

## On Dual Hyperbolic Generalized Pierre Numbers

**Abstract.** In this paper, we introduce the concept of generalized dual hyperbolic Pierre numbers. As particular cases, we examine the dual hyperbolic Pierre numbers and their Lucas-type counterparts. Furthermore, we establish several identities and matrix representations associated with these sequences. The study also includes recurrence relations, Binet-type formulas, generating functions, Simson's formulas, and summation identities, offering a comprehensive exploration of their mathematical properties.

**Keywords.** Pierre numbers, Pierre-Lucas numbers, dual hyperbolic numbers, dual hyperbolic Pierre numbers.

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### 1. Introduction

The hypercomplex numbers systems, [14], are extensions of real numbers. Some commutative examples of hypercomplex number systems are complex numbers,

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\},$$

hyperbolic (double, split-complex) numbers, [17],

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\},$$

and dual numbers, [11],

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\},$$

Some non-commutative examples of hypercomplex number systems are quaternions, [12],

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\},$$

octonions [2] and sedenions [18]. The algebras  $\mathbb{C}$  (complex numbers),  $\mathbb{H}_{\mathbb{Q}}$  (quaternions),  $\mathbb{O}$  (octonions) and  $\mathbb{S}$  (sedenions) are real algebras obtained from the real numbers  $\mathbb{R}$  by a doubling procedure called the Cayley-Dickson Process. This doubling process can be extended beyond the sedenions to form what are known as the  $2^n$ -ions (see for example [3], [13], [15]).

Quaternions were invented by Irish mathematician W. R. Hamilton (1805-1865) [12] as an extension to the complex numbers. Hyperbolic numbers with complex coefficients are introduced by J. Cockle in 1848, [5]. H. H. Cheng and S. Thompson [4] introduced dual numbers with complex coefficients and called complex dual numbers. Akar, Yüce and Şahin [1] introduced dual hyperbolic numbers.

A dual hyperbolic number is a hyper-complex number and is defined by

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3,$$

where  $a_0, a_1, a_2$  and  $a_3$  are real numbers.

The set of all dual hyperbolic numbers are denoted by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The base elements  $\{1, j, \varepsilon, \varepsilon j\}$  of dual hyperbolic numbers satisfy the following properties (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j, \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0, j^2 = j.j = 1, \\ \varepsilon.j &= j.\varepsilon, \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0, j(\varepsilon j) = (\varepsilon j)j = \varepsilon, \end{aligned}$$

where  $\varepsilon$  denotes the pure dual unit ( $\varepsilon^2 = 0, \varepsilon \neq 0$ ),  $j$  denotes the hyperbolic unit ( $j^2 = 1$ ), and  $\varepsilon j$  denotes the dual hyperbolic unit ( $(j\varepsilon)^2 = 0$ ).

The product of two dual hyperbolic numbers  $q = a_0 + ja_1 + \varepsilon a_2 + j\varepsilon a_3$  and  $p = b_0 + jb_1 + \varepsilon b_2 + j\varepsilon b_3$  is

$$qp = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0) + \varepsilon(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + j\varepsilon(a_0b_3 + a_1b_2 + a_2b_1 + b_0a_3)$$

and addition of dual hyperbolic numbers is defined as componentwise.

The dual hyperbolic numbers form a commutative ring, real vector space and an algebra. But  $\mathbb{H}_{\mathbb{D}}$  is not field because every dual hyperbolic numbers doesn't have an inverse. For more information on the dual hyperbolic numbers, see [1].

Now let us recall the definition of generalized Pierre numbers.

A generalized Pierre sequence  $\{W_n\}_{n \geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n \geq 0}$  is defined by the fourth-order recurrence relations

$$W_n = 2W_{n-1} - W_{n-4}, \tag{1.1}$$

with the initial values  $W_0, W_1, W_2, W_3$  not all being zero. The sequence  $\{W_n\}_{n \geq 0}$  can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-3)} - W_{-(n-4)},$$

for  $n = 1, 2, 3, \dots$ . Therefore, recurrence (1.1) holds for all integer  $n$ .

The first few generalized Pierre numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Pierre numbers

$n$	$W_n$	$W_{-n}$
0	$W_0$	$W_0$
1	$W_1$	$2W_2 - W_3$
2	$W_2$	$2W_1 - W_2$
3	$W_3$	$2W_0 - W_1$
4	$2W_3 - W_0$	$4W_2 - W_0 - 2W_3$
5	$4W_3 - W_1 - 2W_0$	$4W_1 - 4W_2 + W_3$
6	$8W_3 - 2W_1 - W_2 - 4W_0$	$4W_0 - 4W_1 + W_2$
7	$15W_3 - 4W_1 - 2W_2 - 8W_0$	$W_1 - 4W_0 + 8W_2 - 4W_3$
8	$28W_3 - 8W_1 - 4W_2 - 15W_0$	$W_0 + 8W_1 - 12W_2 + 4W_3$
9	$52W_3 - 15W_1 - 8W_2 - 28W_0$	$8W_0 - 12W_1 + 6W_2 - W_3$
10	$96W_3 - 28W_1 - 15W_2 - 52W_0$	$6W_1 - 12W_0 + 15W_2 - 8W_3$
11	$177W_3 - 52W_1 - 28W_2 - 96W_0$	$6W_0 + 15W_1 - 32W_2 + 12W_3$
12	$326W_3 - 96W_1 - 52W_2 - 177W_0$	$15W_0 - 32W_1 + 24W_2 - 6W_3$
13	$600W_3 - 177W_1 - 96W_2 - 326W_0$	$24W_1 - 32W_0 + 24W_2 - 15W_3$

If we set  $W_0 = 0, W_1 = 1, W_2 = 2, W_3 = 3$  then  $\{W_n\}$  is the well-known Pierre sequence and if we set  $W_0 = 4, W_1 = 2, W_2 = 2, W_3 = 5$  then  $\{W_n\}$  is the well-known Pierre -Lucas sequence. In other words, Pierre sequence  $\{P_n\}_{n \geq 0}$  and Pierre -Lucas sequence  $\{C_n\}_{n \geq 0}$  are defined by the second-order recurrence relations

$$P_n = 2P_{n-1} - P_{n-4}, \quad P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 4, \quad n \geq 4, \tag{1.2}$$

and

$$C_n = 2C_{n-1} - C_{n-4}, \quad C_0 = 4, C_1 = 2, C_2 = 4, C_3 = 8, \quad n \geq 4. \tag{1.3}$$

The sequences  $\{P_n\}_{n \geq 0}$  and  $\{C_n\}_{n \geq 0}$  can be extended to negative subscripts by defining

$$P_{-n} = 2P_{-(n-3)} - P_{-(n-4)},$$

and

$$C_{-n} = 2C_{-(n-3)} - C_{-(n-4)},$$

for  $n = 1, 2, 3, \dots$  respectively. Therefore, recurrences (1.2) and (1.3) hold for all integer  $n$ .

We can list some important properties of generalized Pierre numbers that are needed.

- Binet formula of generalized Pierre sequence can be calculated using its characteristic equation which is given as

$$z^4 - 2z^3 + 1 = (z^3 - z^2 - z - 1)(z - 1) = 0.$$

The roots of characteristic equation are

$$\begin{aligned} \alpha &= \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3}, \\ \beta &= \frac{1 + \omega \sqrt[3]{19 + 3\sqrt{33}} + \omega^2 \sqrt[3]{19 - 3\sqrt{33}}}{3}, \\ \gamma &= \frac{1 + \omega^2 \sqrt[3]{19 + 3\sqrt{33}} + \omega \sqrt[3]{19 - 3\sqrt{33}}}{3}, \\ \delta &= 1, \end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Using these roots and the recurrence relation, Binet formula can be given as

$$\begin{aligned} W_n &= \frac{p_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} + \frac{p_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{p_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{p_4 \delta^n}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)} \quad (1.4a) \\ &= \frac{p_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)(\alpha - 1)} + \frac{p_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)(\beta - 1)} + \frac{p_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)(\gamma - 1)} + \frac{p_4}{(1 - \alpha)(1 - \beta)(1 - \gamma)} \\ &= A_1 \alpha^n + A_2 \beta^n + A_3 \gamma^n + A_4 \end{aligned}$$

where  $p_1, p_2, p_3$  and  $p_4$  are given below

$$\begin{aligned} p_1 &= W_3 - (\beta + \gamma + \delta)W_2 + (\beta\gamma + \beta\delta + \gamma\delta)W_1 - \beta\gamma\delta W_0, \\ p_2 &= W_3 - (\alpha + \gamma + \delta)W_2 + (\alpha\gamma + \alpha\delta + \gamma\delta)W_1 - \alpha\gamma\delta W_0, \\ p_3 &= W_3 - (\alpha + \beta + \delta)W_2 + (\alpha\beta + \alpha\delta + \beta\delta)W_1 - \alpha\beta\delta W_0, \\ p_4 &= W_3 - W_2 - W_1 - W_0 \end{aligned} \quad (1.5)$$

and

$$\begin{aligned} A_1 &= \frac{W_3 - (\beta + \gamma + \delta)W_2 + (\beta\gamma + \beta\delta + \gamma\delta)W_1 - \beta\gamma\delta W_0}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)}, \\ A_2 &= \frac{W_3 - (\alpha + \gamma + \delta)W_2 + (\alpha\gamma + \alpha\delta + \gamma\delta)W_1 - \alpha\gamma\delta W_0}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)}, \\ A_3 &= \frac{W_3 - (\alpha + \beta + \delta)W_2 + (\alpha\beta + \alpha\delta + \beta\delta)W_1 - \alpha\beta\delta W_0}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)}, \\ A_4 &= \frac{W_3 - W_2 - W_1 - W_0}{-2}. \end{aligned} \quad (1.6)$$

