Norms of Pseudo-differential Operator related to coupled fractional Fourier Transform

Abstract

In this paper, we introduce the characterization of norms of the coupled fractional Fourier transform (CFrFT). Few results on norms of pseudo-differential operators (P.D.O) connected with CFrFT are investigated. We conclude the manuscript by applying some of axioms to obtain the proof of the theorems.

1 Introduction and motivation

The well-known Fourier transform of a function $f \in L_1(\mathbb{R})$, represented by $\widehat{f}$, is described as

$$\widehat{f}(\eta) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\eta\zeta} f(\zeta) d\zeta$$

so that its inverse is given by

$$f(\zeta) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\eta\zeta} \widehat{f}(\eta) d\eta$$

provided the integrals exist.

The fractional Fourier transform, denoted by $\mathscr{F}_\theta$, which is a generalization of the Fourier transform, depends on a parameter $0 \le \theta \le \frac{\pi}{2}$, so that when $\theta = 0$, $\mathscr{F}_0$ is the identity transformation and when $\theta = \frac{\pi}{2}$, $\mathscr{F}_{\frac{\pi}{2}}$ is the standard Fourier transform. The transform has been extended to higher dimensions by taking tensor products of one-dimensional transforms. In 2018, Ahmed I. Zayed introduced a novel generalization of the fractional Fourier transform to two dimensions, which is called the coupled fractional Fourier transform [1] and is denoted by $\mathscr{F}_{\alpha,\beta}$. This transform depends on two independent angles α and β, with $0 \le \alpha,\ \beta \le \frac{\pi}{2}$, so that $\mathscr{F}_{0,0}$ is the identity transformation and $\mathscr{F}_{\frac{\pi}{2},\frac{\pi}{2}}$, is the two-dimensional Fourier transform.

More explicitly, the fractional Fourier transform can be viewed as a family of transformations, $\{\mathscr{F}_\alpha\}$ indexed by a parameter α, with $0 \ge \alpha \le 1$, such that $\mathscr{F}_0$ is the identity transformation and $\mathscr{F}_1$, is the standard Fourier transformation. That is $\mathscr{F}_0 f = f$, $\mathscr{F}_1[f] = \hat{f}$, where $\hat{f}$ is the Fourier transform of f, and, in addition, $\mathscr{F}_\alpha \mathscr{F}_\beta = \mathscr{F}_{\alpha+\beta}$. The range of the parameter does not have to be the interval $[0,\ 1]$ because this interval can be mapped by a simple substitution into the interval $[a,\ b]$, $a < b$. Because of its periodicity, the fractional Fourier transform is parameterized by an angle $0 \le \theta \le 2\pi$, where $\mathscr{F}_0$, is the identity transformation and the conventional Fourier transform is obtained when $\theta = \frac{\pi}{2}$. The fractional Fourier Transform or FrFT of a function $f \in L_1(\mathbb{R})$, is defined by [2, 3, 4, 5, 3, 3, 6, 7, 8, 9, 10, 11, 10]

$$(\mathscr{F}_\theta f)(\eta) = \widehat{f_\theta}(\eta) = \int_{\mathbb{R}} K_\theta(\zeta,\eta) f(\zeta) d\zeta \tag{1}$$

where the kernel $K_\theta(\zeta,\eta)$ is given by

$$K_\theta(\zeta,\eta) = \begin{cases} C_\theta e^{\frac{i(\zeta^2+\eta^2)\cot\theta}{2} - i\zeta\eta\csc\theta}, & \theta \neq n\pi, n \in \mathbb{Z} \\ \frac{1}{\sqrt{2\pi}} e^{-i\zeta\eta}, & \theta = \frac{\pi}{2} \\ \delta(\zeta-\eta), & \theta = 2n\pi \\ \delta(\zeta+\eta), & \theta = (2n+1)\pi \\ 1, & \theta = 0 \end{cases}$$

$C_\theta = \sqrt{\frac{1-i\cot\theta}{2\pi}}$ and studied some properties of this transform.

The corresponding inversion formula of $(\mathscr{F}_\theta \phi)(\eta)$ is defined in the following ways

$$f(\zeta) = \int_{\mathbb{R}} \overline{K_\theta(\zeta,\eta)} (\mathscr{F}_\theta f)(\eta) d\eta \tag{2}$$

$$\overline{K_\theta}(\zeta,\eta) = \overline{C_\theta} e^{\frac{-i(\zeta^2+\eta^2)\cot\theta}{2} + i\zeta\eta\csc\theta}$$

and $\overline{C_\theta} = \sqrt{\frac{1+i\cot\theta}{2\pi}} = C_{-\theta}$.
Hence, $\overline{K_\theta}(\zeta,\eta) = K_{-\theta}(\zeta,\eta)$.
It implies that the inverse of a FrFT with the parameter θ is the FrFT with the parameter $-\theta$.
The fractional Fourier transform in n-variables is defined by taking the tensor product of n copies of the one-dimensional fractional Fourier transforms [4]. That is

$$\begin{aligned} & [\mathscr{F}_{\theta_1,\theta_2,.........,\theta_n} f](\omega_1,\omega_2,.......,\omega_n) \\ = & \int_{\mathbb{R}}\int_{\mathbb{R}}....\int_{\mathbb{R}} K_{\theta_1}(t_1,\omega_1)K_{\theta_2}(t_2,\omega_2)...........K_{\theta_n}(t_n,\omega_n)f(t_1,t_2,...,t_{n-1}) \\ & \times dt_1dt_2.........dt_n, \end{aligned}$$

where $K_{\theta_i}(t_i,\omega_i),\ i=1,2,3,.........,n$, is the kernel of the one-dimensional fractional Fourier transform given by .... In particular, in two dimensions we have

$$[\mathscr{F}_{\theta_1,\theta_2} f](\omega_1,\omega_2) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\theta_1}(t_1,\omega_1)K_{\theta_2}(t_2,\omega_2)f(t_1,t_2)dt_1dt_2 \tag{3}$$

in 2020 a novel two-dimensional fractional Fourier trans- form $\mathscr{F}_{\alpha,\beta}$ that is not a tensor product of two one-dimensional fractional Fourier transforms was introduced [5], [6]. Unlike in the tensor product case, this transform does not depend on the two angles α and β separately but depends on the the sum and the difference of the two angles. This is the reason this transform is sometime called the coupled two-dimensional fractional Fourier transform (CFrFT) [**?**]. Now we introduce the definition of the two-dimensional fractional Fourier transform.

Definition 1. *We assume that* $\theta=(\theta_1,\theta_2)$, $\mathbf{x}=(x,\eta)$, $\mathbf{y}=(y,\zeta)$, $K_\theta(\mathbf{x},\mathbf{y}) = K_{\theta_1}(x,\eta).K_{\theta_2}(y,\zeta) = K_{\theta_1,\theta_2}(x,y,\eta,\zeta)$, *where* $K_{\theta_1}(x,\eta)$ *and* $\mathscr{K}_{\theta_2}(y,\zeta)$ *defined as above.*
The two-dimensional fractional Fourier transform [12, 1, 13] is defined as follows:

$$\begin{aligned} [\mathscr{F}_\theta\phi](\eta,\zeta) = [\mathscr{F}_{\theta_1,\alpha_1}\phi](\eta,\zeta) &= \int_{\mathbb{R}}\int_{\mathbb{R}} K_\theta(\mathbf{x},\mathbf{y})\phi(x,y)dxdy \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\theta_1}(x,\eta)K_{\theta_2}(y,\zeta)\phi(x,y)dxdy \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\theta_1,\theta_2}(x,y,\eta,\zeta)\phi(x,y)dxdy. \end{aligned} \tag{4}$$

The corresponding inversion formula of (4) *is defined as follows:*

$$\phi(x,y) = \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{K_{\theta_1,\theta_2}(x,y,\eta,\zeta)}[\mathscr{F}_{\theta_1,\theta_1}\phi](\eta,\zeta)d\eta d\zeta. \tag{5}$$

It is easy to observe that for $\theta_1 = \theta_2 = \frac{\pi}{2}$, the two-dimensional fractional Fourier transform $\mathscr{F}_{\theta_1,\theta_2}$ becomes a classical two-dimensional Fourier transform.

Definition 2. *A tempered distribution ϕ belongs to the Sobolev type spce $\mathscr{H}^s(\mathbb{R}\times\mathbb{R})$, and $s\in\mathbb{R}$ if its coupled fractional Fourier transform $\mathscr{F}_{\theta_1,\theta_2}\phi$ corresponding to a locally integrable function $(\mathscr{F}_{\theta_1,\theta_2}\phi)(\xi,\eta)$ over $\mathbb{R}\times\mathbb{R}$ such that*

$$^{(\theta_1,\theta_2)}\|\phi\|_s = \left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\{(1+|\xi|^2)(1+|\eta|^2)\}^{\frac{s}{2}}(\mathscr{F}_{\theta_1,\theta_2}\phi)(\xi,\eta)\right|^2 d\eta d\xi\right)^{\frac{1}{2}} < \infty \quad (6)$$

and

$$^{(0,0)}\|\phi\|_0 = \left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\phi(\xi,\eta)\right|^2 d\eta d\xi\right)^{\frac{1}{2}} < \infty. \quad (7)$$

This space is complete with respect to the norm $^{(\theta_1,\theta_2)}\|\phi\|_s$.

Definition 3. *The space $\mathscr{S}(\mathbb{R}\times\mathbb{R})$ is the collection of all complex valued infinitely differentiable functions $\phi(\xi,\eta)\in\mathbb{R}\times\mathbb{R}$ for every choice of $l_1,\ l_2,\ m_1,\ m_2\in\mathbb{N}_0$ which for*

$$\Gamma^{l_1,\ l_2}_{m_1,\ m_2}(\phi) = \sup_{(x,y)\in\mathbb{R}\times\mathbb{R}}\left|x^{l_1}y^{l_2}\frac{\partial^{m_1}}{\partial x^{m_1}}\frac{\partial^{m_2}}{\partial y^{m_2}}\phi(x,y)\right| < \infty. \quad (8)$$

The dual of $\mathscr{S}(\mathbb{R}\times\mathbb{R})$ is denoted by $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$.
If φ is a locally integrable and polynomial growth function on $\mathbb{R}\times\mathbb{R}$, then φ generates a distribution in $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$ as follows:

$$\langle\varphi,\ \phi\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(\xi,\eta)\phi(\xi,\eta)d\xi d\eta, \quad \forall\phi\in\mathscr{S}(\mathbb{R}\times\mathbb{R}). \quad (9)$$

The elements of $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$ are known as tempered distributions.

