The nth Power of 2x2 Matrix: Applications to the Linear Recurring Sequence of Order 2 and to The Birth-Death Process.

Abstract

After exploring a robust method for calculating the *nth* power of a 2x2 matrix A^n [2] without resorting to diagonalization techniques, i.e., we provided a unique formula for the *nth* power of any type of 2x2 matrix (see Arbai [2]), whether diagonalizable or not, whether it has two distinct eigenvalues or not, and whether these eigenvalues are real or not.

Our contributions in this paper would be applications of the above, such as:

1) Application of the nth power of 2x2 matrices to recurrent linear sequences of order 2

2) The (unsolved) problem of the birth and death process (BDP) with constant coefficients and an infinite number of states.

1 A New Approach to Matrix Powers with Wide Applications (Introduction):

This paper presents a unique and efficient method for calculating the nth power of a 2x2 matrix (A^n) without resorting to the traditional diagonalization method. Although diagonalization is a widely used and effective technique for certain classes of matrices, its dependence on distinct and real eigenvalues can be limiting. Our approach, inspired by the work of Arbai [2], provides a single, unified formula applicable to "any" 2x2 matrix, regardless of its characteristics. This includes non-diagonalizable matrices, matrices with repeated eigenvalues, and those with complex eigenvalues. By providing a general and universally applicable solution, we eliminate the need for case-by-case analysis, thus simplifying and streamlining a fundamental operation in linear algebra.

- Key Contributions and Applications:

Our main contribution is the demonstration of the practical utility of this new formula in two important areas:

1. Solving Second-Order Linear Recurrence Sequences:

We show how the shorthand solution of the nth term of a second-order linear recurrence relation, such as the Fibonacci sequence, can be directly derived using our general formula for A^n. By representing the recurrence relation as a matrix equation, we transform a recursive problem into a direct and explicit computation. This method offers a more elegant and computationally efficient alternative to traditional characteristic equation methods, especially for large values of n.

2. Treating the Birth-Death Process (BDP) with an Infinite Number of States:

We apply our methodology to the notoriously complex problem of the Birth-Death Process (BDP) with constant coefficients and an infinite number of states. Although the BDP is a cornerstone of stochastic modeling in fields such as queuing theory, population dynamics, and epidemiology, finding an analytical solution to its state probabilities in the case of an infinite number of states remains an open problem. Our approach to the computation of matrix powers, a key element in solving the BDP system of differential equations (as is also the matrix exponential), offers a new perspective and a potential route to an analytical solution. We believe this work could open new avenues of research on this long-unsolved problem.

2 The power of any (diagonalizable or non-diagonalizable) 2x2 matrix:

Theorem 1 *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *a 2X2 matrix, so* $\forall n \geq 2$ $(n \in IN)$

$$A^n = \begin{pmatrix} a\sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} & b\sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \\ c\sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} & \sum_{k=0}^{n} \alpha_1^k \alpha_2^{n-k} - a\sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \end{pmatrix} \quad (i)$$

with α_1 *and* α_2 *solutions of* $|A - \alpha I| = 0$.

Proof. We will demonstrate our formula using recurrence.

1) For $n=2$, we will show that we actually have

$$A^2 = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{pmatrix}$$
$$A^2 = \begin{pmatrix} a\sum_{k=0}^{1}\alpha_1^k\alpha_2^{1-k}-\alpha_1\alpha_2 & b\sum_{k=0}^{1}\alpha_1^k\alpha_2^{1-k} \\ c\sum_{k=0}^{1}\alpha_1^k\alpha_2^{1-k} & \sum_{k=0}^{2}\alpha_1^k\alpha_2^{2-k}-a\sum_{k=0}^{1}\alpha_1^k\alpha_2^{1-k} \end{pmatrix}$$

we must therefore show the following 4 equalities

$$\begin{cases} a^2+bc = a\sum_{k=0}^{1}\alpha_1^k\alpha_2^{1-k}-\alpha_1\alpha_2 \\ d^2+bc = \sum_{k=0}^{2}\alpha_1^k\alpha_2^{2-k}-a\sum_{k=0}^{1}\alpha_1^k\alpha_2^{1-k} \\ b(a+d) = b\sum_{k=0}^{1}\alpha_1^k\alpha_2^{1-k} \\ c(a+d) = c\sum_{k=0}^{1}\alpha_1^k\alpha_2^{1-k} \end{cases}$$

what does it mean

$$\begin{cases} a^2+bc = a(\alpha_2+\alpha_1)-\alpha_1\alpha_2 \\ d^2+bc = \alpha_2^2+\alpha_1\alpha_2+\alpha_1^2-a(\alpha_2+\alpha_1) \\ b(a+d) = b(\alpha_2+\alpha_1) \\ c(a+d) = c(\alpha_2+\alpha_1) \end{cases}$$