Binet formula of Pierre and Pierre Lucas sequences are

$$P_n = \frac{\alpha^{n+2}}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} + \frac{\beta^{n+2}}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{\gamma^{n+2}}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} - \frac{1}{2},$$

and

$$C_n = \alpha^n + \beta^n + \gamma^n + 1,$$

respectively.

The generating function for generalized Pierre numbers is

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 2W_0)x + (W_2 - 2W_1)x^2 + (W_3 - 2W_2)x^3}{1 - 2x + x^4}. \tag{1.7}$$

In this paper, we define the dual hyperbolic generalized Pierre numbers in the next section and give some properties of them.

Next, we give the exponential generating function of  $\sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$  of the sequence  $W_n$ .

LEMMA 1. [16, Lemma 1.4]. Suppose that  $f_{GW_n}(x) = \sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$  is the exponential generating function of the generalized Pierre sequence  $\{W_n\}$ . Then

$$\begin{aligned} \sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \frac{(\alpha W_3 - \alpha(2 - \alpha)W_2 + (-\alpha^2 + \alpha + 1)W_1 - W_0)}{2\alpha^2 + 2\alpha - 2} e^{\alpha x} \\ &+ \frac{(\beta W_3 - \beta(2 - \beta)W_2 + (-\beta^2 + \beta + 1)W_1 - W_0)}{2\beta^2 + 2\beta - 2} e^{\beta x} \\ &+ \frac{(\gamma W_3 - \gamma(2 - \gamma)W_2 + (-\gamma^2 + \gamma + 1)W_1 - W_0)}{2\gamma^2 + 2\gamma - 2} e^{\gamma x} + \left(\frac{W_3 - W_2 + W_1 - W_0}{-2}\right) e^x. \end{aligned}$$

The previous Lemma gives the following results as particular examples.

COROLLARY 2. Exponential generating function of Pierre and Pierre-Lucas numbers are

$$\begin{aligned} \mathbf{a):} \quad \sum_{n=0}^{\infty} P_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty} \left( \frac{(\alpha^2 + \alpha + 1)\alpha^n}{2(\alpha^2 + \alpha - 1)} + \frac{(\beta^2 + \beta + 1)\beta^n}{2(\beta^2 + \beta - 1)} + \frac{(\gamma^2 + \gamma + 1)\gamma^n}{2(\gamma^2 + \gamma - 1)} - \frac{1}{2} \right) \frac{x^n}{n!} \\ &= \frac{(\alpha^2 + \alpha + 1)}{2(\alpha^2 + \alpha - 1)} e^{\alpha x} + \frac{(\beta^2 + \beta + 1)}{2(\beta^2 + \beta - 1)} e^{\beta x} + \frac{(\gamma^2 + \gamma + 1)\gamma^n}{2(\gamma^2 + \gamma - 1)} e^{\gamma x} - \frac{1}{2} e^x. \\ \mathbf{b):} \quad \sum_{n=0}^{\infty} C_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty} (\alpha^n + \beta^n + \gamma^n + 1) \frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^{\gamma x} + e^x. \end{aligned}$$

Next, we give some information on published papers related to hyperbolic and Dual hyperbolic numbers in literature.

- Cockle [5] presented the hyperbolic numbers with complex coefficients.
- Akar at al [1] introduced the dual hyperbolic numbers.
- Cheng and Thompson[4] studied dual numbers with complex coefficients.

Next, we give some information related to dual hyperbolic sequences presented in literature.

- Soykan at al [7] introduced dual hyperbolic generalized Pell numbers given by

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}$$

where generalized Pell numbers are given by  $V_n = 2V_{n-1} + V_{n-2}$ ,  $V_0 = a, V_1 = b$  ( $n \geq 2$ ) with the initial values  $V_0, V_1$  not all being zero.

- Cihan et al [8] studied dual hyperbolic Fibonacci and Lucas numbers given by, respectively,

$$DHF_n = F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3},$$

$$DHL_n = L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3},$$

where Fibonacci and Lucas numbers, respectively, given by  $F_n = F_{n-1} + F_{n-2}$ ,  $F_0 = 0, F_1 = 1, L_n = L_{n-1} + L_{n-2}$ ,  $L_0 = 2, L_1 = 1$ .

- Soykan et al [20] introduced dual hyperbolic generalized Jacopsthal numbers given by

$$\widehat{J}_n = J_n + jJ_{n+1} + \varepsilon J_{n+2} + j\varepsilon J_{n+3}$$

where  $J_n = J_{n-1} + 2J_{n-2}$ ,  $J_0 = a, J_1 = b$ .

- Bród et al [6] studied dual hyperbolic generalized Balancing numbers are

$$DHB_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}$$

where  $B_n = 6B_{n-1} - B_{n-2}$ ,  $B_0 = 0, B_1 = 1$ .

- Yılmaz and Soykan [10] introduced dual hyperbolic generalized Guglielmo numbers are

$$\widehat{T}_0 = T_0 + jT_1 + \varepsilon T_2 + j\varepsilon T_3$$

where  $T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}$ ,  $T_0 = 0, T_1 = 1, T_2 = 3$ .

- Dikmen [9] introduced dual hyperbolic generalised Leonardo numbers given by

$$\widehat{l}_0 = l_0 + jl_1 + \varepsilon l_2 + j\varepsilon l_3$$

where  $l_n = 2l_{n-1} - l_{n-3}$ ,  $l_0 = 1, l_1 = 1, l_2 = 3$ .

In this paper, we define the dual hyperbolic generalized Pierre numbers in the next section and give some properties of them.

## 2. Dual Hyperbolic Generalized Pierre Numbers and their Generating Functions and Binet's Formulas

In this section, we define dual hyperbolic generalized Pierre numbers and present generating functions and Binet formulas for them. We now define dual hyperbolic generalized Pierre numbers over  $\mathbb{H}_{\mathbb{D}}$ . The  $n$ th dual hyperbolic generalized Pierre number is

$$\widehat{W}_n = W_n + jW_{n+1} + \varepsilon W_{n+2} + j\varepsilon W_{n+3}. \tag{2.1}$$

The sequence  $\{\widehat{W}_n\}_{n \geq 0}$  can be extended to negative subscripts by defining

$$\widehat{W}_{-n} = W_{-n} + jW_{-n+1} + \varepsilon W_{-n+2} + j\varepsilon W_{-n+3},$$

for  $n = 1, 2, 3, \dots$  respectively. Therefore, recurrence (2.2) holds for all integer  $n$ .

Note that

$$\begin{aligned} \widehat{W}_0 &= W_0 + jW_1 + \varepsilon W_2 + j\varepsilon W_3 = W_0 + jW_1 + \varepsilon W_2 + j\varepsilon W_3, \\ \widehat{W}_1 &= W_1 + jW_2 + \varepsilon W_3 + j\varepsilon W_4 = W_1 + j\widehat{W}_2 + \varepsilon\widehat{W}_3 + j\varepsilon(2\widehat{W}_3 - \widehat{W}_0), \\ \widehat{W}_2 &= W_2 + jW_3 + \varepsilon W_4 + j\varepsilon W_5 = W_2 + j\widehat{W}_3 + \varepsilon(2\widehat{W}_3 - \widehat{W}_0) + j\varepsilon(4\widehat{W}_3 - \widehat{W}_1 - 2\widehat{W}_0), \\ \widehat{W}_3 &= W_3 + jW_4 + \varepsilon W_5 + j\varepsilon W_6 = W_3 + j(2\widehat{W}_3 - \widehat{W}_0) + \varepsilon(4\widehat{W}_3 - \widehat{W}_1 - 2\widehat{W}_0) \\ &\quad + j\varepsilon(8\widehat{W}_3 - \widehat{W}_2 - 2\widehat{W}_1 - 4\widehat{W}_0). \end{aligned}$$

It can be easily shown that

$$\widehat{W}_n = 2\widehat{W}_{n-1} - \widehat{W}_{n-4} \tag{2.2}$$

and

$$\widehat{W}_{-n} = 2\widehat{W}_{-(n-3)} - \widehat{W}_{-(n-4)}.$$

The first few dual hyperbolic generalized Pierre numbers with positive subscript and negative subscript are given in the following Table 2.

