

# Positive periodic solutions to Liénard equation with indefinite weights

**Abstract**—In this paper, we propose some sufficient conditions for the existence of a positive periodic solution to the following Liénard equation with indefinite weights

$$x'' = \frac{g(t)}{x^\mu} - \frac{h(t)}{x^\lambda} + p(t),$$

where  $\mu, \lambda$  are positive constants,  $g(t), h(t), p(t) \in L^1(\mathbb{R}/T\mathbb{Z})$  are  $T$ -periodic sign-changing functions. The main tools are Krasnoselskii's-Guo fixed point theorem and the positivity of the associated Green function.

**Keywords**—Periodic solution; Liénard equation; Indefinite singularity; Krasnoselskii's-Guo fixed point theorem.

## 1 Introduction

Singular differential equations have wide applications in mechanics, electronics, modern biology, and many other fields [1, 9]. Among them, the study of the Liénard equation is of great theoretical value and plays an important role in physical modeling. For example, the bubble physics model proposed by Professor Torres in his monograph [11], as well as the Micro-electro-mechanical systems (MEMS) mass-spring model, both demonstrate the modeling significance of this equation in concrete problems. Therefore, research on the Liénard equation has long attracted considerable attention.

The origin of the Liénard equation can be traced back to the 1920s. In 1926, van der Pol [5] proposed the equation

$$x'' + \mu(x^2 - 1)x' + x = 0, \quad \mu \neq 0,$$

in his study of isochronous oscillations in triode circuits. This equation shares the same structural form as the one later introduced by the French engineer Liénard [10] in 1928, who generalized it to the more comprehensive form

$$x'' + f(x)x' + g(x) = 0,$$

which is now referred to as the classical Liénard equation.

With the development of this subject, singular Liénard equations have received continuous attention, and many existence results for periodic solutions have been obtained using a variety of techniques, including topological degree theory, fixed point theorems, and the method of upper and lower solutions [8, 13, 15, 16]. More recently, in 2021, Xin and Cheng [14] considered the existence of positive periodic solutions for generalized singular  $\phi$ -Laplacian type Liénard equations of the form

$$(\phi(x'))' + f(t, x)x' + \frac{b(t)}{x^\rho} = h(t)x^m,$$

where  $\rho$  is a positive constant and  $m$  is a constant. By applying the method of upper and lower solutions, they established existence results. Cheng et al. [4] investigated the following Lennard Jones potential

$$x'' = \frac{a(t)}{x^\rho} - \frac{b(t)}{x^\delta},$$

where  $\rho$  and  $\delta$  are two positive constants, and  $a(t), b(t) \in L^1(\mathbb{R}/T\mathbb{Z})$  are  $T$ -periodic sign-changing functions, By applying the method of Krasnoselskii's-Guo fixed point theorem, they proved the existence of periodic solutions for this equation.

Motivated by the above works, the present paper investigates the existence of periodic solutions for the following Liénard equation with indefinite weight

$$x'' = \frac{g(t)}{x^\mu} - \frac{h(t)}{x^\lambda} + p(t). \tag{1.1}$$

where  $\mu, \lambda > 0$  are constants, and  $g(t), h(t), p(t) \in L^1(\mathbb{R}/T\mathbb{Z})$  are  $T$ -periodic sign-changing functions. The sign-changing nature of the coefficients makes the analysis more intricate. Our approach is mainly based on the Krasnoselskii's-Guo fixed point theorem, combined with the positivity of the associated Green function, to establish existence results for periodic solutions.

## 2 Preliminary and notations

In this section, we will primarily concentrate on the Krasnoselskii's-Guo fixed point theorem, which will be used in the proofs of our theorems below.

**Lemma 2.1.** (*Krasnoselskii's-Guo fixed point theorem [6]*) *Let  $X$  be a Banach space and  $\mathcal{K}$  a cone in  $X$ . Assume that  $\Omega_1$  and  $\Omega_2$  are open subsets of  $X$  with  $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$ . Let*

$$\Phi : \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow \mathcal{K}$$

*be a completely continuous operator such that one of the following conditions holds:*

- (i)  $\|\Phi x\| \geq \|x\|$  for  $x \in \mathcal{K} \cap \partial\Omega_1$  and  $\|\Phi x\| \leq \|x\|$  for  $x \in \mathcal{K} \cap \partial\Omega_2$ ;
- (ii)  $\|\Phi x\| \leq \|x\|$  for  $x \in \mathcal{K} \cap \partial\Omega_1$  and  $\|\Phi x\| \geq \|x\|$  for  $x \in \mathcal{K} \cap \partial\Omega_2$ .

