x) &= \sum_{n=0}^{\infty} HWx^n - 3x\sum_{n=0}^{\infty} HWx^n + x^2\sum_{n=0}^{\infty} HWx^n + x^4\sum_{n=0}^{\infty} HWx^n, \\
&= \sum_{n=0}^{\infty} HWx^n - 3\sum_{n=0}^{\infty} HWx^{n+1} + \sum_{n=0}^{\infty} HWx^{n+2} + \sum_{n=0}^{\infty} HWx^{n+4}, \\
&= \sum_{n=0}^{\infty} HWx^n - 3\sum_{n=1}^{\infty} HW_{(n-1)}x^n + \sum_{n=2}^{\infty} HW_{(n-2)}x^n + \sum_{n=4}^{\infty} HW_{(n-4)}x^n, \\
&= (HW_0 + HW_1x + HW_2x^2 + HW_3x^3) - 3(HW_0x + HW_1x^2 + HW_2x^3) \\
&\quad + (HW_0x^2 + HW_1x^3) + \sum_{n=4}^{\infty}(HW_n - 3HW_{n-1} + HW_{n-2} + HW_{n-4})x^n, \\
&= HW_0 + (HW_1 - 3HW_0)x + (HW_2 - 3HW_1 + HW_0)x^2 \\
&\quad + (HW_3 - 3HW_2 + HW_1)x^3.
\end{aligned}$$

Note that, using the recurrence relation $\widehat{A} = 3\widehat{A}_{n-1} - \widehat{A}_{n-2} - \widehat{A}_{n-4}$ and rearranging above equation the (2.11) has been obtained. □

Now we can write the generating functions of the hyperbolic Adrien and Adrien-Lucas numbers as:

(a): $f_{\widehat{A}_n}(x) = \sum_{n=0}^{\infty} \widehat{A}_n x^n = \frac{j+x}{1-3x+x^2+x^4}$,

(b): $f_{\widehat{B}_n}(x) = \sum_{n=0}^{\infty} \widehat{B}_n x^n = \frac{-4jx^3+2x^2+(-2j-9)x+3j+4}{1-3x+x^2+x^4}$.

Lemma 5. *Suppose that* $f_{HW_n}(x) = \sum\limits_{n=0}^{\infty} HW_n \frac{x^n}{n!}$ *is the exponential generating function of the hyperbolic generalized Adrien sequence* $\{ HW_n \}$.

Then $\sum\limits_{n=0}^{\infty} HW_n \frac{x^n}{n!}$ is given by

$$\sum_{n=0}^{\infty} HW_n \frac{x^n}{n!} = S_1 e^{\alpha x}\widetilde{\alpha} + S_2 e^{\beta x}\widetilde{\beta} + S_3 e^{\gamma x}\widetilde{\gamma} + S_4 e^{x}\widetilde{1}.$$

where $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}$ are given as (2.4), (2.5), (2.6) ,(2.7).

Proof. Using Binet' s formula

$$W_n = S_1\alpha^n + S_2\beta^n + S_3\gamma^n + S_4,$$

where S_1, S_2, S_3, S_4 are given in (1.3) we get

$$\begin{aligned}
\sum_{n=0}^{\infty} HW \frac{x^n}{n!} &= \sum_{n=0}^{\infty} HW_n \frac{x^n}{n!} + j\sum_{n=0}^{\infty} HW_{n+1}\frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty}(S_1\alpha^n + S_2\beta^n + S_3\gamma^n + S_4)\frac{x^n}{n!} + j\sum_{n=0}^{\infty}(S_1\alpha^{n+1} + S_2\beta^{n+1} + S_3\gamma^{n+1} + S_4)\frac{x^n}{n!} \\
&= (S_1e^{\alpha x} + S_2e^{\beta x} + S_3e^{\gamma x} + S_4e^{x}) + j(S_1\alpha e^{\alpha x} + S_2\beta e^{\beta x} + S_3\gamma e^{\gamma x} + S_4e^{x}) \\
&= S_1e^{\alpha x}(1 + j\alpha) + S_2e^{\beta x}(1 + j\beta) + S_3e^{\gamma x}(1 + j\gamma) + S_4e^{x}(1 + j) \\
&= S_1e^{\alpha x}\widetilde{\alpha} + S_2e^{\beta x}\widetilde{\beta} + S_3e^{\gamma x}\widetilde{\gamma} + S_4e^{x}\widetilde{1}. \ \square
\end{aligned}$$

Proof: Note that we have

$$\sum_{n=0}^{\infty} HW_n \frac{x^n}{n!} = \sum_{n=0}^{\infty}(HW_n + jHW_{n+1})\frac{x^n}{n!}.$$

From the previous lemma, we derive the following outcomes as particular instances.