Theorem 1. *Let $K_{\theta_1,\theta_2}(x,y,\eta,\zeta)$ be the kernel of the two-dimensional fractional Fourier transform. Then, for all $\varphi(x,y)\in\mathscr{S}(\mathbb{R}\times\mathbb{R})$, we have*
(i) $D^r_{x,y}K_{\theta_1,\theta_2}(x,y,\eta,\zeta) = \{i(\eta\csc\theta_1+\zeta\csc\theta_2)\}^r K_{\theta_1,\theta_2}(x,y,\eta,\zeta)$,
(ii) $\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(x,y)D^r_{x,y}K_{\theta_1,\theta_2}(x,y,\eta,\zeta)dxdy = \int_{\mathbb{R}}\int_{\mathbb{R}}K_{\theta_1,\theta_2}(x,y,\eta,\zeta)(D'_{x,y})^r\varphi(x,y)dxdy$,
(iii) $\mathscr{F}_{\theta_1,\theta_2}\{(D'_{x,y})^r\varphi(x,y)\}(\eta,\zeta) = \{i(\eta\csc\theta_1+\zeta\csc\theta_2)\}^r(\mathscr{F}_{\theta_1,\theta_2}\varphi(x,y))(\eta,\zeta)$,
for all $r\in\mathbb{N}$, where $D_{x,y} = [\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+i(x\cot\theta_1+y\cot\theta_2)]$
and $D'_{x,y} = -[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}-i(x\cot\theta_1+y\cot\theta_2)]$.

Proof. See [12]. □

2 Symbol Classes

Let $l(s,t,u,v)$ be a complex valued function defined in [?] for $s,t,u \neq 0, v \neq 0 \in \mathbb{R}$. The function $l(s,t,u,v) \in \mathbb{C}^{\infty}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}-\{0\} \times \mathbb{R}-\{0\})$ is said to be an element of the class Λ if and only if $l(s,t,t_1u,t_2v) = l(s,t,u,v)$ for $t_1 > 0,\ t_2 > 0$, and assume also that

$$\lim_{(|s|,|t|)\to(\infty,\infty)} l(s,t,u,v) = l(\infty,\infty,u,v)$$

exists for $u \neq 0, v \neq 0 \in \mathbb{R}$ and $l(\infty,\infty,u,v)$ is a mapping $\mathbb{C}^{\infty}$-function.

Now we define $l'(s,t,u,v) = l(s,t,u,v) - l(\infty,\infty,u,v)$, and assume the estimates

$$(1+s^2+t^2)^p \left| \frac{\partial^k}{\partial s^k} \frac{\partial^l}{\partial t^l} \frac{\partial^m}{\partial u^m} \frac{\partial^n}{\partial v^n} l'(s,t,u,v) \right| \leq \mathbb{C}_{p,k,l,m,n}, \quad \forall s,\ t,\ u \neq 0,\ v \neq 0 \in \mathbb{R} \tag{10}$$

here p=1,2,3,.....,k, l, m, n are natural numbers.

Theorem 2. *(i) We get*
$|l(\infty,\infty,\xi,\zeta) - l(\infty,\infty,\delta,\eta)| \leq \mathbb{C}\big((|\xi-\delta|+|\zeta-\eta|)/\big(|\xi|+|\zeta|+|\delta|+|\eta|\big)\big)$,
$\forall \xi,\ \zeta,\ \delta,\ \eta$ *arbitray in* $\mathbb{R}-\{0\}$.
(ii) The estimates $(1+x^2csc^2\theta_1+y^2csc^2\theta_2)^p|\mathscr{F}_{\theta_1,\theta_2}(l')(x,y,\xi,\zeta)| \leq \mathbb{M}_p$
$\forall x,\ y,\ \xi \neq 0,\ \zeta \neq 0 \in \mathbb{R},\ p = 1,\ 2,\ 3,\ 4,\ 5...;$
(iii) $(1+x^2csc^2\theta_1+y^2csc^2\theta_2)^p|\mathscr{F}_{\theta_1,\theta_2}(l')(x,y,\xi,\zeta) - \mathscr{F}_{\theta_1,\theta_2}(l')(x,y,\delta,\eta)|$
$\leq \mathbb{M}_p(|\xi-\delta|+|\zeta-\eta|)(|\xi|+|\zeta|+|\delta|+|\eta|)^{-1},\quad \forall \xi,\ \zeta,\ \delta,\ \eta \in \mathbb{R}-\{0\}$,
$\forall x,y \in \mathbb{R},\ p = 1,2,...$*to* ∞ *being*
$\mathscr{F}_{\theta_1,\theta_2}(l')(x,y,\xi,\zeta) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\theta_1,\theta_2}(t,u,x,y)l'(t,u,\xi,\zeta)dtdu$,
$\forall x,y,\xi \neq 0, \zeta \neq 0 \in \mathbb{R}$ *are verified.*

Proof. (i) Similar proof of Theorem 1 (a)[14].

(ii) The proof is available in [?].

(iii) It can be easily proved from (ii), [?].

The term "pseudo-differential operators"[15, 16, 17, 18] has a fairly broad definition and covers such topics as harmonic analysis, partial differential equation, geometry, mathematical physics, microlocal analysis, time-frequency analysis, imaging, computations, and quantum mechanics. In mathematics, natural sciences, medicine, scientific computing, and engineering, current trends and novel applications are highlighted. The emphasis is on contemporary developments in different branches of engineering, mathematical sciences, the natural sciences, medicine, scientific computers.

In reality, Kohn-Nirenberg and Hörmander [19] were the ones who first introduced the pseudo-differential calculus, and later authors expanded on it, primarily in a local context, to examine local regularity and local solvability of PDEs.

Pseudo-differential operators on $\mathbb{R}_+$ are standard or conventional generalizations of partial differential operators or ordinary differential operators and singular integrals.

Many faculties, scientists, Ph.D students and researchers of other field developed the theory of pseudo-differential operators with the help of different types of integral operators like Fourier transforms (see [20, 14]), Hankel transform (see [21, 22, 23]), Fourier Bessel Transform on $\mathbb{R}_+$ (see [6, 7]), Weinstein transform (see [24]), Laguerre hypergroups (see [25]) and Jacobi differential operators (see [26]).

3 Pseudo-Differential Operator $L(x,y,D'_{x,y})$ related to $\mathscr{F}_{\theta_1,\theta_2}$

Let $l(x,y,\xi,\zeta) = l'(x,y,\xi,\zeta) + l(\infty,\infty,\xi,\zeta)$ be a symbol, and, as previously,

$$\mathscr{F}_{\theta_1,\theta_2}(l')(x,y,\xi,\zeta) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\theta_1,\theta_2}(t,u,x,y)l'(t,u,\xi,\zeta)dtdu, \quad \forall x,y,\xi \neq 0, \zeta \neq 0 \in \mathbb{R}.$$

Let us define, for any $\phi \in S^(\mathbb{R}\times\mathbb{R})$ and $x,y \in \mathbb{R}$, a function $\mu(x,y) = (L(x,y,D'_{x,y})\phi)(x,y)$, in [**?**] by

$$(L(x,y,D'_{x,y})\phi)(x,y) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\theta_1,\theta_2}(t,u,x,y)G_{\theta_1,\theta_2}(t,u)dtdu, \tag{11}$$

where the function $G_{\theta_1,\theta_2}(t,u)$ is given by

$$G_{\theta_1,\theta_2}(t,u) = l(\infty,\infty,t,u)\widehat{\phi}_{\theta_1,\theta_2}(t,u) + \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{l'}_{\theta_1,\theta_2}(t-\xi,u-\eta,t,u)\widehat{\phi}_{\theta_1,\theta_2}(\xi,\eta)d\xi d\eta. \tag{12}$$

4 The pseudo-differential operator $\mathscr{L}(x,y,D'_{x,y})$

We consider a symbol $l(x,y,\xi,\zeta)$. We introduce an operator $\mathscr{L}(x,y,D'_{x,y})$ from [**?**], of $\mathscr{S}(\mathbb{R}\times\mathbb{R})$ in $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$ by means of the formula

$$[\mathscr{L}(x,y,D'_{x,y})\phi](x,y) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\theta_1,\theta_2}(t,u,x,y)\mathscr{H}_{\theta_1,\theta_2}(t,u)dtdu,$$

where, for $\phi \in \mathscr{S}$, the function $\mathscr{H}_{\theta_1,\theta_2}(t,u)$ is defined by the relation

$$\begin{aligned}\mathscr{H}_{\theta_1,\theta_2}(t,u) &= l(\infty,\infty,t,u)\widehat{\phi}_{\theta_1,\theta_2}(t,u) \\ &+\overline{C_{\theta_1}C_{\theta_2}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t\lambda_1-\lambda_1^2)cot\theta_1+i(u\lambda_2-\lambda_2^2)cot\theta_2} \\ &\times\widehat{l'}_{\alpha_1,\theta_2}(t-\lambda_1,u-\lambda_2,t,u)\widehat{\phi}_{\theta_1,\theta_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2,\end{aligned}$$

$\forall\phi \in \mathscr{S}$ and $t \neq 0,\ u \neq 0 \in \mathbb{R}$.

Theorem 3. *Let $l(s,t,u,v)$ be a symbol, $L(s,t,D'_{s,t})$ the associated pseudo-differential operator. Then, for every $\varepsilon > 0$, there ia a semi-norm $_{\varepsilon}\|.\|$ on $L^2(\mathbb{R}^2)$, dependent of ε, such that every L^2- bounded sequence contains a subsequence convergent in $_{\varepsilon}\|.\|$, such that the inequality*

$$^{(0,0)}||L(s,t,D'_{s,t})||_0 \leq (\mathscr{W}+\varepsilon)^{(0,0)}||\phi||_0 + {}_{\varepsilon}\|\phi\|, \quad \forall\phi \in L^2(\mathbb{R}^2) \tag{13}$$

is satisfied.

Proof. In fact, let us put $m_\varepsilon(s,t,u,v) = \sqrt{\mathscr{W}^2 + \overline{l(s,t,u,v)}l(s,t,u,v)+\varepsilon}$ which is still a homogeneous symbol as we can easily see, and besides is

$$m_\varepsilon(s,t,u,v) = \overline{m_\varepsilon(s,t,u,v)}, \quad \varepsilon > 0,\ s,\ t,\ u \neq 0,\ v \neq 0 \in \mathbb{R}.$$

Let us consider the operator $M_\varepsilon(s,t,D'_{s,t})$, $\mathscr{M}_\varepsilon(s,t,D'_{s,t})$ associated with $m_\varepsilon(s,t,u,v)$ and $\mathscr{L}(s,t,D'_{s,t})$ associated with $\overline{l(s,t,u,v)}$. We have then the following □

Lemma 1. *The linear operator*

$$\mathscr{T}_\varepsilon = (\mathscr{W}^2+\varepsilon)I - \overline{\mathscr{L}}\cdot L - \mathscr{M}_\varepsilon \cdot M_\varepsilon : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$$

is a compact operator.