α_1 and α_2 solutions of $|A-\alpha I|=0$, so

$$\begin{cases} \alpha_1+\alpha_2 = Tr(A) = a+d \\ \alpha_1\alpha_2 = |A| = ad-bc \end{cases}$$

So

$a(\alpha_2+\alpha_1)-\alpha_1\alpha_2 = a(a+d)-(ad-bc) = a^2+ad-ad+bc = a^2+bc$

$\alpha_2^2+\alpha_1\alpha_2+\alpha_1^2-a(\alpha_2+\alpha_1) = (\alpha_1+\alpha_2)^2-\alpha_1\alpha_2-a(\alpha_2+\alpha_1) \Longrightarrow$

$\alpha_2^2+\alpha_1\alpha_2+\alpha_1^2-a(\alpha_2+\alpha_1) = (a+d)^2-(ad-bc)-a(a+d) = d^2+bc$

$b(\alpha_2+\alpha_1) = b(a+d)$

$c(\alpha_2+\alpha_1) = c(a+d)$

hence the result is true for $n=2$.

2) Let's suppose

$$A^n = \begin{pmatrix} a\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}-\sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k} & b\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} \\ c\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} & \sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}-a\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} \end{pmatrix}$$

3) Let us show that

$A^{n+1} = A^n.A$

$$A^{n+1} = \begin{pmatrix} a\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}-\sum_{k=0}^{n-1}\alpha_1^{k+1}\alpha_2^{n-k} & b\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k} \\ c\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k} & \sum_{k=0}^{n+1}\alpha_1^k\alpha_2^{n+1-k}-a\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k} \end{pmatrix}$$

We have $A^{n+1} = A^n.A =$

$$= \begin{pmatrix} a\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} - \sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k} & b\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} \\ c\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} & \sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k} - a\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} \end{pmatrix} A$$

$$A^{n+1} = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

$$x = a^2\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} - a\sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k} + cb\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}$$

$$y = ba\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} - b\sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k} + db\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}$$

$$z = ac\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} + c\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k} - ca\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}$$

$$t = bc\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} + d\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k} - da\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}$$

$$\Longrightarrow \begin{cases} x = \left(a^2 + cb\right)\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} - a\sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k} \\ y = b\left[(a+d)\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} - \sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k}\right] \\ z = c\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k} \\ t = (bc - da)\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} + d\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k} \end{cases}$$

$\Longrightarrow$

$$x = \left[a\left(\alpha_1 + \alpha_2\right) - \alpha_1\alpha_2\right]\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} - a\sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k}$$

$$y = b\left[(\alpha_1 + \alpha_2)\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} - \sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k}\right]$$

$$z = c\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}$$

$$t = -\alpha_1\alpha_2\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} + d\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k}$$

because $\alpha_1 + \alpha_2 = a + d$, $\alpha_1\alpha_2 = ad - bc$, $a\left(\alpha_1 + \alpha_2\right) - \alpha_1\alpha_2 = a^2 + cb$

$$A^{n+1} = \begin{pmatrix} a\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k} - \alpha_1\alpha_2\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} & b\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k} \\ c\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k} & \sum_{k=0}^{n+1}\alpha_1^k\alpha_2^{n+1-k} - a\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k} \end{pmatrix}$$

because

$$x = \left[a\left(\alpha_1 + \alpha_2\right) - \alpha_1\alpha_2\right]\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} - a\sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k}$$

$$= a\left(\alpha_1 + \alpha_2\right)\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} - \alpha_1\alpha_2\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} - a\sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k}$$

$$= a\alpha_1\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} + a\alpha_2\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} - \alpha_1\alpha_2\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} - a\sum_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k}$$

$$= a\alpha_1^n + a\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-k} - \alpha_1\alpha_2\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k}$$