Corollary 6. *Exponential generating function of hiperbolic Adrien and hiperbolic Adrien-Lucas numbers are given by:*

a):

$$\begin{aligned}
\sum_{n=0}^{\infty} \tilde{A}_n \frac{x^n}{n!} &= (\frac{(2\alpha^2+\alpha+1)}{4\alpha^2+3\alpha-1}e^{\alpha x} + \frac{(2\beta^2+\beta+1)}{4\beta^2+3\beta-1}e^{\beta x} + \frac{(2\gamma^2+\gamma+1)}{4\gamma^2+3\gamma-1}e^{\gamma x} - \frac{1}{3}e^{x}) \\
&\quad + j(\frac{(2\alpha^2+\alpha+1)}{4\alpha^2+3\alpha-1}e^{\alpha x} + \frac{(2\beta^2+\beta+1)}{4\beta^2+3\beta-1}e^{\beta x} + \frac{(2\gamma^2+\gamma+1)}{4\gamma^2+3\gamma-1}e^{\gamma x} - \frac{1}{3}e^{x}) \\
&= \frac{(2\alpha^2+\alpha+1)\widetilde{\alpha}}{4\alpha^2+3\alpha-1}e^{\alpha x} + \frac{(2\beta^2+\beta+1)\widetilde{\beta}}{4\beta^2+3\beta-1}e^{\beta x} + \frac{(2\gamma^2+\gamma+1)\widetilde{\gamma}}{4\gamma^2+3\gamma-1}e^{\gamma x} - \widetilde{1}e^{x}
\end{aligned}$$

b):

$$\begin{aligned}
\sum_{n=0}^{\infty} \tilde{B}_n \frac{x^n}{n!} &= e^{\alpha x} + e^{\beta x} + e^{\gamma x} + e^{x} + j(\alpha e^{\alpha x} + \beta e^{\beta x} + \gamma e^{\gamma x} + e^{x}) \\
&= e^{\alpha x}\widetilde{\alpha} + e^{\beta x}\widetilde{\beta} + e^{\gamma x}\widetilde{\gamma} + e^{x}\widetilde{1}
\end{aligned}$$

3. Obtaining Binet Formula From Generating Function

Next, by the using generating function for HW_n find Binet formula of hyperbolic generalized Adrien number $\{HW_n\}$.

THEOREM 7. *(Binet formula of hyperbolic generalized Adrien numbers)*

$$HW_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \tag{3.1}$$

where

$$\begin{aligned}
p_1 &= HW_0\alpha^3 + (HW_1 - 3HW_0)\alpha^2 + (HW_2 + HW_1 + HW_0)\alpha + (HW_3 + HW_2 + HW_1),\\
p_2 &= HW_0\beta^3 + (HW_1 - 3HW_0)\beta^2 + (HW_2 + HW_1 + HW_0)\beta + (HW_3 + HW_2 + HW_1),\\
p_3 &= HW_0\gamma^3 + (HW_1 - 3HW_0)\gamma^2 + (HW_2 + HW_1 + HW_0)\gamma + (HW_3 + HW_2 + HW_1),\\
p_4 &= HW_0\delta^3 + (HW_1 - 3HW_0)\delta^2 + (HW_2 + HW_1 + HW_0)\delta + (HW_3 + HW_2 + HW_1).
\end{aligned}$$

Proof. Let

$$h(x) = 1 - 3x + x^2 + x^4.$$

Then for some α, β, γ and δ we write

$$h(x) = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x),$$

i.e.,

$$1 - 3x + x^2 = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x), \tag{3.2}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ and $\frac{1}{\delta}$ are the roots of $h(x)$. This gives α, β, γ and δ as the roots of

$$h(\frac{1}{x}) = 1 - \frac{3}{x} + \frac{1}{x^2} + \frac{1}{x^4} = 0.$$

This implies $x^4 - 3x^3 + x^2 + u = 0$. Now, it follows that

$$\sum_{n=0}^{\infty} HWx^n = \frac{HW_0 + (HW_1 - 3HW_0)x + (HW_2 - 3HW_1 + HW_0)x^2 + (HW_3 - 3HW_2 + HW_1)x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}.$$