Table 2. A few dual hyperbolic generalized Pierre numbers

$n$	$\widehat{W}_n$	$\widehat{W}_{-n}$
0	$\widehat{W}_0$	$\widehat{W}_0$
1	$\widehat{W}_1$	$2\widehat{W}_2 - \widehat{W}_3$
2	$\widehat{W}_2$	$2\widehat{W}_1 - \widehat{W}_3$
3	$\widehat{W}_3$	$2W_0 - W_1$
4	$2\widehat{W}_3 - \widehat{W}_0$	$4\widehat{W}_2 - \widehat{W}_0 - 2\widehat{W}_3$
5	$4\widehat{W}_3 - \widehat{W}_1 - 2\widehat{W}_0$	$4\widehat{W}_1 - 4\widehat{W}_2 + \widehat{W}_3$
6	$8\widehat{W}_3 - \widehat{W}_2 - 2\widehat{W}_1 - 4\widehat{W}_0$	$4\widehat{W}_0 - 4\widehat{W}_1 + \widehat{W}_2$
7	$15\widehat{W}_3 - 2\widehat{W}_2 - 4\widehat{W}_1 - 8\widehat{W}_0$	$\widehat{W}_1 - 4\widehat{W}_0 + 8\widehat{W}_2 - 4\widehat{W}_3$
8	$28\widehat{W}_3 - 4\widehat{W}_2 - 8\widehat{W}_1 - 15\widehat{W}_0$	$\widehat{W}_0 + 8\widehat{W}_1 - 12\widehat{W}_2 + 4\widehat{W}_3$
9	$52\widehat{W}_3 - 8\widehat{W}_2 - 15\widehat{W}_1 - 28\widehat{W}_0$	$8\widehat{W}_0 - 12\widehat{W}_1 + 6\widehat{W}_2 - \widehat{W}_3$
10	$96\widehat{W}_3 - 15\widehat{W}_2 - 28\widehat{W}_1 - 52\widehat{W}_0$	$6\widehat{W}_1 - 12\widehat{W}_0 + 15\widehat{W}_2 - 8\widehat{W}_3$
11	$177\widehat{W}_3 - 15\widehat{W}_2 - 28\widehat{W}_1 - 96\widehat{W}_0$	$6\widehat{W}_0 + 15\widehat{W}_1 - 32\widehat{W}_2 + 12\widehat{W}_3$
12	$326\widehat{W}_3 - 52\widehat{W}_2 - 96\widehat{W}_1 - 177\widehat{W}_0$	$15\widehat{W}_0 - 32\widehat{W}_1 + 24\widehat{W}_2 - 6\widehat{W}_3$
13	$600\widehat{W}_3 - 96\widehat{W}_2 - 177\widehat{W}_1 - 326\widehat{W}_0$	$24\widehat{W}_1 - 32\widehat{W}_0 + 24\widehat{W}_2 - 15\widehat{W}_3$

As special cases, the  $n$ th dual hyperbolic Pierre numbers and the  $n$ th dual hyperbolic Pierre Lucas numbers are given as

$$\widehat{P}_n = P_n + jP_{n+1} + \varepsilon P_{n+2} + j\varepsilon P_{n+3} \tag{2.3}$$

and

$$\widehat{C}_n = C_n + jC_{n+1} + \varepsilon C_{n+2} + j\varepsilon C_{n+3} \tag{2.4}$$

respectively. The sequences  $\{\widehat{P}_n\}_{n \geq 0}$  and  $\{\widehat{C}_n\}_{n \geq 0}$  can be extended to negative subscripts by defining

$$\widehat{P}_{-n} = 2P_{-(n-3)} - P_{-(n-4)},$$

and

$$\widehat{C}_{-n} = 2C_{-(n-3)} - C_{-(n-4)},$$

for  $n = 1, 2, 3, \dots$  respectively. Therefore, recurrence (2.3) and (2.4) holds for all integer  $n$ .

For dual hyperbolic Pierre numbers (taking  $W_n = P_n$ ,  $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 4$ .) we get

$$\widehat{P}_0 = j + 2\varepsilon + 4j\varepsilon,$$

$$\widehat{P}_1 = 2j + 4\varepsilon + 8j\varepsilon + 1,$$

$$\widehat{P}_2 = 4j + 8\varepsilon + 15j\varepsilon + 2,$$

and for dual hyperbolic Pierre Lucas numbers (taking  $W_n = C_n$ ,  $C_0 = 4, C_1 = 2, C_2 = 4, C_3 = 8$ .) we get

$$\begin{aligned}\widehat{C}_0 &= 2j + 4\varepsilon + 8j\varepsilon + 4, \\ \widehat{C}_1 &= 4j + 8\varepsilon + 12j\varepsilon + 2, \\ \widehat{C}_2 &= 8j + 12\varepsilon + 22j\varepsilon + 4.\end{aligned}$$

A few dual hyperbolic Pierre numbers and dual hyperbolic Pierre Lucas numbers with positive subscript and negative subscript are given in the following Table 3 and Table 4.

Table 3. Dual hyperbolic Pierre numbers

$n$	$\widehat{P}_n$	$\widehat{P}_{-n}$
0	$j + 2\varepsilon + 4j\varepsilon$	$j + 2\varepsilon + 4j\varepsilon$
1	$2j + 4\varepsilon + 8j\varepsilon + 1$	$\varepsilon + 2j\varepsilon$
2	$4j + 8\varepsilon + 15j\varepsilon + 2$	$j\varepsilon$
3	$8j + 15\varepsilon + 28j\varepsilon + 4$	$-1$
4	$15j + 28\varepsilon + 52j\varepsilon + 8$	$-j$
5	$28j + 52\varepsilon + 96j\varepsilon + 15$	$-\varepsilon$

Table 4. Dual hyperbolic Pierre- Lucas numbers

$n$	$\widehat{C}_n$	$\widehat{C}_{-n}$
0	$2j + 4\varepsilon + 8j\varepsilon + 4$	$2j + 4\varepsilon + 8j\varepsilon + 4$
1	$4j + 8\varepsilon + 12j\varepsilon + 2$	$4j + 4j\varepsilon + 2\varepsilon$
2	$8j + 12\varepsilon + 22j\varepsilon + 4$	$4\varepsilon + 2j\varepsilon$
3	$12j + 22\varepsilon + 40j\varepsilon + 8$	$4j\varepsilon + 6$
4	$22j + 40\varepsilon + 72j\varepsilon + 12$	$-4 + 6j$
5	$40j + 72\varepsilon + 132j\varepsilon + 22$	$-4j + 6\varepsilon$

Now, we will state Binet's formula for the dual hyperbolic generalized Pierre numbers and in the rest of the paper, we fix the following notations:

$$\widehat{\alpha} = 1 + j\alpha + \varepsilon\alpha^2 + j\varepsilon\alpha^3, \tag{2.5}$$

$$\widehat{\beta} = 1 + j\beta + \varepsilon\beta^2 + j\varepsilon\beta^3. \tag{2.6}$$

$$\widehat{\gamma} = 1 + j\gamma + \varepsilon\gamma^2 + j\varepsilon\gamma^3 \tag{2.7}$$

$$\widehat{\delta} = \widehat{1} = 1 + j + \varepsilon + j\varepsilon, \tag{2.8}$$

Note that we have the following identities:

$$\begin{aligned} \widehat{\alpha}^2 &= 1 + \alpha^2 + 2\alpha j + 2\alpha^2 (\alpha^2 + 1) \varepsilon + 4\alpha^3 j \varepsilon \\ \widehat{\beta}^2 &= 1 + \beta^2 + 2j\beta + (2\beta^4 + 2\beta^2)\varepsilon + 4j\varepsilon\beta^3, \\ \widehat{\alpha}\widehat{\beta} &= 1 + \alpha\beta + (\alpha + \beta)j + (\alpha^2 + \beta^2 + 2\alpha\beta^3 + \alpha^3\beta) \varepsilon + (\alpha + \beta) (\alpha^2 + \beta^2) j \varepsilon, \\ \widehat{\gamma}^2 &= 1 + \gamma^2 + 2j\gamma + (2\gamma^4 + 2\gamma^2)\varepsilon + 4j\varepsilon\gamma^3, \\ \widehat{\delta}^2 &= \widehat{1}^2 = 1 + 2\gamma + 4\varepsilon + 4j\varepsilon, \\ \widehat{\gamma}\widehat{\delta} &= 1 + \gamma + (\gamma^3 + \gamma)j + (\gamma^3 + \gamma^2 + \gamma + 2) \varepsilon + (\gamma^2 + \gamma + 1) j \varepsilon \end{aligned}$$

