A GEOMETRIC PERSPECTIVE ON SOME SCHRÖDER SEQUENCE SPACES

Abstract. Investigation of some structral properties of some sequence spaces constructed using Schröder numbers has recently become important. Some of the fundamental works on the investigation of the properties of these sequence spaces are given in [3, 4] and [5]. In these work we are going to investigate some geometric properties such as rotundity and uniformly smoothness of the sequence spaces $\lambda(\mathcal{S})$ where λ is c_0, ℓ_∞ or ℓ_p,and $1 \leq p < \infty$.

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1. Introduction

The properties of rotundity and smoothness are important indicators that enable us to understand the geometric structure of Banach spaces. The classical Banach spaces known to possess these properties have been listed previously. For further information, see [1]. The investigation of these types of properties in recently defined quantum sequence spaces has been presented in [6]. The investigation of the properties of some sequence spaces constructed using Schröder numbers has recently become important. Some of the fundamental works on the investigation of the properties of these sequence spaces are [3, 4] and [5]. For furthe similar works one can see [7, 8].

Schröder numbers were named after the German mathematician Ernest Schröder and there are two kinds of Schröder numbers, the large Schröder numbers and the little Schröder numbers. First few numbers of large Schröder numbers are 1, 2, 6, 22, 90, 394, 1806, 8558, ... With the recurrence relation

$$S_{n+1} = S_n + \sum_{k=0}^{n} S_k S_{n-k}, \text{ for } n \geq 0 \text{ and } S_0 = 1,$$

we can write the large Schröder sequence as (S_n). S_n also can be represent by the functional formula

$$S_n = 2 . {}_2F_1(-n+1, n+2; 2; -1),$$

where ${}_2F_1(-n+1, n+2; 2; -1)$ is known as the *hypergeometric function* and it is defined by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}.$$

Here we should specify the Pochhammer symbol $(x)_n = x(x+1)...(x+n-1)$ for $n \geq 1$, and $(x)_n = 1$ for $n = 0$. By using large Schröder sequence in [4] Cihat introduced a sub triangular matrix $\mathcal{S} = (\mathcal{S}_{nk})$ such that

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$$\mathcal{S}_{nk} = \begin{cases} \frac{S_k S_{n-k}}{S_{n+1}-S_n}, & if \;\; 0 \le k \le n \\ 0, & if \;\; k > n \end{cases}.$$

More explicitly,

$$\mathcal{S} = \begin{bmatrix} \frac{S_0S_0}{S_1-S_0} & 0 & 0 & 0 & \cdots \\ \frac{S_0S_1}{S_2-S_1} & \frac{S_1S_0}{S_2-S_1} & 0 & 0 & \cdots \\ \frac{S_0S_2}{S_3-S_2} & \frac{S_1S_1}{S_3-S_2} & \frac{S_2S_0}{S_3-S_2} & 0 & \cdots \\ \frac{S_0S_3}{S_4-S_3} & \frac{S_1S_2}{S_4-S_3} & \frac{S_2S_1}{S_4-S_3} & \frac{S_3S_0}{S_4-S_3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{2}{5} & \frac{2}{5} & 0 & 0 & \cdots \\ \frac{6}{16} & \frac{4}{16} & \frac{6}{16} & 0 & \cdots \\ \frac{22}{68} & \frac{12}{68} & \frac{12}{68} & \frac{22}{68} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

By using $\mathcal{S}$ in [4] he introduced the sequence spaces $\ell_p(\mathcal{S})$, where $1 \le p < \infty$, and $\ell_\infty(\mathcal{S})$ such that

$$\ell_p(\mathcal{S}) = \left\{ u = (u_n) \in w : \sum_{n=0}^{\infty} \left| \frac{1}{S_{n+1}-S_n} \sum_{k=0}^{n} S_k S_{n-k} u_n \right|^p < \infty \right\}$$

and

$$\ell_\infty(\mathcal{S}) = \left\{ u = (u_n) \in w : \sup_n \left| \frac{1}{S_{n+1}-S_n} \sum_{k=0}^{n} S_k S_{n-k} u_n \right| < \infty \right\}.$$