Then we write

$$\begin{aligned}
&\frac{HW_0 + (HW_1 - 3HW_0)x + (HW_2 - 3HW_1 + HW_0)x^2 + (HW_3 - 3HW_2 + HW_1)x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}\\
&= \frac{B_1}{(1-\alpha x)} + \frac{B_2}{(1-\beta x)} + \frac{B_3}{(1-\gamma x)} + \frac{B_4}{(1-\delta x)}.
\end{aligned} \tag{3.3}$$

So

$$\begin{aligned} & HW_0 + (HW_1 - 3HW_0)x + (HW_2 - 3HW_1 + HW_0)x^2 + (HW_3 - 3HW_2 + HW_1)x^3 \\ = \; & B_1(1-\beta x)(1-\gamma x)(1-\delta x) + B_2(1-\alpha x)(1-\gamma x)(1-\delta x) \\ & + B_3(1-\alpha x)(1-\beta x)(1-\delta x) + B_3(1-\alpha x)(1-\beta x)(1-\gamma x). \end{aligned}$$

If we consider $x = \frac{1}{\alpha}$, we get $HW_0 + (HW_1 - 3HW_0)\frac{1}{\alpha} + (HW_2 - 3HW_1 + HW_0)\frac{1}{\alpha^2} + (HW_3 - 3HW_2 + HW_1)\frac{1}{\alpha^3}$ $= B_1(1 - \frac{\beta}{\alpha})(1 - \frac{\gamma}{\alpha})(1 - \frac{\delta}{\alpha})$.

This gives

$$\begin{aligned} B_1 &= \frac{\alpha^3(HW_0 + (HW_1 - 3HW_0)\frac{1}{\alpha} + (HW_2 - 3HW_1 + HW_0)\frac{1}{\alpha^2} + (HW_3 - 6HW_2 + HW_1)\frac{1}{\alpha^3})}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} \\ &= \frac{HW_0\alpha^3 + (HW_1 - HW_0)\alpha^2 + (HW_2 - 3HW_1 + HW_0)\alpha + (HW_3 - 3HW_2 + HW_1)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} B_2 &= \frac{HW_0\beta^3 + (HW_1 - 3HW_0)\beta^2 + (HW_2 - 3HW_1 + HW_0)\beta + (HW_3 - 3HW_2 + HW_1)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\ B_3 &= \frac{HW_0\gamma^3 + (HW_1 - 3HW_0)\gamma^2 + (HW_2 - 3HW_1 + HW_0)\gamma + (HW_3 - 3HW_2 + HW_1)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\ B_4 &= \frac{HW_0\delta^3 + (HW_1 - 3HW_0)\delta^2 + (HW_2 - 3HW_1 + HW_0)\delta + (HW_3 - 3HW_2 + HW_1)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}. \end{aligned}$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty} HWx^n = B_1(1-\alpha x)^{-1} + B_2(1-\beta x)^{-1} + B_3(1-\gamma x)^{-1} + B_4(1-\delta x)^{-1}.$$

This gives

$$\begin{aligned} \sum_{n=0}^{\infty} HWx^n &= B_1\sum_{n=0}^{\infty}\alpha^n x^n + B_2\sum_{n=0}^{\infty}\beta^n x^n + B_3\sum_{n=0}^{\infty}\gamma^n x^n + B_4\sum_{n=0}^{\infty}\delta^n x^n \\ &= \sum_{n=0}^{\infty}(B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n)x^n. \end{aligned}$$

Therefore, equating coefficients from both sides of the above expression yields the following formulation.

$$HW = B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n.$$

and then we get (3.1). □

We can get an identitiy related to hyperbolic Adrien numbers given below.

THEOREM 8. *For all integers m, n the following identities hold:*

$HW_{m+n} = A_{m-2}HW_{n+3} + (-A_{m-3} - A_{m-5})HW_{n+2} + (-A_{m-4})HW_{n+1} - A_{m-3}HW_n.$

Proof. First we assume that $m, n \geq 0$ then (8) can be proved by mathematical induction on m. If $m = 0$ we get

$$HW_n = A_{-2}HW_{n+3} + (-A_{-3} - A_{-5})HW_{n+2} + (-A_{-4})HW_{n+1} - A_{-3}HW_n.$$

which is true since $A_{-2} = 0$, $A_{-3} = -1$, $A_{-4} = 0$, $A_{-5} = 1$. Assume that the equality holds for $m \leq k$. For $m = k + 1$, we get

$$\begin{aligned} HW_{k+1+n} &= 3HW_{n+k} - HW_{n+k-1} - HW_{n+k-3}, \\ & 3(A_{k-2}HW_{n+3} + (-A_{k-3} - A_{k-5})HW_{n+2} + (-A_{k-4})HW_{n+1} - A_{k-3}HW_n) \\ & -(A_{k-3}HW_{n+3} + (-A_{k-4} - A_{k-6})HW_{n+2} + (-A_{-5})HW_{n+1} - A_{k-4}HW_n) \\ & -(A_{k-5}HW_{n+3} + (-A_{k-6} - A_{k-8})HW_{n+2} + (-A_{k-6})HW_{n+1} - A_{k-6}HW_n). \end{aligned}$$

Consequently, by mathematical induction on m, this proves Theorem (8).