*Then  $\Phi$  has a fixed point in the set  $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$ .*

To write the periodic problem as an equivalent fixed point problem, we will use the notion of the Green function. Additionally, the Green function is a crucial mathematical tool and can be found in many papers( such as [2,3,7,12] ).

**Lemma 2.2.** (see [7, Lemma 2.4]) If  $M > 0$  is such that  $M \neq \frac{2k\pi}{T}$  for any natural  $k$ , then for any  $e \in L^1(\mathbb{R}/T\mathbb{Z})$  the equation

$$x'' + M^2x = e(t)$$

has a unique  $T$ -periodic solution given by

$$x(t) = \int_0^T G_1(t, s)e(s)ds,$$

where the Green function  $G_1(t, s)$  has the following form

$$G_1(t, s) = \begin{cases} \frac{\cos M(t - s - \frac{T}{2})}{2M \sin \frac{MT}{2}}, & 0 \leq s \leq t \leq T, \\ \frac{\cos M(t - s + \frac{T}{2})}{2M \sin \frac{MT}{2}}, & 0 \leq t < s \leq T. \end{cases}$$

In addition, if  $M < \frac{\pi}{T}$ , then  $G_1(t, s) > 0$  for all  $(t, s) \in [0, T] \times [0, T]$  and  $\int_0^T G_1(t, s)M^2ds \equiv 1$ .

**Lemma 2.3.** (see [12, Corollary 2.2]) If  $M > 0$ , then for each  $e \in L^1(\mathbb{R}/T\mathbb{Z})$  the equation

$$-x'' + M^2x = e(t)$$

has a unique  $T$ -periodic solution provided by

$$x(t) = \int_0^T G_2(t, s)e(s)ds,$$

where the Green function  $G_2(t, s)$  has the following form

$$G_2(t, s) = \begin{cases} \frac{\exp(-M(s - t)) + \exp(M(s - t - T))}{2M(1 - \exp(-MT))}, & 0 \leq t < s \leq T, \\ \frac{\exp(-M(s - t + T)) + \exp(M(s - t))}{2M(1 - \exp(-MT))}, & 0 \leq s \leq t \leq T. \end{cases}$$

Besides,  $G_2(t, s) > 0$  for  $(t, s) \in [0, T] \times [0, T]$  and  $\int_0^T G_2(t, s)M^2ds \equiv 1$ .

Define

$$\begin{aligned} \mathcal{A}_1 &:= \min_{0 \leq s, t \leq T} G_1(t, s) = \frac{1}{2M} \cot \frac{MT}{2}, \quad \mathcal{B}_1 := \max_{0 \leq s, t \leq T} G_1(t, s) = \frac{1}{2M \sin \frac{MT}{2}}, \\ \mathcal{A}_2 &:= \min_{0 \leq s, t \leq T} G_2(t, s) = \frac{\exp(-\frac{MT}{2})}{M(1 - \exp(-MT))}, \quad \mathcal{B}_2 := \max_{0 \leq s, t \leq T} G_2(t, s) = \frac{1 + \exp(-MT)}{2M(1 - \exp(-MT))}, \\ \sigma_1 &:= \frac{\mathcal{A}_1}{\mathcal{B}_1}, \quad \sigma_2 := \frac{\mathcal{A}_2}{\mathcal{B}_2}. \end{aligned} \tag{2.1}$$

It is clear that  $0 < \mathcal{A}_i \leq \mathcal{B}_i$  and  $0 < \sigma_i \leq 1$ ,  $i = 1, 2$  from Lemmas 2.2 and 2.3. Define

$$\mathcal{K}_i := \{x \in C_T : \min_{t \in \mathbb{R}} x(t) \geq \sigma_i \|x\|\}, \quad i = 1, 2,$$

where  $C_T := \{x \in C(\mathbb{R}, \mathbb{R}) : x(t+T) = x(t), \text{ for all } t \in \mathbb{R}\}$  and  $\|x\| := \max_{t \in \mathbb{R}} |x(t)|$ . It is easy to verify that  $\mathcal{K}_i$  is a cone in  $C_T$ .