**THEOREM 3.** (Binet's Formula) For any integer  $n$ , the  $n$ th dual hyperbolic generalized Pierre number is

$$\widehat{W}_n = \widehat{\alpha}A_1\alpha^n + \widehat{\beta}A_2\beta^n + \widehat{\gamma}A_3\gamma^n + \widehat{\delta}A_4 \tag{2.9}$$

where  $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\delta}$  are given as (2.5)-(2.8)

Proof. Using Binet's formula of the generalized Pierre numbers given below

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4$$

where  $A_1, A_2, A_3, A_4$  are given as in (1.6) we get

$$\begin{aligned} \widehat{W}_n &= W_n + jW_{n+1} + \varepsilon W_{n+2} + j\varepsilon W_{n+3}, \\ &= A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4 + (A_1\alpha^{n+1} + A_2\beta^{n+1} + A_3\gamma^{n+1} + A_4)j \\ &\quad + (A_1\alpha^{n+2} + A_2\beta^{n+2} + A_3\gamma^{n+2} + A_4)\varepsilon + (A_1\alpha^{n+3} + A_2\beta^{n+3} + A_3\gamma^{n+3} + A_4)j\varepsilon. \\ &= \widehat{\alpha}A_1\alpha^n + \widehat{\beta}A_2\beta^n + \widehat{\gamma}A_3\gamma^n + \widehat{\delta}A_4. \end{aligned}$$

This proves (2.9).

As special cases, for any integer  $n$ , the Binet's Formula of  $n$ th dual hyperbolic Pierre number is

$$\widehat{P}_n = \frac{(\alpha^2 + \alpha + 1)\alpha^n\widehat{\alpha}}{2(\alpha^2 + \alpha - 1)} + \frac{(\beta^2 + \beta + 1)\beta^n\widehat{\beta}}{2(\beta^2 + \beta - 1)} + \frac{(\gamma^2 + \gamma + 1)\gamma^n\widehat{\gamma}}{2(\gamma^2 + \gamma - 1)} - \frac{\widehat{1}}{2} \tag{2.10}$$

and the Binet's Formula of  $n$ th dual hyperbolic Pierre Lucas number is

$$\widehat{C}_n = \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n + \widehat{\gamma}\gamma^n + \widehat{1}. \tag{2.11}$$

Next, we present generating function.

**THEOREM 4.** The generating function for the dual hyperbolic generalized Pierre numbers is

$$f_{\widehat{W}_n}(x) = \sum_{n=0}^{\infty} \widehat{W}_n x^n = \frac{\widehat{W}_0 + (\widehat{W}_1 - 2\widehat{W}_0)x + (\widehat{W}_2 - 2\widehat{W}_1)x^2 + (\widehat{W}_3 - 2\widehat{W}_2)x^3}{1 - 2x + x^4}.$$

*Proof.* We assume that  $f_{\widehat{W}_n}(x)$  is the generating function of the dual hyperbolic generalized Pierre numbers and then we can write

$$f_{\widehat{W}_n}(x) = \sum_{n=0}^{\infty} \widehat{W}_n x^n.$$

Then, using the definition of the dual hyperbolic generalized Pierre numbers, and subtracting  $xf(x)$  and  $x^2f(x)$  from  $f(x)$ , we obtain (note the shift in the index  $n$  in the third line)

$$\begin{aligned} (1 - 2x + x^4)f_{\widehat{W}_n}(x) &= \sum_{n=0}^{\infty} \widehat{W}_n x^n - 2x \sum_{n=0}^{\infty} \widehat{W}_n x^n + x^4 \sum_{n=0}^{\infty} \widehat{W}_n x^n \\ &= \sum_{n=0}^{\infty} \widehat{W}_n x^n - 2 \sum_{n=0}^{\infty} \widehat{W}_n x^{n+1} + \sum_{n=0}^{\infty} \widehat{W}_n x^{n+4} \\ &= \sum_{n=0}^{\infty} \widehat{W}_n x^n - 2 \sum_{n=1}^{\infty} \widehat{W}_{(n-1)} x^n + \sum_{n=4}^{\infty} \widehat{W}_{(n-4)} x^n \\ &= (\widehat{W}_0 + \widehat{W}_1 x + \widehat{W}_2 x^2 + \widehat{W}_3 x^3) - 2(\widehat{W}_0 x + \widehat{W}_1 x^2 + \widehat{W}_2 x^3) \\ &\quad + \sum_{n=4}^{\infty} (\widehat{W}_n - 2\widehat{W}_{n-1} + \widehat{W}_{n-4}) x^n \\ &= \widehat{W}_0 + (\widehat{W}_1 - 2\widehat{W}_0)x + (\widehat{W}_2 - 2\widehat{W}_1)x^2 + (\widehat{W}_3 - 2\widehat{W}_2)x^3. \end{aligned}$$

As special cases, the generating functions for the dual hyperbolic Pierre and dual hyperbolic Pierre Lucas numbers are

$$\sum_{n=0}^{\infty} \widehat{P}_n x^n = \frac{(j + 2\varepsilon + 4j\varepsilon) + (1)x + (-j\varepsilon)x^2 + (-\varepsilon - 2j\varepsilon)x^3}{1 - 2x + x^4}$$

and

$$\sum_{n=0}^{\infty} \widehat{C}_n x^n = \frac{(2j + 4\varepsilon + 8j\varepsilon + 4) + (-6 - 4j\varepsilon)x + (-4\varepsilon - 2j\varepsilon)x^2 + (-4j - 2\varepsilon - 4j\varepsilon)x^3}{1 - 2x + x^4}$$

respectively.

Next, we give the exponential generating function of  $\sum_{n=0}^{\infty} \widehat{W}_n \frac{x^n}{n!}$  of the sequence  $\widehat{W}_n$ .

LEMMA 5. *Suppose that  $f_{\widehat{W}_n}(x) = \sum_{n=0}^{\infty} \widehat{W}_n \frac{x^n}{n!}$  is the exponential dual hyperbolic generating function of the generalized Pierre sequence  $\{\widehat{W}_n\}$ .*

Then  $\sum_{n=0}^{\infty} \widehat{W}_n \frac{x^n}{n!}$  is given by

$$\sum_{n=0}^{\infty} \widehat{W}_n \frac{x^n}{n!} = A_1 e^{\alpha x} \widehat{\alpha} + A_2 e^{\beta x} \widehat{\beta} + A_3 e^{\gamma x} \widehat{\gamma} + A_4 e^{x\widehat{1}}.$$

where  $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\delta}$  are given as (2.5)-(2.8)

*Proof.* Using Binet's formula

$$W_n = A_1 \alpha^n + A_2 \beta^n + A_3 \gamma^n + A_4.$$

where  $A_1, A_2, A_3, A_4$  are given as in (1.6) we get

$$\begin{aligned}
 \sum_{n=0}^{\infty} \widehat{W}_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty} W_n \frac{x^n}{n!} + j \sum_{n=0}^{\infty} W_{n+1} \frac{x^n}{n!} + \varepsilon \sum_{n=0}^{\infty} W_{n+2} \frac{x^n}{n!} + j\varepsilon \sum_{n=0}^{\infty} W_{n+3} \frac{x^n}{n!} \\
 &= \sum_{n=0}^{\infty} (A_1 \alpha^n + A_2 \beta^n + A_3 \gamma^n + A_4) \frac{x^n}{n!} + j \sum_{n=0}^{\infty} (A_1 \alpha^{n+1} + A_2 \beta^{n+1} + A_3 \gamma^{n+1} + A_4) \frac{x^n}{n!} \\
 &\quad + \varepsilon \sum_{n=0}^{\infty} (A_1 \alpha^{n+2} + A_2 \beta^{n+2} + A_3 \gamma^{n+2} + A_4) \frac{x^n}{n!} \\
 &\quad + j\varepsilon \sum_{n=0}^{\infty} (A_1 \alpha^{n+3} + A_2 \beta^{n+3} + A_3 \gamma^{n+3} + A_4) \frac{x^n}{n!} \\
 &= (A_1 e^{\alpha x} + A_2 e^{\beta x} + A_3 e^{\gamma x} + A_4 e^x) + j(A_1 \alpha e^{\alpha x} + A_2 \beta e^{\beta x} + A_3 \gamma e^{\gamma x} + A_4 e^x) \\
 &\quad + \varepsilon(A_1 \alpha^2 e^{\alpha x} + A_2 \beta^2 e^{\beta x} + A_3 \gamma^2 e^{\gamma x} + A_4 e^x) + j\varepsilon(A_1 \alpha^3 e^{\alpha x} + A_2 \beta^3 e^{\beta x} + A_3 \gamma^3 e^{\gamma x} + A_4 e^x) \\
 &= A_1 e^{\alpha x} (1 + j\alpha + \varepsilon\alpha^2 + j\varepsilon\alpha^3) + A_2 e^{\beta x} (1 + j\beta + \varepsilon\beta^2 + j\varepsilon\beta^3) \\
 &\quad + A_3 e^{\gamma x} (1 + j\gamma + \varepsilon\gamma^2 + j\varepsilon\gamma^3) + A_4 e^x (1 + j + \varepsilon + j\varepsilon) \\
 &= A_1 e^{\alpha x} \widehat{\alpha} + A_2 e^{\beta x} \widehat{\beta} + A_3 e^{\gamma x} \widehat{\gamma} + A_4 e^x \widehat{1}
 \end{aligned}$$

This proves (5).  $\square$

The previous Lemma gives the following results as particular examples.