He also prove that $\ell_p(\mathcal{S})$ and $\ell_\infty(\mathcal{S})$ are BK- spaces with norms

$$\|u\|_{\ell_p(\mathcal{S})} = \left(\sum_{n=0}^{\infty} \left| \frac{1}{S_{n+1}-S_n} \sum_{k=0}^{n} S_k S_{n-k} u_n \right|^p \right)^{1/p}$$

and

$$\|u\|_{\ell_\infty(\mathcal{S})} = \sup_n \left| \frac{1}{S_{n+1}-S_n} \sum_{k=0}^{n} S_k S_{n-k} u_n \right|,$$

respectively. We can add the sequence space

$$c_0(\mathcal{S}) = \left\{ u = (u_n) \in w : \lim_n \left| \frac{1}{S_{n+1}-S_n} \sum_{k=0}^{n} S_k S_{n-k} u_n \right| = 0 \right\}$$

to these spaces which is a BK-space with the norm $\|u\|_{\ell_\infty(\mathcal{S})}$. $\mathcal{S}$ is an invertible matrix, that is $\mathcal{S}^{-1}$ exists and it is a bijection from λ to $\lambda(\mathcal{S})$ where λ is c_0, ℓ_∞ or ℓ_p. In these work we are going to investigate some geometric properties of these new sequence spaces.

Previously let us presents some preliminary definitions and results.

Definition 1. [2]. *A normed space X is* rotund *or strictly convex or strictly normed if $\|tx_1 + (1-t)x_2\| < 1$ whenever x_1 and x_2 are different points of unit sphere $S_X = \{x \in X : \|x\| = 1\}$ and $0 < t < 1$.*

An easier and more useful characterization of rotundity is the following theorem.

Theorem 1. [1] *1) Suppose that X is a normed space. Then X is rotund if and only if $\left\|\frac{1}{2}(x_1 + x_2)\right\| < 1$ whenever x_1 and x_2 are different points of S_X.*

2) A normed space is rotund if and only if each of its two-dimensional subspaces is rotund.

Definition 2. [1] *Suppose that x_0 is an element of the unit sphere S_X of a normed space X. Then x_0 is a point of smoothness of the unit ball B_X if there is no more than one support hyperplane for B_X that supports B_X at x_0. The space X is smooth if each point of S_X is a point of smoothness of B_X.*

Suppose that X is a normed space, that $x \in S_X$ and that $y \in X$. Let

$$G_-(x,y) = \lim_{t\to 0^-} \frac{\|x+ty\| - \|x\|}{t}$$

and

$$G_+(x,y) = \lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t}$$

Then $G_-(x,y)$ and $G_+(x,y)$ are, respectively, the left-hand and right-hand *Gateaux derivative* of the norm at x in the direction y. The norm is *Gateaux differentiable* at at x in the direction y if $G_-(x,y) = G_+(x,y)$, in which case the common value of $G_-(x,y)$ and $G_+(x,y)$ is denoted by $G(x,y)$ and is called the Gateaux derivative of the norm at x in the direction y. If the norm is Gateaux differentiable at x in every direction y, then the norm is Gateaux differentiable at x. Finally, if the norm is Gateaux differentiable at every point of the unit sphere S_X, then it is simply said that the norm is *Gateaux differentiable* [1].

Theorem 2. *1) A normed space is smooth if and only if its norm is Gateaux differentiable* [1].

2) A normed space is smooth if and only if each of its two-dimensional subspaces is smooth [1].

2. MAIN RESULTS

Now let us discuss some geometric properties of $\ell_p(\mathcal{S})$. Recall that the unit sphere of the n-dimensional Euclidean space is rotund. Also, we know that all $\ell_p, 1 < p < \infty$, spaces tells us that the unit spheres of them are rotund. Is it also true for $\ell_p(\mathcal{S})$ spaces?

Theorem 3. *For $1 < p < \infty$, all $\ell_p(\mathcal{S})$ spaces are rotund.*

Proof. By the Theorem 1 it is sufficient to prove the rotundity of the space *span* $\{e_1, e_2\} = Z$ in $\ell_p(\mathcal{S})$ where e_1, e_2 are elements of the unit vector basis of ℓ_p. In other words, we will think about the two-dimensional subspace

$$Z = \{(x_1, x_2, 0, 0...) : (x_1, x_2, 0, 0...)\}$$

of $\ell_p(\mathcal{S})$. Let x and y be arbitrary elements of S_Z and then $x+y = (x_1+y_1, x_2+y_2, 0, 0, \cdots)$.

$$\left\|\frac{1}{2}(x+y)\right\|^p_{\ell_p(\mathcal{S})} = \left\|\left(\left(\mathcal{S}\left(\frac{1}{2}(x+y)\right)\right)_n\right)_{n=1}^{\infty}\right\|^p_{\ell_p}$$

$$= \left\|\frac{1}{2}\left[\left((\mathcal{S}(x))_n\right)_{n=1}^{\infty} + \left((\mathcal{S}(y))_n\right)_{n=1}^{\infty}\right]\right\|^p_{\ell_p}.$$