Finally, for given periodic functions  $h(t)$  and  $e(t)$ , we denote

$$h^+(t) := \max\{0, h(t)\}, \quad h^-(t) := -\min\{0, h(t)\}, \quad \bar{h} := \frac{1}{T} \int_0^T h(t) dt, \quad e^* := \max_{t \in \mathbb{R}} e(t), \quad e_* := \min_{t \in \mathbb{R}} e(t).$$

### 3 Main results

We prove the result by considering two cases, and the main conclusions are as follows.

#### 3.1 The case $\mu > \lambda$

**Theorem 3.1.** *Assume that there exist four constants  $N \in (0, \frac{\pi}{T})$ ,  $a, b, c \in (0, 1)$  and  $a + b + c = 1$  such that*

$$\bar{h}^- < \sigma_1^{1+\lambda} \bar{h}^+, \quad \bar{p}^+ \leq \sigma_1 \bar{p}^-, \quad (\mathcal{A}_1 T \bar{g}^+)^{\frac{1}{1+\mu}} > \frac{1}{\sigma_1} \max \left\{ \left( \frac{\|g^-\|_\infty}{aN^2} \right)^{\frac{1}{1+\mu}}, \left( \frac{\|h^+\|_\infty}{bN^2} \right)^{\frac{1}{1+\lambda}}, \frac{\|p^-\|_\infty}{cN^2} \right\}. \quad (3.1)$$

If  $\mu > \lambda$ , then equation (1.1) admits at least one positive  $T$ -periodic solution.

**Proof.** We can write equation (1.1) as

$$x'' + N^2 x = \frac{g(t)}{x^\mu} - \frac{h(t)}{x^\lambda} + N^2 x + p(t), \quad (3.2)$$

a  $T$ -periodic solution of equation (3.2) is just a fixed point of the map  $\Phi$  defined by

$$(\Phi x)(t) := \int_0^T G_1(t, s) \left( \frac{g(s)}{x(s)^\mu} - \frac{h(s)}{x(s)^\lambda} + N^2 x(s) + p(s) \right) ds, \quad (3.3)$$

and we know that  $G_1(t, s) > 0$  for all  $(t, s) \in [0, T] \times [0, T]$  from Lemma 2.2.

Now we define two open sets

$$\Omega_1 := \{x \in C_T : \|x\| < r_1\} \quad \text{and} \quad \Omega_2 := \{x \in C_T : \|x\| < R_1\}.$$

Note that  $\Phi$  is well-defined in  $\mathcal{K}_1 \cap (\bar{\Omega}_2 \setminus \Omega_1)$  and is a completely continuous operator via Ascoli-Arzelà Theorem.

By (3.1), the two positive constants  $r_1$  and  $R_1$  can be fixed such that

$$R_1 > r_1 = (\mathcal{A}_1 T \bar{g}^+)^{\frac{1}{1+\mu}} > \frac{1}{\sigma_1} \max \left\{ \left( \frac{\|g^-\|_\infty}{aN^2} \right)^{\frac{1}{1+\mu}}, \left( \frac{\|h^+\|_\infty}{bN^2} \right)^{\frac{1}{1+\lambda}}, \frac{\|p^-\|_\infty}{cN^2} \right\}.$$

First, we claim that  $\Phi(\mathcal{K}_1 \cap (\bar{\Omega}_2 \setminus \Omega_1)) \subset \mathcal{K}_1$ . In fact, for any  $x \in \mathcal{K}_1 \cap (\bar{\Omega}_2 \setminus \Omega_1)$ , we arrive at

$$\sigma_1 r_1 \leq x(t) \leq R_1, \quad \text{for all } t \in \mathbb{R}.$$

Since  $r_1 > \frac{1}{\sigma_1} \max \left\{ \left( \frac{\|g^-\|_\infty}{aN^2} \right)^{\frac{1}{1+\mu}}, \left( \frac{\|h^+\|_\infty}{bN^2} \right)^{\frac{1}{1+\lambda}}, \frac{\|p^-\|_\infty}{cN^2} \right\}$ , and  $a + b + c = 1$  we obtain