The other cases of m, n can be proved smilarly for all integers m, n. □

Taking $HW_n = HA_n$ or $HW_n = HB_n$ in above Theorem, respectively, we get:

Corollary 9.

$$\begin{aligned} HA_{m+n} &= A_{m-2}HA_{n+3} + (-A_{m-3} - A_{m-5})HA_{n+2} + (-A_{m-4})HA_{n+1} - A_{m-3}HA_n, \\ HB_{m+n} &= A_{m-2}HB_{n+3} + (-A_{m-3} - A_{m-5})HB_{n+2} + (-A_{m-4})HB_{n+1} - A_{m-3}HB_n. \end{aligned}$$

4. Simson's Formulas

This section introduces Simson's formula as applied to the hyperbolic generalized Adrien numbers. This is a special case of [24,Theorem 4.1].

Theorem 10. *For all integers n, we have*
$\begin{vmatrix} HW_{n+3} & HW_{n+2} & HW_{n+1} & HW_n \\ HW_{n+2} & HW_{n+1} & HW_n & HW_{n-1} \\ HW_{n+1} & HW_n & HW_{n-1} & HW_{n-2} \\ HW_n & HW_{n-1} & HW_{n-2} & HW_{n-3} \end{vmatrix} = (HW_0 + HW_1 + 2HW_2 - HW_3)(-HW_3^3 + 5HW_2^3 + HW_1^3 + HW_0^3 - (HW_0 + 3HW_1 - 7HW_2)HW_3^2$

$+(3HW_0 - 4HW_1 - 14HW_3)HW_2^2 + (2HW_0 + HW_2 - 6HW_3)HW_1^2 - (HW_1 + 2HW_3)HW_0^2 + 13HW_1HW_2HW_3 + HW_0HW_2HW_3 + 5HW_0HW_1HW_3 - 7HW_0HW_1HW_2).$

Proof. Take $r = 3, s = -1, t = 0, u = -1$. □

Corollary 11. *For all integers n, the Simson's formulas of hyperbolic generalized Adrien number and hyperbolic generalized Adrien-Lucas numbers are given as:*

$$\begin{vmatrix} \tilde{A}_{n+3} & \tilde{A}_{n+2} & \tilde{A}_{n+1} & \tilde{A}_{n} \\ \tilde{A}_{n+2} & \tilde{A}_{n+1} & \tilde{A}_{n} & \tilde{A}_{n-1} \\ \tilde{A}_{n+1} & \tilde{A}_{n} & \tilde{A}_{n-1} & \tilde{A}_{n-2} \\ \tilde{A}_{n} & \tilde{A}_{n-1} & \tilde{A}_{n-2} & \tilde{A}_{n-3} \end{vmatrix} = 3+3j,$$

$$\begin{vmatrix} \tilde{B}_{n+3} & \tilde{B}_{n+2} & \tilde{B}_{n+1} & \tilde{B}_{n} \\ \tilde{B}_{n+2} & \tilde{B}_{n+1} & \tilde{B}_{n} & \tilde{B}_{n-1} \\ \tilde{B}_{n+1} & \tilde{B}_{n} & \tilde{B}_{n-1} & \tilde{B}_{n-2} \\ \tilde{B}_{n} & \tilde{B}_{n-1} & \tilde{B}_{n-2} & \tilde{B}_{n-3} \end{vmatrix} = -2349-2349j,$$

respectively.

5. Linear Sums

This section presents the summation formulas for the hyperbolic generalized Adrien numbers, encompassing both positive and negative subscripts. We then proceed to introduce the summation formulas for the generalized Adrien numbers.

Theorem 12. *For the generalized Adrien numbers with positive and negative subscript, we have the following formulas:*

(a): $\sum_{k=0}^{n} W_k = \frac{1}{3}(-(n+3)W_{n+3} + (2n+7)W_{n+2} + (n+2)W_{n+1} + (n+4)W_n$
$+3W_3 - 7W_2 - 2W_1 - W_0)$.