Remember that

$$\|x\|^p_{\ell_p(\mathcal{S})} = \left\|\left((\mathcal{S}(x))_n\right)_{n=1}^{\infty}\right\|^p_{\ell_p} = 1 = \|y\|^p_{\ell_p(\mathcal{S})} = \left\|\left((\mathcal{S}(y))_n\right)_{n=1}^{\infty}\right\|^p_{\ell_p}$$

and ℓ_p is rotund. Then

$$\left\|\frac{1}{2}\left[\left((\mathcal{S}(x))_n\right)_{n=1}^{\infty} + \left((\mathcal{S}(y))_n\right)_{n=1}^{\infty}\right]\right\|^p_{\ell_p} < 1.$$

□

Theorem 4. *$c_0(\mathcal{S})$ and $\ell_\infty(\mathcal{S})$ are not rotund.*

Proof. Let us prove it only for $c_0(\mathcal{S})$. Because the proof for $\ell_\infty(\mathcal{S})$ is done in the same way. Consider two special elements

$$\begin{aligned} x &= e_1 + e_2 = (1, 1, 0, 0, ...) \\ \text{and } y &= e_1 - e_2 = (1, -1, 0, 0, ...) \end{aligned}$$

in $c_0(\mathcal{S})$ and let us see that x and $y \in S_{c_0(\mathcal{S})}$. Indeed,

$$\begin{aligned} \|y\|_{c_0(\mathcal{S})} &= \|(1, -1, 0, 0, ...)\|_{c_0(\mathcal{S})} \\ &= \|\mathcal{S}y\|_\infty \\ &= \|y\|_\infty \\ &= 1 \end{aligned}$$

and similarly $\|x\|_{c_0(\mathcal{S})} = 1$. Because the triangular infinite matrix $\mathcal{S}$ defines an isometry between c_0 and $c_0(\mathcal{S})$. Now

$$\begin{aligned} \left\|\frac{1}{2}(x+y)\right\| &= \left\|\frac{1}{2}(2, 0, 0, ...)\right\|_{c_0(\mathcal{S})} \\ &= \|(1, 0, 0, ...)\|_{c_0(\mathcal{S})} \\ &= \left\|\left(\frac{S_0 S_0}{S_1 - S_0}, 0, 0, ...\right)\right\|_\infty \\ &= \|(1, 0, 0, ...)\|_\infty \\ &= 1. \end{aligned}$$

This means $c_0(\mathcal{S})$ is not rotund. □

Uniform smoothness of a Banach spaces is an indication that the geometry of the balls in the space does not contain sharp edges or cliffs. First let us give the definition.

Definition 3. [1]*Suppose that X is a normed- space. Define a function $\rho_X : (0, \infty) \to [0, \infty)$ by the formula*

$$\rho_X(t) = \sup\left\{\frac{1}{2}(\|x + ty\| + \|x - ty\|) - 1 : x, y \in S_X\right\}$$

if $X \neq \{0\}$, and by the formula

$$\rho_X(t) = \begin{cases} 0 & \text{if } 0 < t < 1 \\ t - 1 & \text{if } t \geq 1 \end{cases}$$

if $X = \{0\}$. Then ρ_X is the modulus of smoothness *of X. The space X is* uniformly smooth *if $\lim_{t \to 0^+} \rho_X(t)/t = 0$.*

Remark 1. *The condition $\lim_{t \to 0^+} \rho_X(t)/t = 0$ includes that the norm of the space is uniformly Gateaux differentiable, which includes the Frechet differentiability of the norm function at every point in every direction. Therefore, uniformly smooth spaces are smooth, but the reverse is not true.*

Let us now see that mostly $\ell_p(\mathcal{S})$ spaces have this nice property.

Theorem 5. *For $1 < p < \infty$, the space $\ell_p(\mathcal{S})$ is uniformly smooth.*

Proof. Let us first remember that

$$\|x+ty\|_{\ell_p(\mathcal{S})} = \|\mathcal{S}\,(x+ty)\|_{\ell_p}$$

and later we should calculate we $\lim_{t\to 0^+} \rho_X\,(t)\,/t$. But

$$\lim_{t\to 0^+} \rho_X\,(t)\,/t$$

gives the $0/0$ uncertainty in the first stage. We can use the L'Hospital rule here and so

$$\lim_{t\to 0^+} \rho_X\,(t)\,/t = \lim_{t\to 0^+} \frac{d}{dt}\left(\rho_X\,(t)\right).$$