$$\begin{aligned}
 \frac{g(t)}{x^\mu} - \frac{h(t)}{x^\lambda} + N^2x + p(t) &= \frac{g^+(t)}{x^\mu} - \frac{g^-(t)}{x^\mu} - \frac{h^+(t)}{x^\lambda} + \frac{h^-(t)}{x^\lambda} + N^2x + p^+(t) - p^-(t) \\
 &\geq -\frac{g^-(t)}{x^\mu} - \frac{h^+(t)}{x^\lambda} - p^-(t) + N^2x \\
 &= -\frac{g^-(t)}{x^\mu} - \frac{h^+(t)}{x^\lambda} - p^-(t) + (a + b + c)N^2x \\
 &\geq -\frac{\|g^-\|_\infty}{(\sigma_1 r_1)^\mu} + aN^2(\sigma_1 r_1) - \frac{\|h^+\|_\infty}{(\sigma_1 r_1)^\lambda} + bN^2(\sigma_1 r_1) - \|p^-\|_\infty + cN^2(\sigma_1 r_1) \\
 &\geq 0,
 \end{aligned} \tag{3.4}$$

for all  $t \in \mathbb{R}$ . It follows from (2.1) and (3.4) that

$$\begin{aligned}
 \min_{t \in \mathbb{R}}(\Phi x)(t) &= \min_{t \in \mathbb{R}} \int_0^T G_1(t, s) \left( \frac{g(s)}{x^\mu} - \frac{h(s)}{x^\lambda} + N^2x + p(s) \right) ds \\
 &\geq \mathcal{A}_1 \int_0^T \left( \frac{g(s)}{x^\mu} - \frac{h(s)}{x^\lambda} + N^2x + p(s) \right) ds \\
 &= \sigma_1 \mathcal{B}_1 \int_0^T \left( \frac{g(s)}{x^\mu} - \frac{h(s)}{x^\lambda} + N^2x + p(s) \right) ds \\
 &\geq \sigma_1 \max_{s, t \in [0, T]} \int_0^T G_1(t, s) \left( \frac{g(s)}{x^\mu} - \frac{h(s)}{x^\lambda} + N^2x + p(s) \right) ds \\
 &\geq \sigma_1 \|\Phi x\|,
 \end{aligned} \tag{3.5}$$

which implies  $\Phi(\mathcal{K}_1 \cap (\bar{\Omega}_2 \setminus \Omega_1)) \subset \mathcal{K}_1$ .

Next, we prove that

$$\|\Phi x\| \geq \|x\|, \text{ for } x \in \mathcal{K}_1 \cap \partial\Omega_1. \tag{3.6}$$

In fact, for any  $x \in \mathcal{K}_1 \cap \partial\Omega_1$ , it is clear  $\|x\| = r_1$  and

$$\sigma_1 r_1 \leq x(t) \leq r_1, \text{ for all } t \in \mathbb{R}.$$

Then

$$\begin{aligned}
 (\Phi x)(t) &= \int_0^T G_1(t, s) \left( \frac{g(s)}{x^\mu} - \frac{h(s)}{x^\lambda} + N^2x + p(s) \right) ds \\
 &= \int_0^T G_1(t, s) \left( \frac{g^+(s)}{x^\mu} - \frac{g^-(s)}{x^\mu} - \frac{h^+(s)}{x^\lambda} + \frac{h^-(s)}{x^\lambda} + N^2x + p^+(s) - p^-(s) \right) ds \\
 &\geq \int_0^T G_1(t, s) \frac{g^+(s)}{x^\mu} ds \\
 &\geq \frac{\mathcal{A}_1 T \bar{g}^+}{r_1^\mu} = r_1,
 \end{aligned} \tag{3.7}$$

since  $r_1 = (\mathcal{A}_1 T \bar{g}^+)^{\frac{1}{1+\mu}}$  from definition of  $r_1$ . Hence, (3.6) is satisfied.

Finally, we prove that

$$\|\Phi x\| \leq \|x\|, \text{ for } x \in \mathcal{K}_1 \cap \partial\Omega_2. \quad (3.8)$$