(b): $\sum_{k=0}^{n} W_{2k} = \frac{1}{3}(-(n+2)W_{2n+2} + (2n+5)W_{2n+1} + (n+3)W_{2n} + (n+2)W_{2n-1}$
$+2W_3 - 4W_2 - 3W_1)$.

(c): $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{3}(-(n+1)W_{2n+2} + (2n+5)W_{2n+1} + (n+2)W_{2n} + (n+2)W_{2n-1}$
$+2W_3 - 5W_2 - 2W_0)$.

(d): $\sum_{k=1}^{n} W_{-k} = \frac{1}{3}(-(n+1)W_{-n+3} + (2n+1)W_{-n+2} + (n+2)W_{-n+1} + (n+3)W_{-n}+$
$W_3 - W_2 - 2W_1 - 3W_0)$.

(e): $\sum_{k=1}^{n} W_{-2k} = \frac{1}{3}(-(n+2)W_{-2n+2} + (2n+3)W_{-2n+1} + (n+4)W_{-2n} + (n+2)W_{-2n-1}$
$+2W_3 - 4W_2 - W_1 - 4W_0)$.

(f): $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{3}(-(n+3)W_{-2n+2} + 2(n+3)W_{-2n+1} + (n+2)W_{-2n} + (n+2)W_{-2n-1}$
$+2W_3 - 3W_2 - 4W_1 - 2W_0)$.

Proof. For the proof, see Soykan [26]. □

As a first special case of the above theorem, we have the following summation formulas for hyperbolic numbers.

THEOREM 13. *For the hyperbolic numbers, we have the following formulas:*

(a): $\sum\limits_{k=0}^{n} HW_k = \frac{1}{3}(-(n+3)HW_{n+3} + (2n+7)HW_{n+2} + (n+2)HW_{n+1} + (n+4)HW_n$
$+3HW_3 - 7HW_2 - 2HW_1 - HW_0)$.

(b): $\sum\limits_{k=0}^{n} HW_{2k} = \frac{1}{3}(-(n+2)HW_{2n+2} + (2n+5)HW_{2n+1} + (n+3)HW_{2n} + (n+2)HW_{2n-1}$
$+2HW_3 - 4HW_2 - 3HW_1)$.

(c): $\sum\limits_{k=0}^{n} HW_{2k+1} = \frac{1}{3}(-(n+1)HW_{2n+2} + (2n+5)HW_{2n+1} + (n+2)HW_{2n} + (n+2)HW_{2n-1}$
$+2HW_3 - 5HW_2 - 2HW_0)$.

(d): $\sum\limits_{k=1}^{n} HW_{-k} = \frac{1}{3}(-(n+1)HW_{-n+3} + (2n+1)HW_{-n+2} + (n+2)HW_{-n+1} + (n+3)HW_{-n}$
$+HW_3 - HW_2 - 2HW_1 - 3HW_0)$.

(e): $\sum\limits_{k=1}^{n} HW_{-2k} = \frac{1}{3}(-(n+2)HW_{-2n+2} + (2n+3)HW_{-2n+1} + (n+4)HW_{-2n} + (n+2)HW_{-2n-1}$
$+2HW_3 - 4HW_2 - HW_1 - 4HW_0)$.

(f): $\sum\limits_{k=1}^{n} HW_{-2k+1} = \frac{1}{3}(-(n+3)HW_{-2n+2} + 2(n+3)HW_{-2n+1} + (n+2)HW_{-2n} + (n+2)HW_{-2n-1}$
$+2HW_3 - 3HW_2 - 4HW_1 - 2HW_0)$.

Proof.

(a): Use Theorem [12] (a), (b), (c), (d), (e), (f) using (2.1), we get

$$\sum_{k=0}^{n} HW_k = HW_k + jHW_{k+1}.$$

Theorem 12 then (a), (b), (c), (d), (e), (f) the proof is easily attainable, respectively. □

(b):

As a first special case of the above theorem, we have the following summation formulas for hyperbolic Adrien numbers.

THEOREM 14. *For $n \geq 0$, hyperbolic generalized Adrien numbers have the following properties:*

(a): $\sum\limits_{k=0}^{n} \tilde{A}_k = \frac{1}{3}(-(n+3)\tilde{A}_{n+3} + (2n+7)\tilde{A}_{n+2} + (n+2)\tilde{A}_{n+1} + (n+4)\tilde{A}_n + 1)$.