Let us now compute $\frac{d}{dt}\left(\rho_X\,(t)\right)$. By the definition of modulus of smoothness we get

$$\begin{aligned} & \frac{d}{dt}\left(\rho_X\,(t)\right) \\ = \quad & \sup\left\{\frac{1}{2}\left(\frac{d}{dt}\|x+ty\| + \frac{d}{dt}\|x-ty\|\right) : x,y \in S_{\ell_p(\mathcal{S})}\right\}. \end{aligned}$$

Now

$$\begin{aligned} & \frac{d}{dt}\left(\|x+ty\|\right) = \frac{d}{dt}\left(\|\mathcal{S}\,(x+ty)\|_{l_p}\right) \\ = \quad & \frac{d}{dt}\left(\sum_{n=1}^{\infty}\left|\left(\mathcal{S}\,(x+ty)\right)_n\right|^p\right)^{1/p} \\ = \quad & \frac{1}{p}\left(\sum_{n=1}^{\infty}\left|\left(\mathcal{S}\,(x+ty)\right)_n\right|^p\right)^{1-p} \frac{d}{dt}\sum_{n=1}^{\infty}\left|\left(\mathcal{S}\,(x+ty)\right)_n\right|^p \\ = \quad & \frac{1}{p}\left(\sum_{n=1}^{\infty}\left|\left(\mathcal{S}\,(x+ty)\right)_n\right|^p\right)^{1-p} \sum_{n=1}^{\infty}\frac{d}{dt}\left|\left(\mathcal{S}\,(x+ty)\right)_n\right|^p. \end{aligned}$$

and similarly

$$\begin{aligned} & \frac{d}{dt}\left(\|x-ty\|\right) \\ = \quad & \frac{1}{p}\left(\sum_{n=1}^{\infty}\left|\left(\mathcal{S}\,(x-ty)\right)_n\right|^p\right)^{1-p} \sum_{n=1}^{\infty}\frac{d}{dt}\left|\left(\mathcal{S}\,(x-ty)\right)_n\right|^p. \end{aligned}$$

In particular,

$$\frac{d}{dt}\left|\left(\mathcal{S}\,(x+ty)\right)_n\right|^p = p\left|\left(\mathcal{S}\,(x+ty)\right)_n\right|^{p-1}\frac{d}{dt}\left|\left(\mathcal{S}\,(x+ty)\right)_n\right|$$

and now

$$\begin{aligned} \frac{d}{dt}\left|\left(\mathcal{S}\,(x+ty)\right)_n\right| &= \begin{cases} \frac{d}{dt}\left(\left(\mathcal{S}\,(x+ty)\right)_n\right), & \text{if } \left(\mathcal{S}\,(x+ty)\right)_n \geq 0 \\ -\frac{d}{dt}\left(\left(\mathcal{S}\,(x+ty)\right)_n\right), & \text{if } \left(\mathcal{S}\,(x+ty)\right)_n < 0 \end{cases} \\ &= \begin{cases} \left(\mathcal{S}y\right)_n, & \text{if } \left(\mathcal{S}\,(x+ty)\right)_n \geq 0 \\ -\left(\mathcal{S}y\right)_n, & \text{if } \left(\mathcal{S}\,(x+ty)\right)_n < 0 \end{cases} \end{aligned}$$

since the matrix $\mathcal{S}$ is a linear operator.

Eventually

$$\frac{d}{dt}\left|\left(\mathcal{S}\,(x+ty)\right)_n\right|^p = p\left|\left(\mathcal{S}\,(x+ty)\right)_n\right|^{p-1}\begin{cases} \left(\mathcal{S}y\right)_n, & \text{if } \left(\mathcal{S}\,(x+ty)\right)_n \geq 0 \\ -\left(\mathcal{S}y\right)_n, & \text{if } \left(\mathcal{S}\,(x+ty)\right)_n < 0 \end{cases}.$$

Now similarly we get

$$\frac{d}{dt}\left|\left(\mathcal{S}\left(x-ty\right)\right)_n\right|^p = p\left|\left(\mathcal{S}\left(x-ty\right)\right)_n\right|^{p-1}\begin{cases} -\left(\mathcal{S}y\right)_n, & \text{if } \left(\mathcal{S}\left(x+ty\right)\right)_n \geq 0 \\ \left(\mathcal{S}y\right)_n, & \text{if } \left(\mathcal{S}\left(x+ty\right)\right)_n < 0 \end{cases}$$