**COROLLARY 6.** *Exponential generating function of dual hiperbolic Pierre and dual hiperbolic Pierre-Lucas numbers are*

**a):**

$$\begin{aligned}
 \sum_{n=0}^{\infty} \widehat{P}_n \frac{x^n}{n!} &= \left( \frac{(\alpha^2 + \alpha + 1)}{2(\alpha^2 + \alpha - 1)} e^{\alpha x} + \frac{(\beta^2 + \beta + 1)}{2(\beta^2 + \beta - 1)} e^{\beta x} + \frac{(\gamma^2 + \gamma + 1)}{2(\gamma^2 + \gamma - 1)} e^{\gamma x} - \frac{1}{2} e^x \right) \\
 &\quad + j \left( \frac{\alpha(\alpha^2 + \alpha + 1)}{2(\alpha^2 + \alpha - 1)} e^{\alpha x} + \frac{\beta(\beta^2 + \beta + 1)}{2(\beta^2 + \beta - 1)} e^{\beta x} + \frac{\gamma(\gamma^2 + \gamma + 1)}{2(\gamma^2 + \gamma - 1)} e^{\gamma x} - \frac{1}{2} e^x \right) \\
 &\quad + \varepsilon \left( \frac{\alpha^2(\alpha^2 + \alpha + 1)}{2(\alpha^2 + \alpha - 1)} e^{\alpha x} + \frac{\beta^2(\beta^2 + \beta + 1)}{2(\beta^2 + \beta - 1)} e^{\beta x} + \frac{\gamma^2(\gamma^2 + \gamma + 1)}{2(\gamma^2 + \gamma - 1)} e^{\gamma x} - \frac{1}{2} e^x \right) \\
 &\quad + j\varepsilon \left( \frac{\alpha^3(\alpha^2 + \alpha + 1)}{2(\alpha^2 + \alpha - 1)} e^{\alpha x} + \frac{\beta^3(\beta^2 + \beta + 1)}{2(\beta^2 + \beta - 1)} e^{\beta x} + \frac{\gamma^3(\gamma^2 + \gamma + 1)}{2(\gamma^2 + \gamma - 1)} e^{\gamma x} - \frac{1}{2} e^x \right).
 \end{aligned}$$

**b):**

$$\begin{aligned}
 \sum_{n=0}^{\infty} \widehat{C}_n \frac{x^n}{n!} &= e^{\alpha x} + e^{\beta x} + e^{\gamma x} + e^x + j(\alpha e^{\alpha x} + \beta e^{\beta x} + \gamma e^{\gamma x} + e^x) \\
 &\quad + \varepsilon(\alpha^2 e^{\alpha x} + \beta^2 e^{\beta x} + \gamma^2 e^{\gamma x} + e^x) \\
 &\quad + j\varepsilon(\alpha^3 e^{\alpha x} + \beta^3 e^{\beta x} + \gamma^3 e^{\gamma x} + e^x).
 \end{aligned}$$

### 3. Obtaining Binet Formula From Generating Function

We next find Binet's formula generalized dual hyperbolic Pierre number  $\{\widehat{W}_n\}$  by the use of generating function for  $\widehat{W}_n$ .

THEOREM 7. *Binet's formula of generalized Gaussian Pierre numbers*

$$\widehat{W}_n = \frac{q_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} + \frac{q_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{q_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{q_4 \delta^n}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}, \quad (3.1)$$

where

$$\begin{aligned} q_1 &= \widehat{W}_0 \alpha^3 + (\widehat{W}_1 - 2\widehat{W}_0) \alpha^2 + (\widehat{W}_2 - 2\widehat{W}_1) \alpha + \widehat{W}_3 - 2\widehat{W}_2, \\ q_2 &= \widehat{W}_0 \beta^3 + (\widehat{W}_1 - 2\widehat{W}_0) \beta^2 + (\widehat{W}_2 - 2\widehat{W}_1) \beta + \widehat{W}_3 - 2\widehat{W}_2, \\ q_3 &= \widehat{W}_0 \gamma^3 + (\widehat{W}_1 - 2\widehat{W}_0) \gamma^2 + (\widehat{W}_2 - 2\widehat{W}_1) \gamma + \widehat{W}_3 - 2\widehat{W}_2, \\ q_4 &= \widehat{W}_0 \delta^3 + (\widehat{W}_1 - 2\widehat{W}_0) \delta^2 + (\widehat{W}_2 - 2\widehat{W}_1) \delta + \widehat{W}_3 - 2\widehat{W}_2. \end{aligned}$$

*Proof.* Let

$$h(x) = x^4 - 2x + 1.$$

Then for some  $\alpha, \beta, \gamma$  and  $\delta$  we write

$$h(x) = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x).$$

i.e.,

$$x^4 - 2x + 1 = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x). \quad (3.2)$$

Hence  $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$  and  $\frac{1}{\delta}$  are the roots of  $h(x)$ . This gives  $\alpha, \beta, \gamma$  and  $\delta$  as the roots of

$$h\left(\frac{1}{x}\right) = h\left(\frac{1}{x}\right) = 1 - \frac{2}{x} + \frac{1}{x^4} = 0.$$

This implies  $x^4 - 2x + 1 = 0$ . Now, by it follows that

$$\sum_{n=0}^{\infty} \widehat{W}_n x^n = \frac{(\widehat{W}_3 - 2\widehat{W}_2) x^3 + (\widehat{W}_2 - 2\widehat{W}_1) x^2 + (\widehat{W}_1 - 2\widehat{W}_0) x + \widehat{W}_0}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)}.$$

Then we write

$$\begin{aligned} \frac{(\widehat{W}_3 - 2\widehat{W}_2) x^3 + (\widehat{W}_2 - 2\widehat{W}_1) x^2 + (\widehat{W}_1 - 2\widehat{W}_0) x + \widehat{W}_0}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)} &= \frac{B_1}{(1 - \alpha x)} + \frac{B_2}{(1 - \beta x)} \\ &+ \frac{B_3}{(1 - \gamma x)} + \frac{B_4}{(1 - \delta x)}. \end{aligned} \quad (3.3)$$

So

$$\begin{aligned} & (\widehat{W}_3 - 2\widehat{W}_2)x^3 + (\widehat{W}_2 - 2\widehat{W}_1)x^2 + (\widehat{W}_1 - 2\widehat{W}_0)x + \widehat{W}_0 \\ = & B_1(1 - \beta x)(1 - \gamma x)(1 - \delta x) + B_2(1 - \alpha x)(1 - \gamma x)(1 - \delta x) \\ & + B_3(1 - \alpha x)(1 - \beta x)(1 - \delta x) + B_3(1 - \alpha x)(1 - \beta x)(1 - \gamma x). \end{aligned}$$

If we consider  $x = \frac{1}{\alpha}$ , we get  $\widehat{W}_0 + \frac{1}{\alpha}(\widehat{W}_1 - 2\widehat{W}_0) + \frac{1}{\alpha^2}(\widehat{W}_2 - 2\widehat{W}_1) + \frac{1}{\alpha^3}(\widehat{W}_3 - 2\widehat{W}_2)$   
 $= -B_1(\frac{1}{\alpha}\beta - 1)(\frac{1}{\alpha}\gamma - 1)(\frac{1}{\alpha}\delta - 1)$ .