Proof. In fact, we have first of all the relation

$$\begin{aligned}\mathscr{M}_\varepsilon \cdot M_\varepsilon &= (\mathscr{M}_\varepsilon - M_\varepsilon)M_\varepsilon + M_\varepsilon^2 \\ &= \mathscr{T}_1 + M_\varepsilon^2,\end{aligned} \tag{14}$$

where $\mathscr{T}_1 = (\mathscr{M}_\varepsilon - M_\varepsilon)M_\varepsilon$ is a compact operator. So we arrive at the relation

$$\mathscr{T}_\varepsilon = (\mathscr{W}^2+\varepsilon)I - \overline{\mathscr{L}}\cdot L - \mathscr{T}_1 - M_\varepsilon^2.$$

On the other hand, we have the equality

$$\begin{aligned}\overline{\mathscr{M}}\cdot M &= (\overline{\mathscr{M}} - \overline{M})M + \overline{M}\cdot M \\ &= \mathscr{T}_2 + \overline{M}\cdot M\end{aligned}$$

where $\mathscr{T}_2 : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is compact, and hence we get

$$\mathscr{T}_\varepsilon = (\mathscr{W}^2 + \varepsilon)I - \overline{L} \cdot L - M_\varepsilon^2 - (\mathscr{T}_1 + \mathscr{T}_2).$$

Finally, we have $\mathscr{M}_\varepsilon \cdot M_\varepsilon - ((\mathscr{W}^2 + \varepsilon)I - (\bar{l})l(s,t,D'_{s,t})) = \mathscr{T}_3 : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is compact and hence we derrive

$$M_\varepsilon(s,t,D'_{s,t})M_\varepsilon(s,t,D'_{s,t}) = M_\varepsilon^2(s,t,D'_{s,t}) = (\mathscr{W}^2 + \varepsilon)I - (\bar{l}l)(s,t,D'_{s,t}) + \mathscr{T}_3$$

and therefore

$$\begin{aligned}
\mathscr{T}_\varepsilon &= (\mathscr{W}^2 + \varepsilon)I - \overline{L} \cdot L - (\mathscr{W}^2 + \varepsilon)I + (\bar{l}l)(s,t,D'_{s,t}) - (\mathscr{T}_1 + \mathscr{T}_2 + \mathscr{T}_3) \\
&= (\bar{l}l)(s,t,D'_{s,t}) - \overline{L} \cdot L - (\mathscr{T}_1 + \mathscr{T}_2 + \mathscr{T}_3) = \mathscr{T}_0
\end{aligned}$$

where $\mathscr{T}_0 : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is compact.
Hence, Lemma 1 is proved. □

Lemma 2. *Given arbitrary $\varepsilon > 0$, we have the relation*

$$Re(\mathscr{T}_\varepsilon \phi, \phi)_0 + \varepsilon({}^{(0,0)}\|\phi\|_0)^2 \geq -\frac{1}{4\varepsilon}({}^{(0,0)}\|\mathscr{T}_\varepsilon \phi\|_0)^2, \quad \forall\ \phi \in L^2(\mathbb{R}^2).$$

Proof. In fact, we have

$$\begin{aligned}
\left| \mathrm{Re}(\mathscr{T}_\varepsilon \phi, \phi)_0 \right| &\leq {}^{(0,0)}\|\mathscr{T}_\varepsilon \phi\|_0 {}^{(0,0)}\|\phi\|_0 \\
&= \frac{1}{2\sqrt{\varepsilon}} {}^{(0,0)}\|\mathscr{T}_\varepsilon \phi\|_0 2\sqrt{\varepsilon} {}^{(0,0)}\|\phi\|_0 \\
&\leq \varepsilon({}^{(0,0)}\|\phi\|_0)^2 + \frac{1}{4\varepsilon}({}^{(0,0)}\|\mathscr{T}_\varepsilon \phi\|_0)^2
\end{aligned}$$

and consequently

$$\left| \mathrm{Re}(\mathscr{T}_\varepsilon \phi, \phi)_0 \right| \geq -\varepsilon({}^{(0,0)}\|\phi\|_0)^2 - \frac{1}{4\varepsilon}({}^{(0,0)}\|\mathscr{T}_\varepsilon \phi\|_0)^2,$$

follows. □

Lemma 3. *We have the relation, $\forall \varepsilon > 0$*

$$({}^{(0,0)}||L(s,t,D'_{s,t})\phi||_0)^2 \leq (\mathscr{W}^2 + 2\varepsilon)({}^{(0,0)}\|\phi\|_0)^2 + \frac{1}{4\varepsilon}({}^{(0,0)}\|\mathscr{T}_\varepsilon \phi\|_0)^2, \quad \forall \phi \in L^2(\mathbb{R}^2).$$

Proof. In fact, this results from Lemma 2. We have

$$\begin{aligned}\langle \mathscr{T}_\varepsilon \phi, \phi \rangle &= (\mathscr{W}^2+\varepsilon)\left({}^{(0,0)}\|\phi\|_0\right)^2 - \left({}^{(0,0)}\|L(s,t,D'_{s,t})\|_0\right)^2 \\ &\quad -\left({}^{(0,0)}\|M_\varepsilon(s,t,D'_{s,t})\|\right)^2;\end{aligned}$$

$\langle \mathscr{T}_\varepsilon \phi, \phi \rangle$ is hence real-valued.
Hence, we deduce therefore, using Lemma 2, the estimate

$$\begin{aligned}\langle \mathscr{T}_\varepsilon \phi, \phi \rangle &= (\mathscr{W}^2+\varepsilon)\left({}^{(0,0)}\|\phi\|_0\right)^2 - \left({}^{(0,0)}\|L(s,t,D'_{s,t})\phi\|_0\right)^2 \\ &\quad -\left({}^{(0,0)}\|M_\varepsilon(s,t,D'_{s,t})\|\right)^2 \geq -\frac{1}{4\varepsilon}\left({}^{(0,0)}\|\mathscr{T}_\varepsilon\phi\|_0\right)^2\end{aligned}$$

and therefore

$$\left({}^{(0,0)}\|L\phi\|_0\right)^2 + \left({}^{(0,0)}\|M_\varepsilon\phi\|\right)^2 \leq (\mathscr{W}^2+\varepsilon)\left({}^{(0,0)}\|\phi\|_0\right)^2 + \frac{1}{4\varepsilon}\left({}^{(0,0)}\|\mathscr{T}_\varepsilon\phi\|_0\right)^2$$

and hence a fortiori (with greater reason or for a still stronger reason)

$$\left({}^{(0,0)}\|L\phi\|_0\right)^2 \leq (\mathscr{W}^2+\varepsilon)\left({}^{(0,0)}\|\phi\|_0\right)^2 + \frac{1}{4\varepsilon}\left({}^{(0,0)}\|\mathscr{T}_\varepsilon\phi\|_0\right)^2$$

which proves Lemma3.

Extracting the square root and for $\sqrt{\kappa+\tau} \leq \sqrt{\kappa}+\sqrt{\tau},\ \kappa > 0,\ \tau > 0$, we have

$${}^{(0,0)}\|L\phi\|_0 \leq (\mathscr{W}+\sqrt{2\varepsilon})^{(0,0)}\|\phi\|_0 + \frac{1}{2\sqrt{\varepsilon}}{}^{(0,0)}\|\mathscr{T}_\varepsilon\phi\|_0.$$

Theorem 3 is proved if we put ${}_\varepsilon\|\phi\| = w_\varepsilon{}^{(0,0)}\|\mathscr{T}_\varepsilon\phi\|_0$ and if we observe that $\mathscr{T}_\varepsilon$ being compact in $L^2(\mathbb{R}^2)$ the semi-norm ${}_\varepsilon\|\phi\| = w_\varepsilon{}^{(0,0)}\|\mathscr{T}_\varepsilon\phi\|_0$ satisfies the required properties. □

Theorem 4. *Let $\mathscr{H}$ be a Hilbertian space, and $L \in \mathscr{L}(\mathscr{H} : \mathscr{H})$. Let us assume that $\forall\ \varepsilon > 0$, There exists a seminorm ${}^{(\theta_1,\theta_2)\varepsilon}\|\cdot\|_s$ on $\mathscr{H}$ such that ${}^{(\theta_1,\theta_2)}\|\cdot\|_s$ is relatively compact with respect to ${}^{(\theta_1,\theta_2)\varepsilon}\|\cdot\|_s$ and such that ${}^{(\theta_1,\theta_2)\varepsilon}\|\phi\|_s \leq w^{(\theta_1,\theta_2)}\|\phi\|_s$, $\forall\ \phi \in \mathscr{H}$ and*

$${}^{(\theta_1,\theta_2)}\|L\phi\|_s \leq (W+\varepsilon)^{(\theta_1,\theta_2)}\|\phi\|_s + {}^{(\theta_1,\theta_2)\varepsilon}\|\phi\|_s, \quad \forall \phi \in \mathscr{H}.$$

Then

$$\inf\{{}^{(\theta_1,\theta_2)}\|L+\mathscr{T}\|_s : \mathscr{T} \in \mathscr{C}_c\} \leq \mathscr{W}.$$

Proof. In fact, it is sufficient to prove that for every $\varepsilon > 0$ we find a compact operator $\mathscr{T}_\varepsilon$ in $\mathscr{H}$, such that

$$^{(\theta_1,\theta_2)}\|(L - \mathscr{T}_\varepsilon)\phi\|_s \leq (\mathscr{W} + \varepsilon)^{(\theta_1,\theta_2)}\|\phi\|_s, \quad \forall\ \phi\ \in \mathscr{H}.$$

Let be $\mathscr{H}_\varepsilon \subset \mathscr{H}$; for $\phi \in \mathscr{H}_\varepsilon$ we have, $^{(\theta_1,\theta_2)\varepsilon}\|\phi\|_s \leq \varepsilon^{(\theta_1,\theta_2)}\|\phi\|_s$ and $\mathscr{H}_\varepsilon^\perp$ of dimension $\mathscr{N}_\varepsilon-$finite.
Let us put $\mathscr{P}_\varepsilon$ the orthogonal Projection on $\mathscr{H}_\varepsilon$; hence, $\mathscr{I} - \mathscr{P}_\varepsilon$ projects on a space of finite dimension and is therefore compact $\mathscr{I} - \mathscr{P}_\varepsilon : \mathscr{H} \to \mathscr{H}$.
Hence, we put $\mathscr{T}_\varepsilon = L(\mathscr{I} - \mathscr{P}_\varepsilon)$; this is obviously compact, and besides we have:

$$^{(\theta_1,\theta_2)}\|(L - \mathscr{T}_\varepsilon)\phi\|_s = {}^{(\theta_1,\theta_2)}\|(L\mathscr{P}_\varepsilon)\phi\|_s, \quad \forall\ \phi \in \mathscr{H}.$$