In fact, for any  $x \in \mathcal{K}_1 \cap \partial\Omega_2$ , it is clear  $\|x\| = R_1$  and

$$\sigma_1 R_1 \leq x(t) \leq R_1, \text{ for all } t \in \mathbb{R}.$$

It follows from (2.1),  $\int_0^T G_1(t, s) M^2 ds \equiv 1$  and  $\overline{p^+} \leq \sigma_1 \overline{p^-}$  that

$$\begin{aligned} (\Phi x)(t) &= \int_0^T G_1(t, s) \left( \frac{g(s)}{x^\mu} - \frac{h(s)}{x^\lambda} + N^2 x + p(s) \right) ds \\ &= \int_0^T G_1(t, s) \left( \frac{g^+(s)}{x^\mu} - \frac{g^-(s)}{x^\mu} - \frac{h^+(s)}{x^\lambda} + \frac{h^-(s)}{x^\lambda} + N^2 x + p^+(s) - p^-(s) \right) ds \\ &\leq \frac{\mathcal{B}_1 T \overline{g^+}}{(\sigma_1 R_1)^\mu} - \frac{\mathcal{A}_1 T \overline{g^-}}{R_1^\mu} - \frac{\mathcal{A}_1 T \overline{h^+}}{R_1^\lambda} + \frac{\mathcal{B}_1 T \overline{h^-}}{(\sigma_1 R_1)^\lambda} + \mathcal{B}_1 T \overline{p^+} - \mathcal{A}_1 T \overline{p^-} + R_1 \\ &\leq \frac{\mathcal{B}_1 T \overline{g^+}}{(\sigma_1 R_1)^\mu} - \frac{\mathcal{A}_1 T \overline{g^-}}{R_1^\mu} - \frac{\mathcal{A}_1 T \overline{h^+}}{R_1^\lambda} + \frac{\mathcal{B}_1 T \overline{h^-}}{(\sigma_1 R_1)^\lambda} + R_1 \\ &\leq R_1, \end{aligned} \quad (3.9)$$

where  $\frac{\mathcal{B}_1 T \overline{g^+}}{(\sigma_1 R_1)^\mu} - \frac{\mathcal{A}_1 T \overline{g^-}}{R_1^\mu} - \frac{\mathcal{A}_1 T \overline{h^+}}{R_1^\lambda} + \frac{\mathcal{B}_1 T \overline{h^-}}{(\sigma_1 R_1)^\lambda} + R_1 \leq R_1$  holds, i.e.,

$$\overline{g^+} - \sigma_1^{1+\mu} \overline{g^-} \leq \left( \sigma_1^{1+\lambda} \overline{h^+} - \overline{h^-} \right) (\sigma_1 R_1)^{\mu-\lambda}$$

for sufficiently large  $R_1$  and  $\overline{h^-} < \sigma_1^{1+\lambda} \overline{h^+}$ . This implies that (3.8) holds. The proof is finished.  $\square$

In the Theorem 3.1, we know that  $0 < N < \frac{\pi}{T}$ . Next we will give a result similar to Theorem 3.1, in the case without any restriction on  $N > 0$ .

**Theorem 3.2.** *Assume that there exist four constants  $N > 0, a, b, c \in (0, 1)$  and  $a + b + c = 1$  such that*

$$\overline{h^+} < \sigma_2^{1+\lambda} \overline{h^-}, \quad \overline{p^-} \leq \sigma_2 \overline{p^+}, \quad (\mathcal{A}_2 T \overline{g^-})^{\frac{1}{1+\mu}} > \frac{1}{\sigma_2} \max \left\{ \left( \frac{\|g^+\|_\infty}{aN^2} \right)^{\frac{1}{1+\mu}}, \left( \frac{\|h^-\|_\infty}{bN^2} \right)^{\frac{1}{1+\lambda}}, \frac{\|p^+\|_\infty}{cN^2} \right\}.$$

*If  $\mu > \lambda$ , then equation (1.1) admits at least one positive  $T$ -periodic solution.*

**Proof.** We can write equation (1.1) as

$$-x'' + N^2 x = -\frac{g(t)}{x^\mu} + \frac{h(t)}{x^\lambda} + N^2 x - p(t), \quad (3.10)$$

a  $T$ -periodic solution of equation (3.10) is just a fixed point of the map  $\Psi$  defined by

$$(\Psi x)(t) := \int_0^T G_2(t, s) \left( -\frac{g(s)}{x(s)^\mu} + \frac{h(s)}{x(s)^\lambda} + N^2 x(s) - p(s) \right) ds,$$

and we know that  $G_2(t, s) > 0$  for all  $(t, s) \in [0, T] \times [0, T]$  from Lemma 2.3. From this point, the proof follows the same steps as Theorem 3.1.