(b): $\sum\limits_{k=0}^{n} \tilde{A}_{2k} = \frac{1}{3}(-(n+2)\tilde{A}_{2n+2} + (2n+5)\tilde{A}_{2n+1} + (n+3)\tilde{A}_{2n} + (n+2)\tilde{A}_{2n-1} + j + 1)$.

(c): $\sum\limits_{k=0}^{n} \tilde{A}_{2k+1} = \frac{1}{3}(-(n+1)\tilde{A}_{2n+2} + (2n+5)\tilde{A}_{2n+1} + (n+2)\tilde{A}_{2n} + (n+2)\tilde{A}_{2n-1} + 1)$.

(d): $\sum\limits_{k=1}^{n} \tilde{A}_{-k} = \frac{1}{3}(-(n+1)\tilde{A}_{-n+3} + (2n+1)\tilde{A}_{-n+2} + (n+2)\tilde{A}_{-n+1} + (n+3)\tilde{A}_{-n} + 4j + 3)$.

(e): $\sum\limits_{k=1}^{n} \tilde{A}_{-2k} = \frac{1}{3}(-(n+2)\tilde{A}_{-2n+2} + (2n+3)\tilde{A}_{-2n+1} + (n+4)\tilde{A}_{-2n} + (n+2)\tilde{A}_{-2n-1} + 3j + 3)$.

(f): $\sum\limits_{k=1}^{n} \tilde{A}_{-2k+1} = \frac{1}{3}(-(n+3)\tilde{A}_{-2n+2} + 2(n+3)\tilde{A}_{-2n+1} + (n+2)\tilde{A}_{-2n} + (n+2)\tilde{A}_{-2n-1} + 4j + 3)$.

In the following, we derive the ordinary generating functions corresponding to certain special cases of the hyperbolic generalized Adrien numbers.

THEOREM 15. *The ordinary generating functions of the sequences $\widehat{W}_{2n}$, $\widehat{W}_{2n+1}$ are given as follows:*

(a): $\sum_{n=0}^{\infty} HW_{2n}x^n = \dfrac{3x^2HW_3 + (x^3 - 8x^2 + x)HW_2 - 3x^3HW_1 + (x^3 + 2x^2 - 7x + 1)HW_0}{x^4 + 2x^3 + 3x^2 - 7x + 1}$

(b): $\sum_{n=0}^{\infty} HW_{2n+1}x^n = \dfrac{(x^3 + x^2 + x)HW_3 - (3x^3 + 3x^2)HW_2 + (x^3 + 2x^2 - 7x + 1)HW_1 - 3x^2HW_0}{x^4 + 2x^3 + 3x^2 - 7x + 1}$

From the last Theorem, we have the following Corollary which gives sum formula of hyperbolic Adrien numbers (Take $HW_n = \tilde{A}_n$ with

$\tilde{A}_0 = j$, $\tilde{A}_1 = 1 + 3j$, $\tilde{A}_2 = 3 + 8j$, $\tilde{A}_3 = 8 + 21j$)

COROLLARY 16. *For $n \geq 0$ hyperbolic Adrien numbers have the following properties.*

(a): $\sum_{n=0}^{\infty} \tilde{A}_{2n}x^n = \frac{j\left(x^3+2x^2-7x+1\right)-3x^3(3j+1)+3x^2(21j+8)+(8j+3)\left(x^3-8x^2+x\right)}{x^4+2x^3+3x^2-7x+1}$.

(b): $\sum_{n=0}^{\infty} \tilde{A}_{2n+1}x^n = -\frac{(8j+3)\left(3x^3+3x^2\right)-(3j+1)\left(x^3+2x^2-7x+1\right)-(21j+8)\left(x^3+x^2+x\right)+3jx^2}{x^4+2x^3+3x^2-7x+1}$.

6. Matrices related with Hyperbolic Generalized Adrien Numbers

This part of the study introduces matrix identities that arise in connection with the hyperbolic Adrien numbers.

By using the $\{A_n\}$ which is defined by the fourth-order recurrence relation as follows:

$$A_n = 3A_{n-1} - A_{n-2} - A_{n-4} \ ,$$

with the initial conditions

$$A_0 = 0,\ A_1 = 1,\ A_2 = 3,\ A_3 = 8. \tag{6.1}$$

We define the square matrix M of order 4 as

$$M = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $detM = 1$. Then, we give the following Lemma.