This gives

$$\begin{aligned} B_1 &= \frac{\alpha^3 \left( \widehat{W}_0 + \frac{1}{\alpha^2}(\widehat{W}_2 - 2\widehat{W}_1) + \frac{1}{\alpha^3}(\widehat{W}_3 - 2\widehat{W}_2) + \frac{1}{\alpha}(\widehat{W}_1 - 2\widehat{W}_0) \right)}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} \\ &= \frac{\widehat{W}_0\alpha^3 + (\widehat{W}_1 - 2\widehat{W}_0)\alpha^2 + (\widehat{W}_2 - 2\widehat{W}_1)\alpha + \widehat{W}_3 - 2\widehat{W}_2}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} B_2 &= \frac{\widehat{W}_0\beta^3 + (\widehat{W}_1 - 2\widehat{W}_0)\beta^2 + (\widehat{W}_2 - 2\widehat{W}_1)\beta + \widehat{W}_3 - 2\widehat{W}_2}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)}, \\ B_3 &= \frac{\widehat{W}_0\gamma^3 + (\widehat{W}_1 - 2\widehat{W}_0)\gamma^2 + (\widehat{W}_2 - 2\widehat{W}_1)\gamma + \widehat{W}_3 - 2\widehat{W}_2}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)}, \\ B_4 &= \frac{\widehat{W}_0\delta^3 + (\widehat{W}_1 - 2\widehat{W}_0)\delta^2 + (\widehat{W}_2 - 2\widehat{W}_1)\delta + \widehat{W}_3 - 2\widehat{W}_2}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}. \end{aligned}$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty} \widehat{W}_n x^n = B_1(1 - \alpha x)^{-1} + B_2(1 - \beta x)^{-1} + B_3(1 - \gamma x)^{-1} + B_4(1 - \delta x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} \widehat{W}_n x^n = B_1 \sum_{n=0}^{\infty} \alpha^n x^n + B_2 \sum_{n=0}^{\infty} \beta^n x^n + B_3 \sum_{n=0}^{\infty} \gamma^n x^n + B_4 \sum_{n=0}^{\infty} \delta^n x^n = \sum_{n=0}^{\infty} (B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n)x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$\widehat{W}_n = B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n.$$

**THEOREM 8.** *For all integers  $m, n$  the following identities holds:*

$$\widehat{W}_{m+n} = P_{m-2}\widehat{W}_{n+3} - P_{m-5}\widehat{W}_{n+2} - P_{m-4}\widehat{W}_{n+1} - P_{m-3}\widehat{W}_n.$$

**Proof.** First we assume that  $m, n \geq 0$  theorem 8 can be proved by mathematical induction on  $m$ . If  $m = 0$  we get

$$\widehat{W}_n = P_{-2}\widehat{W}_{n+3} - P_{-5}\widehat{W}_{n+2} - P_{-4}\widehat{W}_{n+1} - P_{-3}\widehat{W}_n.$$

which is true since  $P_{-2} = 0, P_{-3} = -1, P_{-4} = 0, P_{-5} = 0$ . Suppose that the equality holds for  $m \leq k$ . For  $m = k + 1$ , we obtain

$$\begin{aligned} \widehat{W}_{k+1+n} &= 2\widehat{W}_{n+k} + -\widehat{W}_{n+k-3} \\ &= 2\left(P_{k-2}\widehat{W}_{n+3} - P_{k-5}\widehat{W}_{n+2} - P_{k-4}\widehat{W}_{n+1} - P_{k-3}\widehat{W}_n\right) \\ &\quad - \left(2P_{k-5}\widehat{W}_{n+3} - P_{k-8}\widehat{W}_{n+2} - P_{k-6}\widehat{W}_{n+1} - P_{k-6}\widehat{W}_n\right) \end{aligned}$$

by mathematical induction on  $m$ , this proves Theorem 8.

The other cases of  $m, n$  can be proved similarly for all integers  $m, n$ .  $\square$

Taking  $\widehat{W}_n = \widehat{P}_n$  or  $\widehat{W}_n = \widehat{C}_n$  in above Theorem, respectively, we obtain:

COROLLARY 9.

$$\begin{aligned} \widehat{P}_{m+n} &= P_{m-2}\widehat{P}_{n+3} - P_{m-5}\widehat{P}_{n+2} - P_{m-4}\widehat{P}_{n+1} - P_{m-3}\widehat{P}_n \\ \widehat{C}_{m+n} &= P_{m-2}\widehat{C}_{n+3} - P_{m-5}\widehat{C}_{n+2} - P_{m-4}\widehat{C}_{n+1} - P_{m-3}\widehat{C}_n. \end{aligned}$$

#### 4. SIMSON'S FORMULA

In this section, we present Simpson's formula for the dual hyperbolic generalized Pierre numbers . This is a special case of [22,Theorem 4.1].

THEOREM 10. (*Simpson's formula for dual hyperbolic generalized Pierre numbers*) For all integers  $n$  we have,

$$\begin{aligned} &\begin{vmatrix} \widehat{W}_{n+3} & \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n \\ \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} \\ \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} \\ \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} & \widehat{W}_{n-3} \end{vmatrix} = \begin{vmatrix} \widehat{W}_3 & \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} & \widehat{W}_{-3} \end{vmatrix} \\ &= (\widehat{W}_3 - \widehat{W}_2 - \widehat{W}_1 - \widehat{W}_0)(\widehat{W}_3^3 - \widehat{W}_2^3 - \widehat{W}_1^3 - \widehat{W}_0^3 + (-5\widehat{W}_2 + \widehat{W}_1 + \widehat{W}_0)\widehat{W}_3^2 + (7\widehat{W}_3 - 3\widehat{W}_0 - \widehat{W}_1)\widehat{W}_2^2 \\ &\quad + (3\widehat{W}_3 + \widehat{W}_2 - \widehat{W}_0)\widehat{W}_1^2 + (\widehat{W}_3 + \widehat{W}_2 + \widehat{W}_1)\widehat{W}_0^2 + 4(-\widehat{W}_2\widehat{W}_3 - \widehat{W}_0\widehat{W}_3 + \widehat{W}_0\widehat{W}_2)\widehat{W}_1). \end{aligned}$$

Proof. Using Theorem 10 it can be proved by using induction or use [22,Theorem 4.1]

From the Theorem 10 we get the following Corollary.

COROLLARY 11. For all integers  $n$ , the Simson's formulas of dual hyperbolic Pierre numbers and dual hyperbolic Pierre Lucas numbers are given as,

$$\text{a): } \begin{vmatrix} \widehat{P}_{n+3} & \widehat{P}_{n+2} & \widehat{P}_{n+1} & \widehat{P}_n \\ \widehat{P}_{n+2} & \widehat{P}_{n+1} & \widehat{P}_n & \widehat{P}_{n-1} \\ \widehat{P}_{n+1} & \widehat{P}_n & \widehat{P}_{n-1} & \widehat{P}_{n-2} \\ \widehat{P}_n & \widehat{P}_{n-1} & \widehat{P}_{n-2} & \widehat{P}_{n-3} \end{vmatrix} = 2 + 2j + 8\varepsilon + 8j\varepsilon,$$

$$\mathbf{b):} \begin{vmatrix} \widehat{C}_{n+3} & \widehat{C}_{n+2} & \widehat{C}_{n+1} & \widehat{C}_n \\ \widehat{C}_{n+2} & \widehat{C}_{n+1} & \widehat{C}_n & \widehat{C}_{n-1} \\ \widehat{C}_{n+1} & \widehat{C}_n & \widehat{C}_{n-1} & \widehat{C}_{n-2} \\ \widehat{C}_n & \widehat{C}_{n-1} & \widehat{C}_{n-2} & \widehat{C}_{n-3} \end{vmatrix} = -352 - 352j - 1408\varepsilon - 1408j\varepsilon,$$

respectively.

### 5. Linear Sums

In this section, we give the summation formulas of the dual hyperbolic generalized Pierre numbers with positive and negatif subscripts.

Now, we present the summation formulas of the generalized Pierre numbers.