By the hypothesis of the theorem, we arrive at

$$^{(\theta_1,\theta_2)}\|(L - \mathscr{T}_\varepsilon)\phi\|_s \leq (\mathscr{W} + \varepsilon)^{(\theta_1,\theta_2)}\|\mathscr{P}_\varepsilon\phi\|_s + {}^{(\theta_1,\theta_2)\varepsilon}\|\mathscr{P}_\varepsilon\phi\|_s, \quad \forall\ \phi \in \mathscr{H}.$$

Being now $\mathscr{P}_\varepsilon\phi \in \mathscr{H}_\varepsilon$, we have

$$^{(\theta_1,\theta_2)\varepsilon}\|\mathscr{P}_\varepsilon\phi\|_s \leq \varepsilon \times {}^{(\theta_1,\theta_2)}\|\mathscr{P}_\varepsilon\phi\|_s \leq \varepsilon \times {}^{(\theta_1,\theta_2)}\|\phi\|_s \tag{15}$$

therefore we get,

$$^{(\theta_1,\theta_2)}\|(L - \mathscr{T}_\varepsilon)\phi\|_s \leq (\mathscr{W} + 2\varepsilon)^{(\theta_1,\theta_2)}\|\phi\|_s \tag{16}$$

This completes the proof of the Theorem 4. □

Theorem 5. *Let $l(s,t,u,v)$ be a symbol and $\mathscr{W} = max\{|l(s,t,u,v)| : |u| = |v| = 1,\ s,\ t \in \mathbb{R}\}$; let $L(s,t,D'_{s,t})$ be the associated pseudo-differential operator; let $L(s,t,D'_{s,t})$ be the associated pseudo-differential operator. Let $\mathscr{C}_c$=$\{\mathscr{T} : \mathscr{T} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a linear compact operator$\}$. Then we have the upper estimates*

$$\inf\{\|L(s,t,D'_{s,t}) + \mathscr{T}\| : \mathscr{T} \in \mathscr{C}_c\} \leq \mathscr{W}$$

$$\inf\{\|\mathscr{L}(s,t,D'_{s,t}) + \mathscr{T}\| : \mathscr{T} \in \mathscr{C}_c\} \leq \mathscr{W}.$$

.

Proof. Applying Theorems 3 and 4, Theorem 5 is proved. □

5 A few further estimations

In this section, we consider the latter applications. We shall prove here the following

Theorem 6. *Let $l(s,t,u,v)$ be a symbol defined for $s,\ t,\ u \neq 0,\ v \neq 0 \in \mathbb{R}$, $\mho$ an open set, and $\mathscr{W}_{\mho} = \max\{|l(s,t,u,v)| : s,\ t \in \mho$ and $|u| = |v| = 1\}$. Then, for every $\varepsilon > 0$ there is a constant $\Bbbk_{\varepsilon}$ such that*

$$^{(0,0)}\|L(s,t,D'_{s,t})\phi\|_0 \leq (\mathscr{W}_{\mho}+\varepsilon)^{(0,0)}\|\phi\|_0 + \Bbbk_{\varepsilon}\ ^{(\theta_1,\theta_2)}\|\phi\|_{-\frac{1}{2}}, \quad \forall\ \phi \in C_0^{\infty}(\overline{\mho}) \tag{17}$$

be satisfied.

Proof. Firstly we have to prove the following Lemma.

Lemma 4. *Let $l(s,t,u,v)$ be a symbol, $\mho$ an open subset of $\mathbb{R}$, $\mathscr{W}_{\mho} = \max\{|l(s,t,u,v)| : s,\ t \in \mho$ and $|u| = |v| = 1\}$. Then, $\forall\ \varepsilon > 0$ there is an open set $\overline{\mho} \subset \mho_{\varepsilon}$ such that the relation $\mathscr{W}_{\mho_{\varepsilon}} \leq \mathscr{W}_{\mho} + \varepsilon$ is satisfied.*

Proof. Actually, we have, for every $s_0,\ t_0 \in \mathbb{R}$, $|l(s,t,u,v) - l(s_0,t_0,u,v)| \leq \varepsilon$ if $|s - s_0| < \delta'_{\varepsilon},\ |t - t_0| < \delta''_{\varepsilon}$ and $u \neq 0,\ v \neq 0 \in \mathbb{R}$; here δ'_{ε} and δ''_{ε} are independent of s_0 and t_0 respectively. Let us consider here if $\partial\mho$ is the boundary of $\mho$, for every $s_0,\ t_0 \in \partial\mho$ the square $\{(s,t) : |s - s_0| < \delta'_{\varepsilon}$ and $|t - t_0| < \delta''_{\varepsilon}\}$.
Let us take

$$\mho_{\varepsilon} = \mho \cup \{\mathbb{S}(s_0,t_0,\delta'_{\varepsilon},\delta''_{\varepsilon}) : s_0,t_0 \in \partial\mho\};$$

where $\mathbb{S}(s_0,t_0,\delta'_{\varepsilon},\delta''_{\varepsilon}) = \{(s_0,t_0) : |s - s_0| \leq \delta'_{\varepsilon}$ and $|t - t_0| \leq \delta''_{\varepsilon}\}$.
Therefore, if $\upsilon,\ \vartheta \in \mho_{\varepsilon}$, we have $\upsilon,\ \vartheta \in \mho$ or $\upsilon,\ \vartheta \in \mathbb{S}(s^*,t^*,\delta'_{\varepsilon},\delta''_{\varepsilon})$ for a certain $s^*,\ t^* \in \partial\mho$. In the first case, we have

$$|l(\upsilon,\vartheta,u,v)| \leq \max\{|l(s,t,u,v)| : |u| = |v| = 1, s,t \in \overline{\mho}\} = \mathscr{W}_{\mho}.$$

In the 2^{nd} case, we get

$$|l(\upsilon,\vartheta,u,v)| \leq |l(\upsilon,\vartheta,u,v) - l(\upsilon^*,\vartheta^*,u,v)| + |l(\upsilon^*,\vartheta^*,u,v)| \leq \varepsilon + \mathscr{W}_{\mho}.$$

Hence, for every $\upsilon,\vartheta \in \mho_{\varepsilon},\ u \neq 0,\ v \neq 0 \in \mathbb{R}$ we have $|l(\upsilon,\vartheta,u,v)| \leq \varepsilon + \mathscr{W}_{\mho}$.
Thus, $\mathscr{W}_{\mho_{\varepsilon}} \leq \mathscr{W}_{\mho} + \varepsilon$. □

Proof of the Theorem 6: Given $\varepsilon > 0$, and $\phi \in C_0^{\infty}(\overline{\mho})$ we build $\mho_{\varepsilon}$ given in the Lemma 4. There exists also, a function $\Psi_{\varepsilon}(\upsilon,\vartheta) \in C_0^{\infty}(\mathbb{R}^2)$, such that

$$\Psi_{\varepsilon}(\upsilon,\vartheta) = \begin{cases} 1, & \forall\ (\upsilon,\vartheta) \in supp\phi, \\ 0, & \forall\ (\upsilon,\vartheta) \notin \mho_{\varepsilon}. \end{cases}$$

Obviously $\Psi_\varepsilon(s,t)$ is a symbol, and $\Phi_\varepsilon(s,t,u,v) = \Psi_\varepsilon(s,t)ll(s,t,u,v)$ is another symbol. Furthermore $\Phi_\varepsilon(s,t,u,v) = 0$ if $s,\ t \in \mho_\varepsilon^c$; hence we have

$$\max\{|\Phi_\varepsilon(s,t,u,v)| :\ s,\ t \in \mathbb{R}\ and\ |u| = |v| = 1\}$$
$$\leq \max\{|l(s,t,u,v)| :\ s,\ t \in \mho_\varepsilon\ and\ |u| = |v| = 1\} = \mathscr{W}_\mho \leq \mathscr{W}_{\mho_\varepsilon} + \varepsilon.$$

We define $\Upsilon_\varepsilon(s,t,D'_{s,t})$ the pseudo-differential operator associated with $\Phi_\varepsilon(s,t,u,v)$. We have

$$\Upsilon_\varepsilon(s,t,D'_{s,t}) = L(s,t,D'_{s,t})(\Psi_\varepsilon(s,t)).$$

Actually,

$$\begin{aligned} & \mathscr{F}_{\theta_1,\theta_1}[\Upsilon_\varepsilon(s,t,D'_{s,t})\phi](u,v) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\theta_1,\theta_2}(s,t,u,v)[l(s,t,u,v)\Psi_\varepsilon(s,t)]\phi(s,t)dsdt \\ = \ & \mathscr{F}_{\theta_1,\theta_1}[L(s,t,D'_{s,t})(\Psi_\varepsilon\phi)](u,v),\ \ \forall\ \phi \in \mathscr{S},\ \forall\ u \neq 0,\ v \neq 0 \in \mathbb{R}. \end{aligned}$$

Hence we get

$$\Upsilon_\varepsilon(s,t,D'_{s,t})\phi = L(s,t,D'_{s,t})(\Psi_\varepsilon(s,t)\phi(s,t)),\quad \forall\ \phi \in \mathscr{S}(\mathbb{R}^2).$$

Now we have decomposition

$$\phi(s,t) = \Psi_\varepsilon(s,t)\phi(s,t) + \{1 - \Psi_\varepsilon(s,t)\}\phi(s,t)$$

and

$$\begin{aligned} L(s,t,D'_{s,t})\phi &= L(s,t,D'_{s,t})\Psi_\varepsilon\phi + L(s,t,D'_{s,t})\{1 - \Psi_\varepsilon(s,t)\}\phi \\ &= \Upsilon_\varepsilon(s,t,D'_{s,t})\phi + L(s,t,D'_{s,t})\{1 - \Psi_\varepsilon(s,t)\}\phi, \end{aligned}$$

as it is $1 - \Psi_\varepsilon(s,t) = 0\ on\ supp\phi$, then it is $(1 - \Psi_\varepsilon(s,t))\phi(s,t) = 0\ on\ \mathbb{R}^2$, and therefore

$$L(s,t,D'_{s,t})\phi = \Upsilon_\varepsilon(s,t,D'_{s,t})\phi.$$

Hence, we obtain

$$\begin{aligned} & {}^{(0,0)}\|L(s,t,D'_{s,t})\phi\|_0 = {}^{(0,0)}\|\Upsilon_\varepsilon(s,t,D'_{s,t})\phi\|_0 \\ & \leq (\max\{|\Phi_\varepsilon(s,t,u,v)| : s,\ t \in \mathbb{R}\ and\ |u| = |v| = 1\} + \varepsilon)^{(0,0)}\|\phi\|_0 + \Bbbk_\varepsilon{}^{(\theta_1,\theta_2)}\|\phi\|_{-\frac{1}{2}} \\ & \leq (\mathscr{W}_\varepsilon + 2\varepsilon)^{(0,0)}\|\phi\|_0 + \Bbbk_\varepsilon{}^{(\theta_1,\theta_2)}\|\phi\|_{-\frac{1}{2}}. \end{aligned}$$