$x = a \sum_{k=0}^{n} \alpha_1^k \alpha_2^{n-k} - \alpha_1\alpha_2 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k}$;
$y = b \left[(\alpha_1 + \alpha_2) \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} \right]$
$y = b \left[\sum_{k=0}^{n-1} \alpha_1^{k+1} \alpha_2^{n-1-k} + \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-k} - \sum_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} \right]$
$y = b \left[\alpha_1^n + \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-k} \right] = b \sum_{k=0}^{n} \alpha_1^k \alpha_2^{n-k}$
and
$t = -\alpha_1\alpha_2 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} + d \sum_{k=0}^{n} \alpha_1^k \alpha_2^{n-k}$
$t = -\sum_{k=0}^{n-1} \alpha_1^{k+1} \alpha_2^{n-k} + (\alpha_1 + \alpha_2 - a) \sum_{k=0}^{n} \alpha_1^k \alpha_2^{n-k}$ (as $\alpha_1 + \alpha_2 = a + d$)
$t = -\sum_{k=0}^{n-1} \alpha_1^{k+1} \alpha_2^{n-k} - a \sum_{k=0}^{n} \alpha_1^k \alpha_2^{n-k} + \sum_{k=0}^{n} \alpha_1^{k+1} \alpha_2^{n-k} + \sum_{k=0}^{n} \alpha_1^k \alpha_2^{n+1-k}$
$t = -a \sum_{k=0}^{n} \alpha_1^k \alpha_2^{n-k} + \sum_{k=0}^{n+1} \alpha_1^k \alpha_2^{n+1-k}$
hence the result. ■

Remark 2 *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *a 2X2 matrix and* α_1 *and* α_2 *solutions of* $|A - \alpha I| = 0$.

1) If $\alpha_1 = \alpha_2 = \alpha$, *so*

$$A^n = \left(\frac{a+d}{2}\right)^{n-1} \begin{pmatrix} \frac{(n+1)a-(n-1)d}{2} & nb \\ nc & \frac{(n+1)d-(n-1)a}{2} \end{pmatrix}, \forall n \geq 2 \ (n \in IN)$$

with $\alpha = \frac{a+d}{2}$ *and* $(a+d)^2 = 4(ad - bc)$ $((a-d)^2 = -4bc)$
And

$$A^n = n\alpha^{n-1} A - (n-1)\alpha^n I$$

2) If $\alpha_1 \neq \alpha_2$ *so* $\forall n \geq 2 \ (n \in IN)$

$$\begin{aligned} A^n &= \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} (a-\alpha_2)\alpha_1^n - (a-\alpha_1)\alpha_2^n & b(\alpha_1^n - \alpha_2^n) \\ c(\alpha_1^n - \alpha_2^n) & (a-\alpha_2)\alpha_2^n - (a-\alpha_1)\alpha_1^n \end{pmatrix} \\ A^n - \alpha_2^n I &= \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}(A - \alpha_2 I) \ \text{and} \ A^n - \alpha_1^n I = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}(A - \alpha_1 I) \\ A^n &= \left(\frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}\right) A - \alpha_1\alpha_2 \left(\frac{\alpha_1^{n-1} - \alpha_2^{n-1}}{\alpha_1 - \alpha_2}\right) I \end{aligned}$$

Proof. 1) $A^n = \alpha^{n-1} \begin{pmatrix} na - (n-1)\alpha & nb \\ nc & (n+1)\alpha - na \end{pmatrix}$
And
$A^n = \alpha^{n-1} \begin{pmatrix} na - (n-1)\alpha & nb \\ nc & nd + (n+1)\alpha - n(a+d) \end{pmatrix}$

$$A^n = \alpha^{n-1}\begin{pmatrix} na-(n-1)\alpha & nb \\ nc & nd+(n+1)\alpha-2n\alpha \end{pmatrix}$$
$$A^n = \alpha^{n-1}\begin{pmatrix} na-(n-1)\alpha & nb \\ nc & nd+(1-n)\alpha \end{pmatrix}$$
$$A^n = \alpha^{n-1}\begin{pmatrix} na & nb \\ nc & nd \end{pmatrix} + \alpha^{n-1}\begin{pmatrix} -(n-1)\alpha & 0 \\ 0 & -(n-1)\alpha \end{pmatrix}$$
$$A^n = n\alpha^{n-1}A-(n-1)\alpha^n I$$