When we apply $t \to 0^+$ we get the following equations:

$$\begin{aligned} &\lim_{t\to 0^+}\frac{d}{dt}\left\|x+ty\right\| \\ &= \begin{cases} \frac{1}{p}\left(\sum\limits_{n=1}^{\infty}\left|\left(\mathcal{S}\left(x\right)\right)_n\right|^p\right)^{1-p}\sum\limits_{n=1}^{\infty}p\left|\left(\mathcal{S}\left(x\right)\right)_n\right|^{p-1}\left(\mathcal{S}y\right)_n, & \text{if } \left(\mathcal{S}x\right)_n \geq 0 \\ -\frac{1}{p}\left(\sum\limits_{n=1}^{\infty}\left|\left(\mathcal{S}\left(x\right)\right)_n\right|^p\right)^{1-p}\sum\limits_{n=1}^{\infty}p\left|\left(\mathcal{S}\left(x\right)\right)_n\right|^{p-1}\left(\mathcal{S}y\right), & \text{if } \left(\mathcal{S}x\right)_n < 0 \end{cases} \end{aligned}$$

and

$$\begin{aligned} &\lim_{t\to 0^+}\frac{d}{dt}\left\|x-ty\right\|^p \\ &= \begin{cases} -\frac{1}{p}\left(\sum\limits_{n=1}^{\infty}\left|\left(\mathcal{S}\left(x\right)\right)_n\right|^p\right)^{1-p}\sum\limits_{n=1}^{\infty}p\left|\left(\mathcal{S}\left(x\right)\right)_n\right|^{p-1}\left(\mathcal{S}y\right)_n, & \text{if } \left(\mathcal{S}x\right)_n \geq 0 \\ \frac{1}{p}\left(\sum\limits_{n=1}^{\infty}\left|\left(\mathcal{S}\left(x\right)\right)_n\right|^p\right)^{1-p}\sum\limits_{n=1}^{\infty}p\left|\left(\mathcal{S}\left(x\right)\right)_n\right|^{p-1}\left(\mathcal{S}y\right), & \text{if } \left(\mathcal{S}x\right)_n < 0 \end{cases} \end{aligned}$$

We just see that

$$\lim_{t\to 0^+}\frac{d}{dt}\left\|x+ty\right\| + \lim_{t\to 0^+}\frac{d}{dt}\left\|x-ty\right\| = 0.$$

Hence we get by this result that

$$\lim_{t\to 0^+}\frac{d}{dt}\left(\rho_X\left(t\right)\right) = 0$$

and the proof is now completed. □

Conclusion *From this study, we understand that for* $1 < p < \infty$ *the geometric structure of* $\ell_p\left(\mathcal{S}\right)$ *exhibits a certain degree of similarity to the geometric structure of n-dimensional Euclidean spaces. For example, the closed unit sphere of this space possesses a certain degree of roundness. The best roundness is given by* $p = 2$. *As* $p \to \infty$, *the roundness decreases and the geometry of the unit circle tends to form a corner point. This is just as it is in the* ℓ_p *spaces for* $1 < p < \infty$.

References

[1] R. E. Megginson, An Introduction to Banach Space Theory, Springer-Verlag, New York,1998.

[2] N. I. Akhiezer and M. G. Krein, Some questions in the theory of moments, Gosudarstv. Naucno-Tehn. Izdat. Ukrain., Kharkov, 1938.

[3] Dağlı MC., Matrix mappings and compact operators for Schröder sequence spaces, *Turkish Journal of Mathematics,* (2022) 46: 2304 – 2320, doi:10.55730/1300-0098.3270.

[4] Dağlı MC. A novel conservative matrix arising from Schröder numbers and its properties. *Linear and Multilinear Algebra* 2022; doi: 10.1080/03081087.2022.2061401

[5] Y.Yılmaz, A. N. Tuncer, S. Yalçın, On Structral Properties of Some Banach Space-Valued Schröder Sequence Spaces, *Symmetry* 2025, 17(7), 977; https://doi.org/10.3390/sym17070977

[6] Y. Yılmaz, A. O. Akdemir, On some topological and geometric properties of some q-Cesàro sequence spaces, *Symmetry,* 2023, 15(4), 791; https://doi.org/10.3390/sym15040791

[7] Y. Yılmaz, H. Bozkurt, On Euler Sequence Spaces, *Open Journal of Mathematical Sciences,* Vol.8, 128-136, 2024.

[8] Y. Yılmaz, S. Yalçın, On Some Properties of Banach Space-Valued Fibonacci Sequence Spaces, *Communications in Advanced Mathematical Sciences,* Volume: 7 Issue: 2, 80 - 87, 2024.