This completes the proof of the Throrem 6. □

Theorem 7. *Let $l(s,t,u,v)$ be a symbol, and $l(s_0,t_0,u_0,v_0) = \Bbbk_0$ for a certain $s_0,\ t_0 \in \mathbb{R}$, $|u_0| = |v_0| = 1$. Then $\forall\, \varepsilon > 0$, $\exists\, \phi_\varepsilon(s,t) \in C_0^\infty(\mathbb{R}^2)$, such that ${}^{(0,0)}\|\phi\|_0 \neq 0$ and estimates*

$$\left| {}^{(0,0)}\|L(s,t,D'_{s,t})\phi_\varepsilon\|_0 - \Bbbk_0 {}^{(0,0)}\|\phi_\varepsilon\|_0 \right| \leq \varepsilon\ {}^{(0,0)}\|\phi_\varepsilon\|_0 \tag{18}$$

$${}^{(\theta_1,\theta_2)}\|\phi_\varepsilon\|_{-1} \leq \varepsilon\ {}^{(0,0)}\|\phi_\varepsilon\|_0 \tag{19}$$

are satisfied.

Corollary to the above Theorem 7**:** Let $l(s,t,u,v)$ be a symbol such that the estimate

$${}^{(0,0)}\|\phi\|_0 \leq \Bbbk' \left({}^{(0,0)}\|L(s,t,D'_{s,t})\phi\|_0 + {}^{(\theta_1,\theta_2)}\|\phi\|_{-1} \right),\ \ \forall\, \phi \in \mathscr{S}(\mathbb{R}^2), \tag{20}$$

is satisfied.

Proof. There exists a real number $\delta > 0$, such that

$$|l(s,t,u,v)| > \delta > 0,\ \forall\, s,\ t \in \mathbb{R},\ u \neq 0,\ v \neq 0 \in \mathbb{R}.$$

In fact, otherwise we could find the sequences $\{s_n\},\ \{t_n\} \in \mathbb{R}$ and $\{u_n,\ v_n\}$ on the unit interval, such that $|l(s_n,t_n,u_n,v_n)| \leq \frac{1}{n}$, $n = 1,2,3,\ldots$. Then, $\forall\, n = 1,2,3,\ldots$, take $\phi_n(s,t) \in C_0^\infty(\mathbb{R}^2)$ corresponding to $\varepsilon_n = \frac{1}{n}$. We get

$${}^{(0,0)}\|\phi_n\|_0 \leq \Bbbk' \left({}^{(0,0)}\|L(s,t,D'_{s,t})\phi_n\|_0 + {}^{(\theta_1,\theta_2)}\|\phi_n\|_{-1} \right),\ \ \forall\, \phi \in \mathscr{S}(\mathbb{R}^2),$$

and applying (18) we deduce

$${}^{(0,0)}\|\phi_n\|_0 \leq \Bbbk' \left((|l(s_n,t_n,u_n,v_n)|^{(0,0)}\|\phi_n\|_0 + \frac{1}{n}{}^{(0,0)}\|\phi_n\|_0 + \frac{1}{n}{}^{(0,0)}\|\phi_n\|_0) \right)$$

(when (19) is also used): it follows $1 \leq 3\Bbbk'/n$, $n = 1,2,3,\ldots$, which is impossible.
We have: $\inf\{{}^{(\theta_1,\theta_2)}\|L + \mathscr{T}\|_s : \mathscr{C}_{-1}\} = w^\star < W$, there could be taken w such that $w^\star < w < W$ at least one $\mathscr{T}_n \in \mathscr{C}_{-1}$ so that

$$w^\star \leq {}^{(\theta_1,\theta_2)}\|L + \mathscr{T}_n\|_s \leq w < W$$

and therefore

$$w^\star \leq \sup\left\{ \frac{1}{{}^{(0,0)}\|\phi\|_0} {}^{(0,0)}\|(L + \mathscr{T}_n)\phi\|_0 : \phi \in L^2(\mathbb{R}^2) \right\} \leq w < W$$

whence ${}^{(0,0)}\|(L+\mathscr{T}_n)\phi\|_0 \leq w^{(0,0)}\|\phi\|_0,\ \forall\ \phi \in L^2(\mathbb{R}^2)$.
Being $w < W = \max\{|l(s,t,u,v)| : s,\ t \in \mathbb{R},\ \&\ |u| = |v| = 1\}$ we find at least two $s_0,\ t_0 \in \mathbb{R}$ and $u_0,\ v_0,\ |u_0| = |v_0| = 1$ such that $w < |l(s_0,t_0,u_0,v_0)| = \mathbb{k}'_0 < W$.
We find $\phi_\varepsilon(s,t) \in C_0^\infty(\mathbb{R}^2)$ such that

$$-\varepsilon^{(0,0)}\|\phi_\varepsilon\|_0 \leq {}^{(0,0)}\|L\phi_\varepsilon\|_0 - \mathbb{k}'_0{}^{(0,0)}\|\phi_\varepsilon\|_0$$

or

$$\begin{aligned}
(\mathbb{k}'_0 - \varepsilon)^{(0,0)}\|\phi_\varepsilon\|_0 &\leq {}^{(0,0)}\|L\phi_\varepsilon\|_0 \\
&= {}^{(0,0)}\|(L+\mathscr{T}_n)\phi_\varepsilon - \mathscr{T}_n\phi_\varepsilon\|_0 \\
&\leq {}^{(0,0)}\|(L+\mathscr{T}_n)\phi_\varepsilon\|_0 + {}^{(0,0)}\|\mathscr{T}_n\phi_\varepsilon\|_0 \\
&\leq w^{(0,0)}\|\phi_\varepsilon\|_0 + \mathbb{k}'^{(\theta_1,\theta_2)}\|\phi\|_{-1} \\
&\leq w^{(0,0)}\|\phi_\varepsilon\|_0 + \mathbb{k}' \cdot \varepsilon^{(0,0)}\|\phi_\varepsilon\|_0 \\
&= (w + \mathbb{k}'_\varepsilon)^{(0,0)}\|\phi_\varepsilon\|_0
\end{aligned}$$

and being ${}^{(0,0)}\|\phi_\varepsilon\|_0 \neq 0$ we get, $\forall\ \varepsilon$

$$\mathbb{k}'_0 - \varepsilon < w + \mathbb{k}' \cdot \varepsilon$$

and

$$w < \mathbb{k}'_0$$

we have a contradiction, as easily seen.
We pass now to the

Proof of Theorem 7: Let us take ε'; we have $|l(s,t,u,v) - l(s_0,t_0,u,v)| \leq \varepsilon'$ if $|s-s_0| < \delta'_\varepsilon$ *and* $|t-t_0| < \delta'_\varepsilon,\ u \neq 0,\ v \neq 0 \in \mathbb{R}$. We consider a function $\psi_{\varepsilon'} \in C_0^\infty$ with support contained in the square $\{(s,t) : |s-s_0| \leq \delta_{\varepsilon'}$ *and* $|t-t_0| \leq \delta_{\varepsilon'}\}$, and the sequence

$$\phi_{n,\varepsilon'}(s,t) = e^{\frac{i}{2}[-2n(uu_0 cot\theta_1 + vv_0\cot\theta_2)] + in(su_0 csc\theta_1 + tv_0\csc\theta_2)}\psi_{\varepsilon'}(s,t) \qquad (21)$$

where by hpothesis is

$$|l(s_0,t_0,u_0,v_0)| = \mathbb{k}'_0 \text{ and } |u_0| = |v_0| = 1.$$

Let be $\Psi(\lambda,\mu) \in C_0^\infty(\mathbb{R}^2)$ such that

$$\Psi(\lambda,\mu) = \begin{cases} 1, & |\lambda| \leq 1 \text{ and } |\mu| \leq 1, \\ 0, & \text{for } |\lambda| > 2 \text{ and } |\mu| > 2. \end{cases}$$

Hence we write

$$\Phi_n(u,v) = \Psi\left(\frac{u-nu_0}{\sqrt{n}}, \frac{v-nv_0}{\sqrt{n}}\right).$$

The following estimate is valid:

$$|grad\Phi_n(u,v)| \leq \frac{\mathbb{k}'}{\sqrt{n}}.$$

Let us consider now the operator $\Phi_n(D'_{s,t})$ and observe the obvious decomposition

$$\begin{aligned} L(s,t,D'_{s,t})\phi_{n,\varepsilon'} &= l(s_0,t_0,u_0,v_0)\phi_{n,\varepsilon'} + \Phi_n(D'_{s,t})\Big(L(s,t,D'_{s,t}) - l(s_0,t_0,u_0,v_0)I\Big)\phi_{n,\varepsilon'} \\ &\quad + \Big(I - \Phi_n(D'_{s,t})\Big)\Big(L(s,t,D'_{s,t}) - l(s_0,t_0,u_0,v_0)I\Big)\phi_{n,\varepsilon'} \\ &= l(s_0,t_0,u_0,v_0)\phi_{n,\varepsilon'} + \mathscr{I}_1 + \mathscr{I}_2 \quad (\text{ say }), \end{aligned}$$

where I being the identity mapping and therefore we get

$$^{(0,0)}\|L(s,t,D'_{s,t})\phi_{n,\varepsilon'}\|_0 = {}^{(0,0)}\|l(s_0,t_0,u_0,v_0)\phi_{n,\varepsilon'} + \mathscr{I}_1 + \mathscr{I}_2\|_0$$

ans hence

$$\begin{aligned} &\left|{}^{(0,0)}\|L(s,t,D'_{s,t})\phi_{n,\varepsilon'}\|_0 - \mathbb{k}_0{}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0\right| \\ &= \left|{}^{(0,0)}\|l(s_0,t_0,u_0,v_0)\phi_{n,\varepsilon'} + \mathscr{I}_1 + \mathscr{I}_2\|_0 - {}^{(0,0)}\|l(s_0,t_0,u_0,v_0)\phi_{n,\varepsilon'}\|_0\right| \\ \leq\ & \left|{}^{(0,0)}\|l(s_0,t_0,u_0,v_0)\phi_{n,\varepsilon'}\|_0 + {}^{(0,0)}\|\mathscr{I}_1 + \mathscr{I}_2\|_0 - {}^{(0,0)}\|l(s_0,t_0,u_0,v_0)\phi_{n,\varepsilon'}\|_0\right| \\ \leq\ & {}^{(0,0)}\|\mathscr{I}_1 + \mathscr{I}_2\|_0 \leq {}^{(0,0)}\|\mathscr{I}_1\|_0 + {}^{(0,0)}\|\mathscr{I}_2\|_0. \end{aligned}$$