2)

$$A^n = \frac{1}{\alpha_1-\alpha_2}\begin{pmatrix} a(\alpha_1^n-\alpha_2^n)-\alpha_1\alpha_2(\alpha_1^{n-1}-\alpha_2^{n-1}) & b(\alpha_1^n-\alpha_2^n) \\ c(\alpha_1^n-\alpha_2^n) & (\alpha_1^{n+1}-\alpha_2^{n+1})-a(\alpha_1^n-\alpha_2^n) \end{pmatrix}$$

■

3 Applications:

3.1 Linear recurrent sequence order 2:

A second-order linear recurrence sequence is a sequence in which each term is defined as a linear combination of the two preceding terms. In simpler terms, to find the next number in the sequence, simply multiply the two preceding numbers by constants and add the results. A classic example is the Fibonacci sequence, where each number is the sum of the two preceding numbers.

$$u_{n+1} = au_n + bu_{n-1} \text{ with } (u_1, u_0) \in IK^2 \text{ and } (a,b) \in IK^2$$

$IK = IR$ or $\mathbb{C}$

$$\Longleftrightarrow \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} \text{ and } (u_1,u_0) \in IK^2$$
$$\Longleftrightarrow \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^n\begin{pmatrix} u_1 \\ u_0 \end{pmatrix} \text{ and } (u_1,u_0) \in IK^2$$
$$\Longleftrightarrow \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = \begin{pmatrix} x_n & y_n \\ z_n & t_n \end{pmatrix}\begin{pmatrix} u_1 \\ u_0 \end{pmatrix} \text{ and } (u_1,u_0) \in IK^2$$

such

$$\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} x_n & y_n \\ z_n & t_n \end{pmatrix}$$

So, according to our formula (i) "the n-th Power of 2x2 Matrix" in the first theorem

$$\begin{cases} x_n = a\sum\limits_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} - \sum\limits_{k=0}^{n-2}\alpha_1^{k+1}\alpha_2^{n-1-k} \\ y_n = b\sum\limits_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} \\ z_n = \sum\limits_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} \\ t_n = \sum\limits_{k=0}^{n}\alpha_1^k\alpha_2^{n-k} - a\sum\limits_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} \end{cases}$$

with α_1 and α_2 are solutions of $\alpha^2 - a\alpha - b = 0$

Remark 3 $\alpha_1 + \alpha_2 = a$ *and* $\alpha_1\alpha_2 = -b$

Theorem 4 *Let* $u_{n+1} = au_n + bu_{n-1}$ *with* $(u_1, u_0) \in IK^2$*, so*

$$u_n = (u_1 - au_0)\sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} + u_0 \sum_{k=0}^{n} \alpha_1^k \alpha_2^{n-k}$$

such α_1 *and* α_2 *are solutions of* $\alpha^2 - a\alpha - b = 0$*.*

Proof. $\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} u_1 \\ u_0 \end{pmatrix} = \begin{pmatrix} x_n & y_n \\ z_n & t_n \end{pmatrix} \begin{pmatrix} u_1 \\ u_0 \end{pmatrix}$

$\Longrightarrow \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = \begin{pmatrix} x_n u_1 + y_n u_0 \\ z_n u_1 + t_n u_0 \end{pmatrix}$

such $\begin{cases} x_n = a \sum\limits_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} - \sum\limits_{k=0}^{n-2} \alpha_1^{k+1} \alpha_2^{n-1-k} \\ y_n = b \sum\limits_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \\ z_n = \sum\limits_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \\ t_n = \sum\limits_{k=0}^{n} \alpha_1^k \alpha_2^{n-k} - a \sum\limits_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \end{cases}$

so, $u_n = z_n u_1 + t_n u_0 = u_1 \sum\limits_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} + u_0 \left(\sum\limits_{k=0}^{n} \alpha_1^k \alpha_2^{n-k} - a \sum\limits_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} \right)$

$\Longrightarrow u_n = (u_1 - au_0) \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} + u_0 \sum_{k=0}^{n} \alpha_1^k \alpha_2^{n-k}$. ■

Remark 5 *1st case: If* $a^2 + 4b = 0$*, so* $\alpha_1 = \alpha_2 = \frac{a}{2}$ *and*

$$\begin{aligned} u_n &= \left[(u_1 + (1-a)\, u_0)\, n + u_0 \alpha \right] \alpha^{n-1} \\ u_n &= \left[n\,(u_0 + u_1) + (1-2n)\, \alpha u_0 \right] \alpha^{n-1} \end{aligned}$$