LEMMA 17. *For $n \geq 0$ the following identity is true*

$$\begin{pmatrix} HW_{n+3} \\ HW_{n+2} \\ HW_{n+1} \\ HW_n \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}. \tag{6.2}$$

Proof. First, we prove the assertion for the case $n \geq 0$. Lemma 17 can be given by mathematical induction on n. If $n = 0$ we get

$$\begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}$$

which is true. We assume that (6.2) is true for $n = k$. Thus the following identity is true.

$$\begin{pmatrix} HW_{k+3} \\ HW_{k+2} \\ HW_{k+1} \\ HW_k \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}.$$

For $n = k + 1$, we get

$$\begin{aligned} \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix} &= \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} HW_3 \\ HW_2 \\ HW_1 \\ HW_0 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} HW_{k+3} \\ HW_{k+2} \\ HW_{k+1} \\ HW_k \end{pmatrix} \\ &= \begin{pmatrix} HW_{k+4} \\ HW_{k+3} \\ HW_{k+2} \\ HW_{k+1} \end{pmatrix}. \end{aligned}$$

Consequently, by mathematical induction on n, the proof is completed. □

Note that

$$A^n = \begin{pmatrix} A_{n+1} & -A_n - A_{n-2} & -A_{n-1} & -A_n \\ A_n & -A_{n-1} - A_{n-3} & -A_{n-2} & -A_{n-1} \\ A_{n-1} & -A_{n-2} - A_{n-4} & -A_{n-3} & -A_{n-2} \\ A_{n-2} & -A_{n-3} - A_{n-5} & -A_{n-4} & -A_{n-3} \end{pmatrix}.$$

For the proof see [25].

We define

$$N_{HW} = \begin{pmatrix} HW_3 & HW_2 & HW_1 & HW_0 \\ HW_2 & HW_1 & HW_0 & HW_{-1} \\ HW_1 & HW_0 & HW_{-1} & HW_{-2} \\ HW_0 & HW_{-1} & HW_{-2} & HW_{-3} \end{pmatrix}, \tag{6.3}$$

$$E_{HW} = \begin{pmatrix} HW_{n+3} & HW_{n+2} & HW_{n+1} & HW_n \\ HW_{n+2} & HW_{n+1} & HW_n & HW_{n-1} \\ HW_{n+1} & HW_n & HW_{n-1} & HW_{n-2} \\ HW_n & HW_{n-1} & HW_{n-2} & HW_{n-3} \end{pmatrix}. \tag{6.4}$$

Now, we have the following theorem for N_{HW} and E_{HW}.

Theorem 18. *Using N_{HW} and E_{HW}, we get*

$$A^n N_{HW} = E_{HW}.$$

Proof. Note that we get

$$\begin{aligned} A^n N_{\widehat{W}} &= \begin{pmatrix} A_{n+1} & -A_n - A_{n-2} & -A_{n-1} & -A_n \\ A_n & -A_{n-1} - A_{n-3} & -A_{n-2} & -A_{n-1} \\ A_{n-1} & -A_{n-2} - A_{n-4} & -A_{n-3} & -A_{n-2} \\ A_{n-2} & -A_{n-3} - A_{n-5} & -A_{n-4} & -A_{n-3} \end{pmatrix} \begin{pmatrix} HW_3 & HW_2 & HW_1 & HW_0 \\ HW_2 & HW_1 & HW_0 & HW_{-1} \\ HW_1 & HW_0 & HW_{-1} & HW_{-2} \\ HW_0 & HW_{-1} & HW_{-2} & HW_{-3} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \end{aligned}$$