**THEOREM 12.** *For the dual hyperbolic Pierre numbers, we have the following formulas:*

$$\begin{aligned} \mathbf{(a):} \quad & \sum_{k=0}^n W_k = \frac{1}{2}(-(n+3)W_{n+3} + (n+4)W_{n+2} + (n+3)W_{n+1} + (n+4)W_n + 3W_3 - 4W_2 - 3W_1 - 2W_0). \\ \mathbf{(b):} \quad & \sum_{k=0}^n W_{2k} = \frac{1}{2}(-(n+2)W_{2n+2} + (n+3)W_{2n+1} + (n+3)W_{2n} + (n+2)W_{2n-1} + 2W_3 - 2W_2 - 3W_1 - W_0). \\ \mathbf{(c):} \quad & \sum_{k=0}^n W_{2k+1} = \frac{1}{2}(-(n+1)W_{2n+2} + (n+3)W_{2n+1} + (n+2)W_{2n} + (n+2)W_{2n-1} + 2W_3 - 3W_2 - W_1 - 2W_0). \end{aligned}$$

Proof. For the proof, see Soykan [19, Theorem 3.10].  $\square$

**THEOREM 13.** *For the dual hyperbolic Pierre numbers, we have the following formulas:*

$$\begin{aligned} \mathbf{(a):} \quad & \sum_{k=0}^n \widehat{W}_k = \frac{1}{2}(-(n+3)\widehat{W}_{n+3} + (n+4)\widehat{W}_{n+2} + (n+3)\widehat{W}_{n+1} + (n+4)\widehat{W}_n + 3\widehat{W}_3 - 4\widehat{W}_2 - 3\widehat{W}_1 - 2\widehat{W}_0). \\ \mathbf{(b):} \quad & \sum_{k=0}^n \widehat{W}_{2k} = \frac{1}{2}(-(n+2)\widehat{W}_{2n+2} + (n+3)\widehat{W}_{2n+1} + (n+3)\widehat{W}_{2n} + (n+2)\widehat{W}_{2n-1} + 2\widehat{W}_3 - 2\widehat{W}_2 - 3\widehat{W}_1 - \widehat{W}_0). \\ \mathbf{(c):} \quad & \sum_{k=0}^n \widehat{W}_{2k+1} = \frac{1}{2}(-(n+1)\widehat{W}_{2n+2} + (n+3)\widehat{W}_{2n+1} + (n+2)\widehat{W}_{2n} + (n+2)\widehat{W}_{2n-1} + 2\widehat{W}_3 - 3\widehat{W}_2 - \widehat{W}_1 - 2\widehat{W}_0). \end{aligned}$$

Proof. Use Theorem 12 and the definition of  $\widehat{W}_n$ .  $\square$

As a special case of the Theorem 13, we present the following Corollary.

**COROLLARY 14.** *For  $n \geq 0$ , dual hyperbolic Pierre numbers have the following properties:*

$$\begin{aligned} \mathbf{(a):} \quad & \sum_{k=0}^n \widehat{P}_k = \frac{1}{2}(-(n+3)\widehat{P}_{n+3} + (n+4)\widehat{P}_{n+2} + (n+3)\widehat{P}_{n+1} + (n+4)\widehat{P}_n + 1 - 8j\varepsilon - 3\varepsilon). \\ \mathbf{(b):} \quad & \sum_{k=0}^n \widehat{P}_{2k} = \frac{1}{2}(-(n+2)\widehat{P}_{2n+2} + (n+3)\widehat{P}_{2n+1} + (n+3)\widehat{P}_{2n} + (n+2)\widehat{P}_{2n-1} + j - 2j\varepsilon + 1). \\ \mathbf{(c):} \quad & \sum_{k=0}^n \widehat{P}_{2k+1} = \frac{1}{2}(-(n+1)\widehat{P}_{2n+2} + (n+3)\widehat{P}_{2n+1} + (n+2)\widehat{P}_{2n} + (n+2)\widehat{P}_{2n-1} + 1 - 5j\varepsilon - 2\varepsilon). \end{aligned}$$

As a second special case of the above theorem, we have the following summation formulas for dual hyperbolic Pierre Lucas numbers:

**COROLLARY 15.** *For  $n \geq 0$ , dual hyperbolic Pierre Lucas numbers have the following properties.*

- (a):  $\sum_{k=0}^n \widehat{C}_k = \frac{1}{2}(- (n+3)\widehat{C}_{n+3} + (n+4)\widehat{C}_{n+2} + (n+3)\widehat{C}_{n+1} + (n+4)\widehat{C}_n - 12j - 14\varepsilon - 20j\varepsilon - 6).$
- (b):  $\sum_{k=0}^n \widehat{C}_{2k} = \frac{1}{2}(- (n+2)\widehat{C}_{2n+2} + (n+3)\widehat{C}_{2n+1} + (n+3)\widehat{C}_{2n} + (n+2)\widehat{C}_{2n-1} + -6j - 8\varepsilon - 8j\varepsilon - 2).$
- (c):  $\sum_{k=0}^n \widehat{C}_{2k+1} = \frac{1}{2}(- (n+1)\widehat{C}_{2n+2} + (n+3)\widehat{C}_{2n+1} + (n+2)\widehat{C}_{2n} + (n+2)\widehat{C}_{2n-1} + -8j - 8\varepsilon - 14j\varepsilon - 6).$

Next, we give the ordinary generating functions of some special cases of dual hyperbolic generalized Pierre numbers.

**THEOREM 16.** *The ordinary generating functions of the sequences  $\widehat{W}_{2n}, \widehat{W}_{2n+1}$  are given as follows:*

- (a):  $\sum_{n=0}^{\infty} \widehat{W}_{2n}x^n = \frac{\widehat{W}_3(2x^2) + \widehat{W}_2(x^3 - 4x^2 + x) - \widehat{W}_1(2x^3) + \widehat{W}_0(x^2 - 4x + 1)}{x^4 + 2x^2 - 4x + 1}.$
- (b):  $\sum_{n=0}^{\infty} \widehat{W}_{2n+1}x^n = \frac{\widehat{W}_3(x^3 + x) - \widehat{W}_2(2x^3) - \widehat{W}_1(x^2 - 4x + 1) - \widehat{W}_0(2x^2)}{x^4 + 2x^2 - 4x + 1}.$

Proof. Similarly, the proof can be constructed as in [4].

From the last Theorem, we have the following Corollary which gives sum formula of dual hyperbolic Pierre numbers (Take  $\widehat{W}_n = \widehat{P}_n$  with  $\widehat{P}_0 = j + 2\varepsilon + 4j\varepsilon, \widehat{P}_1 = 2j + 4\varepsilon + 8j\varepsilon + 1, \widehat{P}_2 = 4j + 8\varepsilon + 15j\varepsilon + 2, \widehat{P}_3 = 8j + 15\varepsilon + 28j\varepsilon + 4$ )

**COROLLARY 17.** *For  $n \geq 0$  dual hyperbolic Pierre numbers have the following properties.*

- (a):  $\sum_{n=0}^{\infty} \widehat{P}_{2n}x^n = \frac{(j+2\varepsilon+4j\varepsilon) + (2-j\varepsilon)x + jx^2 - j\varepsilon x^3}{x^4 + 2x^2 - 4x + 1}.$
- (b):  $\sum_{n=0}^{\infty} \widehat{P}_{2n+1}x^n = \frac{(-2j-4\varepsilon-8j\varepsilon-1) + (8+31\varepsilon+16j+60j\varepsilon)x + (-4j-8\varepsilon-1-16j\varepsilon)x^2 + (-\varepsilon-2j\varepsilon)x^3}{x^4 + 2x^2 - 4x + 1}.$

### 6. Matrices related with Dual Hyperbolic Generalized Pierre Numbers

We define the square matrix  $A$  of order 4 as

$$A = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that  $\det A = 1$ . Note that

$$A^n = \begin{pmatrix} P_{n+1} & -P_{n-2} & -P_{n-1} & -P_n \\ P_n & -P_{n-3} & -P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-4} & -P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-5} & -P_{n-4} & -P_{n-3} \end{pmatrix}$$

for the proof see [21].

Then we give the following lemma.

LEMMA 18. For  $n \geq 0$  the following identity is true:

$$\begin{pmatrix} \widehat{W}_{n+3} \\ \widehat{W}_{n+2} \\ \widehat{W}_{n+1} \\ \widehat{W}_n \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \widehat{W}_3 \\ \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix}.$$

*Proof.* The identity(18) can be proved by mathematical induction on  $n$ . If  $n = 0$  we obtain

$$\begin{pmatrix} \widehat{W}_3 \\ \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} \widehat{W}_3 \\ \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for  $n = k$ . Thus the following identity is true

$$\begin{pmatrix} \widehat{W}_{k+3} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \\ \widehat{W}_k \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \widehat{W}_3 \\ \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix}.$$

For  $n = k + 1$ , we get

$$\begin{aligned} \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} \widehat{W}_3 \\ \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix} &= \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \widehat{W}_3 \\ \widehat{W}_2 \\ \widehat{W}_1 \\ \widehat{W}_0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{W}_{k+3} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \\ \widehat{W}_k \end{pmatrix} \\ &= \begin{pmatrix} \widehat{W}_{k+4} \\ \widehat{W}_{k+3} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \end{pmatrix}. \end{aligned}$$

Hence,the proof completed  $\square$ .