We consider hence the expression

$$^{(0,0)}\|\mathscr{I}_1\|_0 = {}^{(0,0)}\|\Phi_n(D'_{s,t})\Big(L(s,t,D'_{s,t}) - l(s_0,t_0,u_0,v_0)I\Big)\phi_{n,\varepsilon'}\|_0$$

which is estimated by

$$\begin{aligned} &{}^{(0,0)}\|\Phi_n(D'_{s,t})\Big(L(s,t,D'_{s,t}) - L(s_0,t_0,D'_{s_0,t_0})\Big)\phi_{n,\varepsilon'}\|_0 \\ &+{}^{(0,0)}\|\Phi_n(D'_{s,t})\Big(L(s,t,D'_{s,t}) - l(s_0,t_0,u_0,v_0)I\Big)\phi_{n,\varepsilon'}\|_0 \end{aligned}$$

where

$$\mathscr{F}_{\theta_1,\theta_2}[L(s_0,t_0,D'_{s_0,t_0})\phi](u,v) = l(s_0,t_0,u,v)[\mathscr{F}_{\theta_1,\theta_2}\phi](u,v), \quad \forall\phi \in \mathscr{S}(\mathbb{R}^2).$$

Hence, we have

$$\begin{aligned}
& {}^{(0,0)}\|\Phi_n(D'_{s,t})\Big(L(s,t,D'_{s,t}) - l(s_0,t_0,u_0,v_0)I\Big)\phi_{n,\varepsilon'}\|_0 \\
= & {}^{(0,0)}\|\Phi_n(D'_{s,t})\Big(L(s,t,D'_{s,t}) - l(s_0,t_0,nu_0,nv_0)I\Big)\phi_{n,\varepsilon'}\|_0 \\
= & \Big(\int_{\mathbb{R}^2}|\Phi_n(u,v)|^2|l(s_0,t_0,u,v) - l(s_0,t_0,nu_0,nv_0)|^2 \\
& \times|[\mathscr{F}_{\theta_1,\theta_{\theta_2}}]\phi_{n,\varepsilon'}(u,v)|^2 dudv\Big)^{\frac{1}{2}}.
\end{aligned}$$

By the inequality, we have

$$\begin{aligned}
|l(s_0,t_0,u,v) - l(s_0,t_0,nu_0,nv_0)| & \leq \mathbb{k}\frac{|u-nu_0| + |v-nv_0|}{|u| + |nu_0| + |v| + |nv_{0|}} \\
& \leq \mathbb{k}\frac{|u-nu_0| + |v-nv_0|}{2n},
\end{aligned}$$

$p = 1,2,3,\ldots\ to\ \infty,\ u \neq 0,\ v \neq 0 \in \mathbb{R},\ |u_0| = |v_0| = 1,\ s,\ t \in \mathbb{R}.$
Therefore, considering too that

$$\Phi_n(u,v) = 0$$

for $|u-nu_0| > 2\sqrt{n}\ and\ |v-nv_0| > 2\sqrt{n}$, we have

$$\begin{aligned}
& {}^{(0,0)}\|\Phi_n(D'_{s,t})\Big(L(s_0,t_0,D'_{s_0,t_0}) - l(s_0,t_0,u_0,v_0)I\Big)\phi_{n,\varepsilon'}\|_0 \\
\leq & \mathbb{k}\Big(\int_{-2\sqrt{n}<u-nu_0}^{u-nu_0<2\sqrt{n}}\int_{-2\sqrt{n}<v-nv_0}^{v-nv_0<2\sqrt{n}}\frac{1}{4n^2}(|u-nu_0|^2 + |u-nu_0|^2) \\
& \times|[\mathscr{F}_{\theta_1,\theta_{\theta_2}}]\phi_{n,\varepsilon'}(u,v)|^2 dudv\Big)^{\frac{1}{2}} \\
\leq & \frac{\mathbb{k}_1}{2\sqrt{n}}{}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0.
\end{aligned} \tag{22}$$

Besides, we observe that we have also estimate

$$\begin{aligned}
& {}^{(0,0)}\|\Phi_n(D'_{s,t})(L(s,t,D'_{s,t}) - L(s_0,t_0,D'_{s_0,t_0}))\phi_{n,\varepsilon'}\|_0 \\
& \leq {}^{(0,0)}\|(L(s,t,D'_{s,t}) - L(s_0,t_0,D'_{s_0,t_0}))\phi_{n,\varepsilon'}\|_0.
\end{aligned}$$

If $m(s,t,u,v) = l(s,t,u,v) - l(s_0,t_0,u,v)$ is the symbol associated with the operator $L(s,t,D'_{s,t}) - L(s_0,t_0,D'_{s_0,t_0})$, we have $|m(s,t,u,v)| < \varepsilon'$ for $|s-s_0| < \delta_{\varepsilon'},\ |t-t_0| < \delta_{\varepsilon'},\ |u_0| = 1\ \&\ |v_0| = 1$. On the other hand, the functions $\phi_{n,\varepsilon'}$ in (21) belong to $C_0^\infty(\{(s,t) : |s-s_0| < \delta_{\varepsilon'}\ \&\ |t-t_0| < \delta_{\varepsilon'}\})$ and hence (*by Theorem* 6), we have, given $\varepsilon' > 0$, a constant $\Bbbk_{\varepsilon'}$, such that

$$^{(0,0)}\|(L(s,t,D'_{s,t}) - L(s_0,t_0,D'_{s_0,t_0}))\phi_{n,\varepsilon'}\|_0 \le (2\varepsilon')\ ^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 + \Bbbk_{\varepsilon'}{}^{(\theta_1,\theta_2)}\|\phi_{n,\varepsilon'}\|_{-1},$$

$n = 1,2,3,4,5\ldots$ to ∞.
Up to now, we have arrived at estimate

$$^{(0,0)}\|\mathscr{I}_1\|_0 \le \frac{\Bbbk}{\sqrt{n}}\ ^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 + 2\varepsilon'^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 + \Bbbk_{\varepsilon'}{}^{(\theta_1,\theta_2)}\|\phi_{n,\varepsilon'}\|_{-1},$$

$n = 1,2,3,4,5\ldots$ to ∞.
We will consider the expression for $\mathscr{I}_2$. Obviously, we have

$$\begin{aligned}\mathscr{I}_2 &= \Big(L(s,t,D'_{s,t}) - l(s_0,t_0,nu_0,nv_0)(I-\Phi_n(D'_{s,t}))\Big)\phi_{n,\varepsilon'} \\ &\quad - \Big[L(s,t,D'_{s,t}) - l(s_0,t_0,nu_0,nv_0)I,\ I-\Phi_n(D'_{s,t})\Big]\phi_{n,\varepsilon'}.\end{aligned}$$

On the other hand, we see that the considered commutator is equal to the commutator $[L(s,t,D'_{s,t}),\ \Phi_n(D'_{s,t})]$, and therefore

$$\begin{aligned}\mathscr{I}_2 &= \Big(L(s,t,D'_{s,t}) - l(s_0,t_0,nu_0,nv_0)(I-\Phi_n(D'_{s,t}))\Big)\phi_{n,\varepsilon'} \\ &\quad - \Big[L(s,t,D'_{s,t}),\ \Phi_n(D'_{s,t})\Big]\phi_{n,\varepsilon'} = \mathscr{I}_3 + \mathscr{I}_4 \quad \text{(say)}.\end{aligned}$$

Hence, first of all we have (being $|l(s_0,t_0,nu_0,nv_0)| \le \Bbbk'$) that

$$\begin{aligned}^{(0,0)}\|\mathscr{I}_3\|_0 &\le \Bbbk\ ^{(0,0)}\|(\mathscr{I}_3 - \Phi_n(D'_{s,t}))\phi_{n,\varepsilon'}\|_0 \\ &\le \Bbbk\left(\int_{\mathbb{R}}\int_{\mathbb{R}}(1-\Phi_n(u,v))|[\mathscr{F}_{\theta_1,\ \theta_2}\phi_{n,\varepsilon'}](u,v)|^2 dudv\right)^{\frac{1}{2}}\end{aligned}$$

Now we observe that we have $\Phi_n(u,v) = 1$ for $|u-nu_0| < \sqrt{n}\ \&\ |v-nv_0| < \sqrt{n}$; hence $1-\Phi_n(u,v) = 0$ for $|u-nu_0| < \sqrt{n}\ \&\ |v-nv_0| < \sqrt{n}$ and besides it is

$$\begin{aligned}[\mathscr{F}_{\theta_1,\ \theta_2}\phi_{n,\varepsilon'}](u,v) &= \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\theta_1,\ \theta_2}(s,t,u,v)e^{\frac{i}{2}[-2n(uu_0 cot\theta_1 + vv_0\cot\theta_2)]} \\ &\quad \times e^{in(su_0 csc\theta_1 + tv_0\csc\theta_2)}\psi_{\varepsilon'}(s,t)dsdt \\ &= [\mathscr{F}_{\theta_1,\ \theta_2}\psi_{\varepsilon'}](u-nu_0, v-nv_0)\end{aligned}$$

and therefore

$$
\begin{aligned}
{}^{(0,0)}\|\mathscr{I}_3\|_0 &\leq \Bbbk\left(\int_{|u-nu_0|\geq\sqrt{n}}\int_{|v-nv_0|\geq\sqrt{n}}|[\mathscr{F}_{\theta_1,\ \theta_2}\psi_{\varepsilon'}](u-nu_0,v-nv_0)|^2 du dv\right)^{\frac{1}{2}}\\
&= \Bbbk\left(\int_{|U|\geq\sqrt{n}}\int_{|V|\geq\sqrt{n}}|[\mathscr{F}_{\theta_1,\ \theta_2}\psi_{\varepsilon'}](U,V)|^2 dU dV\right)^{\frac{1}{2}},
\end{aligned}
$$

where $u-nu_0=U$ and $v-nv_0=V$, we have:

$$
\begin{aligned}
&\left(\int_{|U|\geq\sqrt{n}}\int_{|V|\geq\sqrt{n}}|[\mathscr{F}_{\theta_1,\ \theta_2}\psi_{\varepsilon'}](U,V)|^2 dU dV\right)^{\frac{1}{2}}\\
&\leq \varepsilon'\left(\int_{\mathbb{R}}\int_{\mathbb{R}}|[\mathscr{F}_{\theta_1,\ \theta_2}\psi_{\varepsilon'}](U,V)|^2 dU dV\right)^{\frac{1}{2}}\\
&= \varepsilon'\ {}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 \quad \text{if } n\geq N_0(\varepsilon',\ \mathscr{F}_{\theta_1,\ \theta_2}\psi_{\varepsilon'}).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
{}^{(0,0)}\|\mathscr{I}_4\|_0 &= {}^{(0,0)}\|\int_{\mathbb{R}}\int_{\mathbb{R}}[\mathscr{F}_{\theta_1,\ \theta_2}l'](u-\eta,v-\zeta,u,v)(\Phi_n(u,v)-\Phi_n(\eta,\zeta))\\
&\quad\times[\mathscr{F}_{\theta_1,\ \theta_2}\phi_{n,\varepsilon'}](\eta,\zeta)d\eta d\zeta\|_0.
\end{aligned}
$$