### 3.2 The case $\mu < \lambda$

**Theorem 3.3.** *Assume that there exist four constants  $N \in (0, \frac{\pi}{T})$ ,  $a, b, c \in (0, 1)$  and  $a + b + c = 1$  such that*

$$(\mathcal{A}_1 T \bar{g}^+)^{\frac{1}{1+\mu}} > \left( \frac{\sigma_1^{1+\lambda} \bar{h}^+ - \bar{h}^-}{(\bar{g}^+ - \sigma_1^{1+\mu} \bar{g}^-) \sigma_1^{\lambda-\mu}} \right)^{\frac{1}{\lambda-\mu}} > \frac{1}{\sigma_1} \max \left\{ \left( \frac{\|g^-\|_\infty}{aN^2} \right)^{\frac{1}{1+\mu}}, \left( \frac{\|h^+\|_\infty}{bN^2} \right)^{\frac{1}{1+\lambda}}, \frac{\|p^-\|_\infty}{cN^2} \right\},$$

$\bar{h}^- < \sigma_1^{1+\lambda} \bar{h}^+$ , and  $\sigma_1 \bar{p}^- \geq \bar{p}^+$ . If  $\mu < \lambda$ , then equation (1.1) admits at least one positive  $T$ -periodic solution.

Now we define two open sets

$$\Omega_3 := \{x \in C_T^1 : \|x\| < r_2\} \text{ and } \Omega_4 := \{x \in C_T^1 : \|x\| < R_2\},$$

where  $r_2$  and  $R_2$  are two constants and

$$R_2 = (\mathcal{A}_2 T \bar{g}^+)^{\frac{1}{1+\mu}} > \left( \frac{\sigma_1^{1+\lambda} \bar{h}^+ - \bar{h}^-}{(\bar{g}^+ - \sigma_1^{1+\mu} \bar{g}^-) \sigma_1^{\lambda-\mu}} \right)^{\frac{1}{\lambda-\mu}} > r_2 > \frac{1}{\sigma_2} \max \left\{ \left( \frac{\|g^-\|_\infty}{aN^2} \right)^{\frac{1}{1+\mu}}, \left( \frac{\|h^+\|_\infty}{bN^2} \right)^{\frac{1}{1+\lambda}}, \frac{\|p^-\|_\infty}{cN^2} \right\}.$$

By an analogous reasoning as in the proof of Theorem 3.1, we get that  $\Phi : \mathcal{K}_1 \cap (\bar{\Omega}_4 \setminus \Omega_3) \in \mathcal{K}_1$ , where  $\Phi$  is defined in (3.3).

Next, we prove that

$$\|\Phi x\| \leq \|x\|, \text{ for } x \in \mathcal{K}_1 \cap \partial\Omega_3. \quad (3.11)$$

In fact, for any  $x \in \mathcal{K}_1 \cap \partial\Omega_3$ , it is clear  $\|x\| = r_2$  and

$$\sigma_1 r_2 \leq x(t) \leq r_2, \text{ for all } t \in \mathbb{R}.$$

It follows from (2.1),  $\int_0^T G_1(t, s) M^2 ds \equiv 1$  and  $\bar{p}^- \leq \sigma_1 \bar{p}^-$  that

$$\begin{aligned} (\Phi x)(t) &= \int_0^T G_1(t, s) \left( \frac{g(s)}{x^\mu} - \frac{h(s)}{x^\lambda} + N^2 x + p(s) \right) ds \\ &= \int_0^T G_1(t, s) \left( \frac{g^+(s)}{x^\mu} - \frac{g^-(s)}{x^\mu} - \frac{h^+(s)}{x^\lambda} + \frac{h^-(s)}{x^\lambda} + N^2 x + p^+(s) - p^-(s) \right) ds \\ &\leq \frac{\mathcal{B}_1 T \bar{g}^+}{(\sigma_1 r_1)^\mu} - \frac{\mathcal{A}_1 T \bar{g}^-}{r_1^\mu} - \frac{\mathcal{A}_1 T \bar{h}^+}{r_1^\lambda} + \frac{\mathcal{B}_1 T \bar{h}^-}{(\sigma_1 r_1)^\lambda} + \mathcal{B}_1 T \bar{p}^+ - \mathcal{A}_1 T \bar{p}^- + r_1 \\ &\leq \frac{\mathcal{B}_1 T \bar{g}^+}{(\sigma_1 r_1)^\mu} - \frac{\mathcal{A}_1 T \bar{g}^-}{r_1^\mu} - \frac{\mathcal{A}_1 T \bar{h}^+}{r_1^\lambda} + \frac{\mathcal{B}_1 T \bar{h}^-}{(\sigma_1 r_1)^\lambda} + r_1 \\ &\leq r_1, \end{aligned} \quad (3.12)$$