where

$$
\begin{aligned}
a_{11} &= A_{n+1}HW_3 + (-A_n - A_{n-2})HW_2 + (-A_{n-1})HW_1 + (-A_n)HW_0, \\
a_{12} &= A_{n+1}HW_2 + (-A_n - A_{n-2})HW_1 + (-A_{n-1})HW_0 + (-A_n)HW_{-1}, \\
a_{13} &= A_{n+1}HW_1 + (-A_n - A_{n-2})HW_0 + (-A_{n-1})HW_{-1} + (-A_n)HW_{-2}, \\
a_{14} &= A_{n+1}HW_0 + (-A_n - A_{n-2})HW_{-1} + (-A_{n-1})HW_{-2} + (-A_n)HW_{-3}, \\
a_{21} &= A_nHW_3 + (-A_{n-1} - A_{n-3})HW_2 + (-A_{n-2})HW_1 + (-A_{n-1})HW_0, \\
a_{22} &= A_nHW_2 + (-A_{n-1} - A_{n-3})HW_1 + (-A_{n-2})HW_0 + (-A_{n-1})HW_{-1}, \\
a_{23} &= A_nHW_1 + (-A_{n-1} - A_{n-3})HW_0 + (-A_{n-2})HW_{-1} + (-A_{n-1})HW_{-2}, \\
a_{24} &= A_nHW_0 + (-A_{n-1} - A_{n-3})HW_{-1} + (-A_{n-2})HW_{-2} + (-A_{n-1})HW_{-3}, \\
a_{31} &= A_{n-1}HW_3 + (-A_{n-2} - A_{n-4})HW_2 + (-A_{n-3})HW_1 + (-A_{n-2})HW_0, \\
a_{32} &= A_{n-1}HW_2 + (-A_{n-2} - A_{n-4})HW_1 + (-A_{n-3})HW_0 + (-A_{n-2})HW_{-1}, \\
a_{33} &= A_{n-1}HW_1 + (-A_{n-2} - A_{n-4})HW_0 + (-A_{n-3})HW_{-1} + (-A_{n-2})HW_{-2}, \\
a_{34} &= A_{n-1}HW_0 + (-A_{n-2} - A_{n-4})HW_{-1} + (-A_{n-3})HW_{-2} + (-A_{n-2})HW_{-3}, \\
a_{41} &= A_{n-2}HW_3 + (-A_{n-3} - A_{n-5})HW_2 + (-A_{n-4})HW_1 + (-A_{n-3})HW_0, \\
a_{42} &= A_{n-2}HW_2 + (-A_{n-3} - A_{n-5})HW_1 + (-A_{n-4})HW_0 + (-A_{n-3})HW_{-1}, \\
a_{43} &= A_{n-2}HW_1 + (-A_{n-3} - A_{n-5})HW_0 + (-A_{n-4})HW_{-1} + (-A_{n-3})HW_{-2}, \\
a_{44} &= A_{n-2}HW_0 + (-A_{n-3} - A_{n-5})HW_{-1} + (-A_{n-4})HW_{-2} + (-A_{n-3})HW_{-3}.
\end{aligned}
$$

Using the theorem (8) the proof is done. □

By taking $W_n = A_n$ with A_0, A_1, A_2, A_3 in (6.3) and (6.4)

and $\widehat{W}_n = \widehat{W}B_n$ with $\widehat{W}B_0$, $\widehat{W}B_1$, $\widehat{W}B_2$, $\widehat{W}B_3$ in (6.3) and (6.4)

respectively, we get:

$$N_{\tilde{A}} = \begin{pmatrix} 8+21j & 3+8j & 1+3j & j \\ 3+8j & 1+3j & j & 0 \\ 1+3j & j & 0 & 0 \\ j & 0 & 0 & -1 \end{pmatrix},$$

$$E_{\tilde{A}} = \begin{pmatrix} \tilde{A}_{n+3} & \tilde{A}_{n+2} & \tilde{A}_{n+1} & \tilde{A}_{n} \\ \tilde{A}_{n+2} & \tilde{A}_{n+1} & \tilde{A}_{n} & \tilde{A}_{n-1} \\ \tilde{A}_{n+1} & \tilde{A}_{n} & \tilde{A}_{n-1} & \tilde{A}_{n-2} \\ \tilde{A}_{n} & \tilde{A}_{n-1} & \tilde{A}_{n-2} & \tilde{A}_{n-3} \end{pmatrix},$$

$$N_{\tilde{B}} = \begin{pmatrix} 18+43j & 7+18j & 3+7j & 4+3j \\ 7+18j & 3+7j & 4+3j & 4j \\ 3+7j & 4+3j & 4j & -2 \\ 4+3j & 4j & -2 & 9-2j \end{pmatrix},$$

$$E_{\tilde{B}} = \begin{pmatrix} \tilde{B}_{n+3} & \tilde{B}_{n+2} & \tilde{B}_{n+1} & \tilde{B}_{n} \\ \tilde{B}_{n+2} & \tilde{B}_{n+1} & \tilde{B}_{n} & \tilde{B}_{n-1} \\ \tilde{B}_{n+1} & \tilde{B}_{n} & \tilde{B}_{n-1} & \tilde{B}_{n-2} \\ \tilde{B}_{n} & \tilde{B}_{n-1} & \tilde{B}_{n-2} & \tilde{B}_{n-3} \end{pmatrix}.$$

From Theorem [18], we can write the following corollary.

COROLLARY 19. *The following identities are hold:*

a): $A^n N_{\widehat{W}A} = E_{\widehat{W}A}$.

b): $A^n N_{\widehat{W}B} = E_{\widehat{W}B}$.

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