We see that

$$
\begin{aligned}
&|\Phi_n(u,v)-\Phi(\eta,\zeta)| \leq (|u-\eta|+|v-\zeta|)|grad\Phi_n(U,V)|\\
&\leq \Bbbk\frac{1}{\sqrt{n}}\left(1+|u-\eta|^2+|v-\zeta|^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\left|\int_{\mathbb{R}}\int_{\mathbb{R}}[\mathscr{F}_{\theta_1,\ \theta_2}l'](u-\eta,v-\zeta,u,v)(\Phi_n(u,v)-\Phi_n(\eta,\zeta))[\mathscr{F}_{\theta_1,\ \theta_2}\phi_{n,\varepsilon'}](\eta,\zeta)d\eta d\zeta\right|\\
&\leq \frac{\Bbbk_q}{\sqrt{n}}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(1+|u-\eta|^2+|v-\zeta|^2\right)^{-q+\frac{1}{2}}|[\mathscr{F}_{\theta_1,\ \theta_2}\phi_{n,\varepsilon'}](\eta,\zeta)|d\eta d\zeta,
\end{aligned}
$$

$\forall\ q=1,2,3,\ldots$
from where we arrive easily at estimate,

$$
{}^{(0,0)}\|\mathscr{I}_4\|_0 \leq \frac{\Bbbk}{\sqrt{n}}{}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0, \quad n=1,2,3,\ldots
$$

Adding the different inequalities obtained up to now, we have

$$\left|{}^{(0,0)}\|L(s,t,D'_{s,t})\phi_{n,\varepsilon'}\|_0 - \Bbbk_0\, {}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0\right| \le \frac{\Bbbk}{\sqrt{n}}{}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 + 2\varepsilon'^{(0,0)}\|\phi_{n,\varepsilon'}\|_0$$
$$+\Bbbk_{\varepsilon'}{}^{(\theta_1,\theta_2)}\|\phi_{n,\varepsilon'}\|_{-1} + \varepsilon'^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 + \frac{\Bbbk}{\sqrt{n}}{}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0, \quad \text{for } n \ge N_0(\varepsilon').$$

Now let us prove that for every $\varepsilon'' > 0$ there is $[\mathscr{F}_{\theta_1,\ \theta_2} n](\varepsilon'', \varepsilon')$ such that we have

$${}^{(\theta_1,\theta_2)}\|\phi_{n,\varepsilon'}\|_{-1} \le \Bbbk\varepsilon''^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 \quad \text{for } n \ge [\mathscr{F}_{\theta_1,\theta_2} n](\varepsilon'', \varepsilon').$$

In fact, we have

$$\begin{aligned}
&\left({}^{(\theta_1,\theta_2)}\|\phi_{n,\varepsilon'}\|_{-1}\right)^2 \\
= &\int\!\!\int (1+|u|^2+|v|^2)^{-1}\left|[\mathscr{F}_{\theta_1,\theta_2}\psi_{\varepsilon'}](u-nu_0, v-nv_0)\right|^2 du dv \\
= &\int_{|u-nu_0|>\rho_1}\int_{|v-nv_0|>\rho_2} \left|[\mathscr{F}_{\theta_1,\theta_2}\psi_{\varepsilon'}](u-nu_0, v-nv_0)\right|^2 du dv \\
+ &\int_{|u-nu_0|<\rho_1}\int_{|v-nv_0|<\rho_2} (1+|u|^2+|v|^2)^{-1}\left|[\mathscr{F}_{\theta_1,\theta_2}\psi_{\varepsilon'}](u-nu_0, v-nv_0)\right|^2 du dv,
\end{aligned}$$

for every $\rho_1 > 0$ & $\rho_2 > 0$.
Given now $\varepsilon'' > 0$ there is $\rho_1^*(\varepsilon', \varepsilon'')$ & $\rho_2^*(\varepsilon', \varepsilon'')$ such that

$$\int_{|U|>\rho_1^*}\int_{|V|>\rho_2^*} [\mathscr{F}_{\theta_1,\ \theta_2}\psi_{\varepsilon'}](U,V) dU dV \le \varepsilon''\left({}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0\right)^2.$$

We observe that if $|u - nu_0| < \rho_1^*$ and $|v - nv_0| < \rho_2^*$, it results $|u| \ge n - \rho_1^*$ and $|v| \ge n - \rho_2^*$ and therefore, for $n > \rho_1^* + \rho_2^* + 1$, we get

$$\begin{aligned}
&\int_{|u-nu_o|\le\rho_1^*}\int_{|v-nv_o|\le\rho_2^*} (1+|u|^2+|v|^2)^{-1}|[\mathscr{F}_{\theta_1,\theta_2}\psi_{\varepsilon'}](u-nu_o, v-nv_o)|^2 du dv \\
&\le \left(1+(n-\rho_1^*-\rho_2^*)^2\right)^{-1}\left(\int\!\!\int |[\mathscr{F}_{\theta_1,\theta_2}\psi_{\varepsilon'}](u-nu_o, v-nv_o)|^2 du dv\right) \\
&= \left(1+(n-\rho_1^*-\rho_2^*)^2\right)^{-1}\left({}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0\right)^2 \\
&\le (\varepsilon'')^2\left({}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0\right)^2, \quad \text{if } n > \max\{\rho_1^*+\rho_2^*+1,\ N_{\varepsilon'}\}
\end{aligned}$$

and therefore, for $n \ge N_1(\varepsilon', \varepsilon'')$, we get

$${}^{(\theta_1,\theta_2)}\|\phi_{n,\varepsilon'}\|_{-1} \le 2\varepsilon''^{(0,0)}\|\phi_{n,\varepsilon'}\|_0\ .$$

Hence we arrive at inequalities

$$\left| {}^{(0,0)}\|L(s,t,D'_{s,t})\phi_{n,\varepsilon'}\|_0 - \Bbbk_o{}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 \right| \le \frac{\Bbbk}{\sqrt{n}}{}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 + 2\varepsilon'^{(0,0)}\|\phi_{n,\varepsilon'}\|_0$$

for $n > N(\varepsilon', \varepsilon'')$ and ${}^{(\theta_1,\theta_2)}\|\phi_{n,\varepsilon'}\|_{-1} \le \Bbbk^{(0,0)}\|\phi_{n,\varepsilon'}\|_0$ for $n \ge N_1(\varepsilon', \varepsilon'')$.
Let us take $\varepsilon''(\varepsilon')$ small enough to have $\Bbbk\varepsilon'' < \varepsilon'$ and $2\Bbbk_{\varepsilon'}.\varepsilon'' < \varepsilon'$; hence, for $n \ge \mathcal{N}'(\varepsilon')$.
We have

$${}^{(\theta_1,\theta_2)}\|\phi_{n,\varepsilon'}\|_{-1} \le \varepsilon'^{(0,0)}\|\phi_{n,\varepsilon'}\|_0$$

and

$$\left| {}^{(0,0)}\|L(s,t,D'_{s,t})\phi_{n,\varepsilon'}\|_0 - \Bbbk_o{}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 \right| \le \frac{\Bbbk}{\sqrt{n}}{}^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 + 3\varepsilon'^{(0,0)}\|\phi_{n,\varepsilon'}\|_0$$
$$\le 4\varepsilon'^{(0,0)}\|\phi_{n,\varepsilon'}\|_0 \quad \text{if } n \ge \mathcal{N}_1'^{\,\prime}(\varepsilon').$$

Finally, given $\varepsilon > 0$, let us take $\varepsilon' < \frac{\varepsilon}{4}$ and the result is proven.
This completes the proof of the Theorem 7. □

Theorem 8. *If $l(s,t,u,v)$ is a symbol, $L(s,t,D'_{s,t})$ the associated pseudo-differential operatot, $\mathscr{C}_c = \{\mathscr{T} | \mathscr{T} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a compact operator$\}$, $\mathscr{M} = \max\{|l(s,t,u,v)| : s,\ t \in \mathbb{R}$ and $|u| = |v| = 1\}$, we get*

$$\mathscr{M} \le inf\{\|L(s,t,D'_{s,t})\| : \mathscr{T} \in \mathscr{C}_c\}. \tag{23}$$

Remark 1. *As a simple corollary of* (23) *we obtain also the estimate*

$$\mathscr{M} \le inf\{\|\mathscr{L}(s,t,D'_{s,t}) + \mathscr{T}\| : \mathscr{T} \in \mathscr{C}_c\}. \tag{24}$$

In fact, if we take an arbitrary $\mathscr{T}_o \in \mathscr{C}_c$, we obtain

$$\begin{aligned} \mathscr{L}(s,t,D'_{s,t}) + \mathscr{T}_o &= \mathscr{L}(s,t,D'_{s,t}) - L(s,t,D'_{s,t}) + L(s,t,D'_{s,t}) + \mathscr{T}_o \\ &= L(s,t,D'_{s,t}) + \mathscr{T}_1 \end{aligned}$$

where $\mathscr{T}_1 \in \mathscr{C}_c$. Consequently, applying (23)*, we obtain $\|\mathscr{L}(s,t,D'_{s,t}) + \mathscr{T}_o\| = \|L(s,t,D'_{s,t}) + \mathscr{T}_1\| \ge \mathscr{M}$. As $\mathscr{T}_o$ is arbitrary in $\mathscr{C}_c$, we get the equality*

$$inf\{\|\mathscr{L}(s,t,D'_{s,t}) + \mathscr{T}\| : \mathscr{T} \in \mathscr{C}_c\} = \mathscr{M}. \tag{25}$$