where  $\frac{\mathcal{B}_1 T \bar{g}^+}{(\sigma_1 r_1)^\mu} - \frac{\mathcal{A}_1 T \bar{g}^-}{r_1^\mu} - \frac{\mathcal{A}_1 T \bar{h}^+}{r_1^\lambda} + \frac{\mathcal{B}_1 T \bar{h}^-}{(\sigma_1 r_1)^\lambda} \leq 0$  holds, i.e.,

$$r_1^{\lambda-\mu} \left( \sigma_1^{\lambda-\mu} \bar{g}^+ - \sigma_1^{1+\lambda} \bar{g}^- \right) \leq \sigma_1^{1+\lambda} \bar{h}^+ - \bar{h}^-$$

for  $\bar{h}^- < \sigma_1^{1+\lambda} \bar{h}^+$  and  $r_1 < \left( \frac{\sigma_1^{1+\lambda} \bar{h}^+ - \bar{h}^-}{(\bar{g}^+ - \sigma_1^{1+\mu} \bar{g}^-) \sigma_1^{\lambda-\mu}} \right)^{\frac{1}{\lambda-\mu}}$ . This implies that (3.11) holds.

Finally, we prove that

$$\|\Phi x\| \geq \|x\|, \text{ for } x \in \mathcal{K}_1 \cap \partial\Omega_4. \quad (3.13)$$

In fact, for any  $x \in \mathcal{K}_1 \cap \partial\Omega_4$ , it is clear  $\|x\| = R_1$  and

$$\sigma_1 R_1 \leq x(t) \leq R_1, \text{ for all } t \in \mathbb{R}.$$

Then

$$\begin{aligned} (\Phi x)(t) &= \int_0^T G_1(t, s) \left( \frac{g(s)}{x^\mu} - \frac{h(s)}{x^\lambda} + N^2 x + p(s) \right) ds \\ &= \int_0^T G_1(t, s) \left( \frac{g^+(s)}{x^\mu} - \frac{g^-(s)}{x^\mu} - \frac{h^+(s)}{x^\lambda} + \frac{h^-(s)}{x^\lambda} + N^2 x + p^+(s) - p^-(s) \right) ds \\ &\geq \int_0^T G_1(t, s) \frac{g^+(s)}{x^\mu} ds \\ &\geq \frac{\mathcal{A}_1 T \overline{g^+}}{R_1^\mu} = R_1, \end{aligned} \quad (3.14)$$

since  $R_1 = (\mathcal{A}_1 T \overline{g^+})^{\frac{1}{1+\mu}}$  from definition of  $R_1$ . Hence, (3.13) is satisfied. The proof is finished.

By Theorems 3.2 and 3.3, we can also come to the following conclusion.

**Theorem 3.4.** *Assume that there exist four constants  $N > 0, a, b, c \in (0, 1)$  and  $a + b + c = 1$  such that*

$$(\mathcal{A}_2 T \overline{g^-})^{\frac{1}{1+\mu}} > \left( \frac{\sigma_2^{1+\lambda} \overline{h^-} - \overline{h^+}}{(\overline{g^-} - \sigma_2^{1+\mu} \overline{g^+}) \sigma_2^{\lambda-\mu}} \right)^{\frac{1}{\lambda-\mu}} > \frac{1}{\sigma_2} \max \left\{ \left( \frac{\|g^+\|_\infty}{aN^2} \right)^{\frac{1}{1+\mu}}, \left( \frac{\|h^-\|_\infty}{bN^2} \right)^{\frac{1}{1+\lambda}}, \frac{\|p^+\|_\infty}{cN^2} \right\},$$

$\overline{h^+} < \sigma_2^{1+\lambda} \overline{h^-}$  and  $\overline{p^-} \leq \sigma_2 \overline{p^+}$ . If  $\mu < \lambda$ , then equation (1.1) admits at least one positive  $T$ -periodic solution.

□

## Declarations

The authors declare that they have no conflict of interest concerning the publication of this manuscript.

## Data Availability Statements

The data sets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

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