We define

$$N_{\widehat{W}} = \begin{pmatrix} \widehat{W}_3 & \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} & \widehat{W}_{-3} \end{pmatrix}, \quad (6.1)$$

$$E_{\widehat{W}} = \begin{pmatrix} \widehat{W}_{n+3} & \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n \\ \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} \\ \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} \\ \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} & \widehat{W}_{n-3} \end{pmatrix}. \quad (6.2)$$

Now, we have the following theorem with  $N_{\widehat{W}}$  and  $E_{\widehat{W}}$ ,

**THEOREM 19.** *Using  $N_{\widehat{W}}$  and  $E_{\widehat{W}}$ , we get*

$$A^n N_{\widehat{W}} = E_{\widehat{W}}.$$

*Proof.* Note that we get

$$\begin{aligned} A^n N_{\widehat{W}} &= \begin{pmatrix} P_{n+1} & -P_{n-2} & -P_{n-1} & -P_n \\ P_n & -P_{n-3} & -P_{n-2} & -P_{n-1} \\ P_{n-1} & -P_{n-4} & -P_{n-3} & -P_{n-2} \\ P_{n-2} & -P_{n-5} & -P_{n-4} & -P_{n-3} \end{pmatrix} \begin{pmatrix} \widehat{W}_3 & \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} & \widehat{W}_{-3} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned}
 a_{11} &= P_{n+1}\widehat{W}_3 - P_{n-2}\widehat{W}_2 - P_{n-1}\widehat{W}_1 - P_n\widehat{W}_0 = \widehat{W}_{n+3}, \\
 a_{12} &= P_{n+1}\widehat{W}_2 - P_{n-2}\widehat{W}_1 - P_{n-1}\widehat{W}_0 - P_n\widehat{W}_{-1} = \widehat{W}_{n+2}, \\
 a_{13} &= P_{n+1}\widehat{W}_1 - P_{n-2}\widehat{W}_0 - P_{n-1}\widehat{W}_{-1} - P_n\widehat{W}_{-2} = \widehat{W}_{n+1}, \\
 a_{14} &= P_{n+1}\widehat{W}_0 - P_{n-2}\widehat{W}_{-1} - P_{n-1}\widehat{W}_{-2} - P_n\widehat{W}_{-3} = \widehat{W}_n, \\
 a_{21} &= P_n\widehat{W}_3 - P_{n-3}\widehat{W}_2 - P_{n-2}\widehat{W}_1 - P_{n-1}\widehat{W}_0 = \widehat{W}_{n+2}, \\
 a_{22} &= P_n\widehat{W}_2 - P_{n-3}\widehat{W}_1 - P_{n-2}\widehat{W}_0 - P_{n-1}\widehat{W}_{-1} = \widehat{W}_{n+1}, \\
 a_{23} &= P_n\widehat{W}_1 - P_{n-3}\widehat{W}_0 - P_{n-2}\widehat{W}_{-1} - P_{n-1}\widehat{W}_{-2} = \widehat{W}_n, \\
 a_{24} &= P_n\widehat{W}_0 - P_{n-3}\widehat{W}_{-1} - P_{n-2}\widehat{W}_{-2} - P_{n-1}\widehat{W}_{-3} = \widehat{W}_{n-1}, \\
 a_{31} &= P_{n-1}\widehat{W}_3 - P_{n-4}\widehat{W}_2 - P_{n-3}\widehat{W}_1 - P_{n-2}\widehat{W}_0 = \widehat{W}_{n+1}, \\
 a_{32} &= P_{n-1}\widehat{W}_2 - P_{n-4}\widehat{W}_1 - P_{n-3}\widehat{W}_0 - P_{n-2}\widehat{W}_{-1} = \widehat{W}_n, \\
 a_{33} &= P_{n-1}\widehat{W}_1 - P_{n-4}\widehat{W}_0 - P_{n-3}\widehat{W}_{-1} - P_{n-2}\widehat{W}_{-2} = \widehat{W}_{n-1}, \\
 a_{34} &= P_{n-1}\widehat{W}_0 - P_{n-4}\widehat{W}_{-1} - P_{n-3}\widehat{W}_{-2} - P_{n-2}\widehat{W}_{-3} = \widehat{W}_{n-2}, \\
 a_{41} &= P_{n-2}\widehat{W}_3 - P_{n-5}\widehat{W}_2 - P_{n-4}\widehat{W}_1 - P_{n-3}\widehat{W}_0 = \widehat{W}_n, \\
 a_{42} &= P_{n-2}\widehat{W}_2 - P_{n-5}\widehat{W}_1 - P_{n-4}\widehat{W}_0 - P_{n-3}\widehat{W}_{-1} = \widehat{W}_{n-1}, \\
 a_{43} &= P_{n-2}\widehat{W}_1 - P_{n-5}\widehat{W}_0 - P_{n-4}\widehat{W}_{-1} - P_{n-3}\widehat{W}_{-2} = \widehat{W}_{n-2}, \\
 a_{44} &= P_{n-2}\widehat{W}_0 - P_{n-5}\widehat{W}_{-1} - P_{n-4}\widehat{W}_{-2} - P_{n-3}\widehat{W}_{-3} = \widehat{W}_{n-3}.
 \end{aligned}$$

Using the theorem (8) the proof is done.  $\square$

By taking  $\widehat{W}_n = \widehat{P}_n$  with  $\widehat{P}_0, \widehat{P}_1, \widehat{P}_2, \widehat{P}_3$  in (6.1) and (6.2)  
 $\widehat{W}_n = \widehat{C}_n$  with  $\widehat{C}_0, \widehat{C}_1, \widehat{C}_2, \widehat{C}_3$  in (6.1) and (6.2)

respectively, we get:

$$\begin{aligned}
 N_{\widehat{P}} &= \begin{pmatrix} 8j + 15\varepsilon + 28j\varepsilon + 4 & 4j + 8\varepsilon + 15j\varepsilon + 2 & 2j + 4\varepsilon + 8j\varepsilon + 1 & j + 2\varepsilon + 4j\varepsilon \\ 4j + 8\varepsilon + 15j\varepsilon + 2 & 2j + 4\varepsilon + 8j\varepsilon + 1 & j + 2\varepsilon + 4j\varepsilon & \varepsilon + 2j\varepsilon \\ 2j + 4\varepsilon + 8j\varepsilon + 1 & j + 2\varepsilon + 4j\varepsilon & \varepsilon + 2j\varepsilon & j\varepsilon \\ j + 2\varepsilon + 4j\varepsilon & \varepsilon + 2j\varepsilon & j\varepsilon & -1 \end{pmatrix}, \\
 E_{\widehat{P}} &= \begin{pmatrix} \widehat{P}_{n+3} & \widehat{P}_{n+2} & \widehat{P}_{n+1} & \widehat{P}_n \\ \widehat{P}_{n+2} & \widehat{P}_{n+1} & \widehat{P}_n & \widehat{P}_{n-1} \\ \widehat{P}_{n+1} & \widehat{P}_n & \widehat{P}_{n-1} & \widehat{P}_{n-2} \\ \widehat{P}_n & \widehat{P}_{n-1} & \widehat{P}_{n-2} & \widehat{P}_{n-3} \end{pmatrix}, \\
 N_{\widehat{C}} &= \begin{pmatrix} 12j + 22\varepsilon + 40j\varepsilon + 8 & 8j + 12\varepsilon + 22j\varepsilon + 4 & 4j + 8\varepsilon + 12j\varepsilon + 2 & 2j + 4\varepsilon + 8j\varepsilon + 4 \\ 8j + 12\varepsilon + 22j\varepsilon + 4 & 4j + 8\varepsilon + 12j\varepsilon + 2 & 2j + 4\varepsilon + 8j\varepsilon + 4 & 4j + 4j\varepsilon + 2\varepsilon \\ 4j + 8\varepsilon + 12j\varepsilon + 2 & 2j + 4\varepsilon + 8j\varepsilon + 4 & -4j + 4j\varepsilon + 2\varepsilon & 4\varepsilon + 2j\varepsilon \\ 2j + 4\varepsilon + 8j\varepsilon + 4 & 4j + 4j\varepsilon + 2\varepsilon & 4\varepsilon + 2j\varepsilon & 4j\varepsilon + 6 \end{pmatrix}, \\
 E_{\widehat{C}} &= \begin{pmatrix} \widehat{C}_{n+3} & \widehat{C}_{n+2} & \widehat{C}_{n+1} & \widehat{C}_n \\ \widehat{C}_{n+2} & \widehat{C}_{n+1} & \widehat{C}_n & \widehat{C}_{n-1} \\ \widehat{C}_{n+1} & \widehat{C}_n & \widehat{C}_{n-1} & \widehat{C}_{n-2} \\ \widehat{C}_n & \widehat{C}_{n-1} & \widehat{C}_{n-2} & \widehat{C}_{n-3} \end{pmatrix}.
 \end{aligned}$$

From Theorem [19], we can write the following corollary.

**COROLLARY 20.** *The following identities are hold:*

- a):**  $A^n N_{\widehat{P}} = E_{\widehat{P}}$ .
- b):**  $A^n N_{\widehat{C}} = E_{\widehat{C}}$ .

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