Corollary 5.1. *Combining with Theorem* 5 *we get the interesting the result*

$$\inf\{\|L(s,t,D'_{s,t}) + \mathscr{T}\| : \mathscr{T} \in \mathscr{C}_c\} = \mathscr{M}. \tag{26}$$

Proof. First of all, we have the following. □

Lemma 5. *Let $l(s,t,u,v)$ be a symbol and $\mathbb{k}_o = |l(s_o,t_o,u_o,v_o)|$ for a certain s_o, $t_o \in \mathbb{R}$ and $|u_o| = |v_o| = 1$. There is then, for every $\varepsilon > 0$ a sequence $\phi_n(s,t) \in C_o^\infty(\mho_n \times \mho'_n)$; $\mho_n \times \mho'_n = \{(s,t) : |s-s_o| \leq \frac{1}{n}$ and $|t-t_o| \leq \frac{1}{n}\}$ with ${}^{(0,0)}\|\phi_n\|_0 = 1$ and $\mathbb{k}_o - \varepsilon \leq {}^{(0,0)}\|L\phi_n\|_0$.*

Proof. As we have seen in Theorem 7, given $\varepsilon > 0$, the function ϕ_ε is obtained $= e^{in(s.u_o+t.v_o)}\psi_\varepsilon(s,t)$, where $\psi_\varepsilon \in C_o^\infty\{(s,t) : |s-s_o| < \delta_\varepsilon$ & $|t-t_o| < \delta'_\varepsilon\}$. Hence, for $n \geq n_o$ we get $\frac{1}{n} \leq \delta_\varepsilon$, and all the functions

$$\phi_{n,\varepsilon}(s,t) = e^{iq_n(su_o+tv_o)}\psi_n(s,t)$$

(with q_n big enough, fixed, dependent from $\varepsilon > 0$ and from ψ_n), verify estimate

$$(\mathbb{k}_o - \varepsilon)^{(0,0)}\|\phi_{n,\varepsilon}\|_0 \leq {}^{(0,0)}\|L(s,t,D'_{s,t})\phi_{n,\varepsilon}\|_0.$$

Dividing by ${}^{(0,0)}\|\phi_{n,\varepsilon}\|_0$, we can have the sequence of norm 1. We have

$$\mathbb{k}_o - \varepsilon \leq {}^{(0,0)}\|L(s,t,D'_{s,t})\phi_n\|_0.$$

□

Lemma 6. *We have*

$$\lim_{n\to\infty}\iint \phi_{n,\varepsilon}\psi(s,t)dsdt = 0, \quad \forall\psi \in L^2(\mathbb{R}\times\mathbb{R}).$$

Proof. In fact, we have

$$\begin{aligned}\iint \phi_{n,\varepsilon}(s,t)\psi(s,t)dsdt &= \int_{|s-s_o|>\rho}\int_{|t-t_o|>\rho}\phi_{n,\varepsilon}(s,t)\psi(s,t)dsdt \\ &\quad + \int_{|s-s_o|<\rho}\int_{|t-t_o|<\rho}\phi_{n,\varepsilon}(s,t)\psi(s,t)dsdt.\end{aligned}$$

For n big enough, $\phi_{n,\varepsilon} = 0$ when $|s-s_o| > \rho,\ |t-t_o| > \rho$ and therefore

$$\begin{aligned}\iint \phi_{n,\varepsilon}(s,t)\psi(s,t)dsdt &= \int_{|s-s_o|<\rho}\int_{|t-t_o|<\rho}\phi_{n,\varepsilon}(s,t)\psi(s,t)dsdt \\ &\leq {}^{(0,0)}\|\phi_{n,\varepsilon}\|_0\left(\int_{|s-s_o|<\rho}\int_{|t-t_o|<\rho}|\psi(s,t)|^2dsdt\right)^{\frac{1}{2}} \\ &= \left(\int_{|s-s_o|<\rho}\int_{|t-t_o|<\rho}|\psi(s,t)|^2dsdt\right)^{\frac{1}{2}}.\end{aligned}$$

Hence, given $\upsilon > 0$, we take $\rho(\upsilon)$ such that

$$\left(\int_{|s-s_o|<\rho}\int_{|t-t_o|<\rho}|\psi(s,t)|^2 dsdt\right)^{\frac{1}{2}} < \upsilon.$$

At last, we take n big enough to have $\phi_{n,\varepsilon}(s,t) = 0$ when $|s-s_o| > \frac{1}{n}$ & $|t-t_o| > \frac{1}{n}$. □

Proof of the Theorem: We assume, that

$$\inf\{\|L(s,t,D'_{s,t}) + \mathscr{T}\| : \mathscr{T} \in \mathscr{C}_c\} = m < \mathscr{M}. \tag{27}$$

Hence, take m' such that $m < m' < M$ there is at least a $\mathscr{T} \in \mathscr{C}_c$ such that $\|L(s,t,D'_{s,t}) + \mathscr{T}\| < m'$. Hence we get

$${}^{(0,0)}\|(L+\mathscr{T})\phi\|_0 \le m' \,{}^{(0,0)}\|\phi\|_0, \quad \phi \in L^2(\mathbb{R}\times\mathbb{R}).$$

Being $m' < M$, we find at least two $s_o,\ t_o \in \mathbb{R},\ u_o \neq 0,\ v_o \neq 0 \in \mathbb{R}$ and $|u_o| = |v_o| = 1$ such that $m' < |l(s_o,t_o,u_o,v_o)| = \Bbbk_o < \mathscr{M}$.
Hence, we get, for $\phi - \phi_{n,\varepsilon}$ (applying Lemma 5), that

$$\begin{aligned}(\Bbbk_o - \varepsilon) \le {}^{(0,0)}\|[L(s,t,D'_{s,t})]\phi_{n,\varepsilon}\|_0 &\le {}^{(0,0)}\|[L+\mathscr{T}]\phi_{n,\varepsilon}\|_0 + {}^{(0,0)}\|\mathscr{T}\phi_{n,\varepsilon}\|_0 \\ &\le m' + {}^{(0,0)}\|\mathscr{T}\phi_{n,\varepsilon}\|_0.\end{aligned}$$

If $n \to \infty$, $\mathscr{T}\phi_{n,\varepsilon} \to 0$ strongly in $L^2(\mathbb{R}\times\mathbb{R})$; hence $\Bbbk_o - \varepsilon \le m'$, absurd for ε small enough.
Taken then $|s_o| \le \mathscr{N}_o$, $|u_o| = 1$, such that $|l(s_o,t_o,u_o,v_o)| = \mathscr{M}_{\mathscr{N}_o}$; then $l(s,t,u,v) \in C_o^\infty(|s| \le \mathscr{N}_o \times |t| \le \mathscr{N}_o)$ and the function $\phi(s,t) \not\equiv 0$ and the sequence

$$\phi_\vartheta(s,t) = \vartheta^{\frac{n}{4}}\phi((s-s_o)\sqrt{\vartheta},(t-t_o)\sqrt{\vartheta})e^{i(s.u_o+t.v_o)\vartheta}, \quad \vartheta = 1,\ 2,\ 3,\ldots \text{to } \infty.$$

It follows $\|\phi_\vartheta\|_{L^2(\mathbb{R}^2)} = \|\phi\|_{L^2(\mathbb{R}^2)}$ and

$$\lim_{\vartheta\to\infty}\phi_\vartheta(s,t) = 0 \text{ in } L^2(\mathbb{R}^2)$$

By direct computation one gets

$$[\mathscr{L}\phi_\vartheta](s,t) = \vartheta^{\frac{n}{4}}\chi_\vartheta((s-s_o)\sqrt{\vartheta},(t-t_o)\sqrt{\vartheta})e^{i(s.u_o+t.v_o)\sqrt{\vartheta}}$$

where

$$\begin{aligned}\chi_\vartheta(s,t) &= \iint l(s_o + \frac{1}{\sqrt{\vartheta}}s,\ t_o + \frac{1}{\sqrt{\vartheta}}t,\ \vartheta u_o + \eta\sqrt{\vartheta},\ \vartheta v_o + \xi\sqrt{\vartheta})[\mathscr{F}_{\theta_1,\theta_2}\phi](\eta,\xi) \\ &\quad \times e^{i(s.\eta+t.\xi)}d\eta d\xi;\end{aligned} \tag{28}$$

it follows $\|\mathscr{L}\phi_\vartheta\|_{L^2(\mathbb{R}^2)} = \|\chi_\vartheta\|_{L^2(\mathbb{R}^2)}$; some simple estimates give also that

$$\lim_{\vartheta\to\infty} |\chi_\vartheta(s,t)|^2 = |l(s_o,\ t_o,u_o,v_o)|^2|\phi(s,t)|^2,$$

uniformly on bounded sets in $\mathbb{R}\times\mathbb{R}$.
Then apply FATOU's lemma to sequence $|\chi_\vartheta(s,t)|^2$. We get

$$\begin{aligned}\iint |l(s_o,t_o,u_o,v_o)|^2|\phi(s,t)|^2 dsdt &= |l(s_o,t_o,u_o,v_o)|^2\|\phi\|^2_{L^2(\mathbb{R}^2)} \\ &\leq \lim_{\vartheta\to\infty}\inf\|\mathscr{L}\phi_\vartheta\|^2_{L^2(\mathbb{R}^2)}.\end{aligned}$$

Take now arbitrary $\mathscr{T}\in\mathscr{C}_c$. Then it follows readily estimate

$$\|\mathscr{L}\phi_\vartheta\|^2_{L^2(\mathbb{R}^2)} \leq \left(\|\mathscr{L}+\mathscr{T}\|_{L^2(\mathbb{R}^2)}\|\phi\| + \|\mathscr{T}\phi_\vartheta\|_{L^2(\mathbb{R}^2)}\right)^2$$

and consequently

$$\lim_{\vartheta\to\infty}\inf\|\mathscr{L}\phi_\vartheta\|^2_{L^2(\mathbb{R}^2)} \leq \|\mathscr{L}+\mathscr{T}\|^2_{L^2(\mathbb{R}^2)}\|\phi\|^2$$

We obtained this way the inequality

$$|l(s_0,t_o,u_o,v_o)|^2\|\phi\|^2_{L^2(\mathbb{R}^2)} \leq \|\mathscr{L}+\mathscr{T}\|^2_{L^2(\mathbb{R}^2)}\|\phi\|^2_{L^2(\mathbb{R}^2)};$$

hence $M_{N_o} \leq \|\mathscr{L}+\mathscr{T}\|$, which gives the desired result.
This completes the proof of the Theorem 7.

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