2nd case: If $a^2 + 4b \neq 0$*, so* $\alpha_1 \neq \alpha_2 = \frac{a}{2}$ *and*

$$u_n = \frac{1}{\alpha_1 - \alpha_2} \left[(u_1 - au_0)\,(\alpha_1^n - \alpha_2^n) + u_0 \left(\alpha_1^{n+1} - \alpha_2^{n+1} \right) \right]$$

Proof. 1)

$u_n = (u_1 - au_0) \sum\limits_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} + u_0 \sum\limits_{k=0}^{n} \alpha_1^k \alpha_2^{n-k} \Longrightarrow$

$u_n = (u_1 - au_0)\, n\alpha^{n-1} + u_0\,(n+1)\,\alpha^n$

since $\alpha_1 = \alpha_2 \Rightarrow \sum\limits_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} = \sum\limits_{k=0}^{n-1} \alpha_2^{n-1} = \alpha_2^{n-1} \sum\limits_{k=0}^{n-1} 1 = n\alpha_2^{n-1}$ and $\sum\limits_{k=0}^{n} \alpha_1^k \alpha_2^{n-k} = (n+1)\,\alpha_2^n$

2)

$$
\begin{aligned}
u_n &= (u_1 - au_0)\sum_{k=0}^{n-1}\alpha_1^k\alpha_2^{n-1-k} + u_0\sum_{k=0}^{n}\alpha_1^k\alpha_2^{n-k} \\
&= (u_1 - au_0)\left(\frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}\right) + u_0\left(\frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2}\right) \\
&= \frac{1}{\alpha_1 - \alpha_2}\left[(u_1 - au_0)\left(\alpha_1^n - \alpha_2^n\right) + u_0\left(\alpha_1^{n+1} - \alpha_2^{n+1}\right)\right]
\end{aligned}
$$

■

3.2 The birth-death process (BDP):

The birth-death process (BDP) is a classic example of a continuous-time, discrete-state Markov process that is used to model population growth over time. It was introduced by Feller (1938) [7] and has been used in various fields, such as biology, physics, ecology, queuing theory, ...

Mathematically, birth and death processes are often modeled using systems of differential equations, called direct Kolmogorov equations. These equations describe the evolution of the probability distribution of the number of individuals in the system over time. However, these equations can be notoriously difficult to solve analytically or even impossible, especially for complex systems with non-constant birth and death rates.

As a result, researchers often resort to approximation methods or numerical simulations to study birth-death processes (see [5]). These methods can provide valuable insights into the behavior of these systems, but they also have limitations. Therefore, there is a constant search for new methods to resolve theses équations.

The use of spectral methods to study birth and death processes was pioneered by S. Karlin and J. McGregor (see [8] and [9]). They defined a sequence of polynomials $Q_k(x)$ such that $Q_0(x) = 1$ and $xQ = AQ$.

This article focuses on general birth and death processes with an infinite number of states:

Postulates. The system changes only through transitions from states to their nearest neighbors (from E_n to E_{n+1} or E_{n-1} if $n \geq 1$, but from E_0 to E_1 only). If at epoch t the system is in state E_n, the probability that between t and $t+h$ the transition $E_n \to E_{n+1}$ occurs equals $\lambda_n h + o(h)$, and the probability of $E_n \to E_{n-1}$ (if $n \geq 1$) equals $\mu_n h + o(h)$. The probabilitynthat during $(t, t+h)$ more than one change occurs is $o(h)$. ([6] page 454).

This paper builds upon our previous work [1], [2], [3], [4] which laid the foundations for this approach. However, the current study extends and refines these methods, making significant new contributions, especially by applying our formula that gives the nth power of any 2x2 matrix.

Let $(X_t)_{t\in IR^+}$ be the discret and homogeneous "*Birth and Death*" stochastique process such $\forall (i, j) \in IN^2$

$$P_i(t) = P(X_t = i), P(t) = \begin{pmatrix} P_1(t) \\ P_2(t) \\ \vdots \\ P_n(t) \\ \vdots \end{pmatrix}$$

with $\sum_{i \geq 1} P(X_t = i) = 1,\ P(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \end{pmatrix}$ and

$$\begin{aligned} \forall (i,j) &\in IN^2 \\ P_{ij}(\Delta t) &= P(X_{t+\Delta t} = j / X_t = i) = \begin{cases} \lambda_i \Delta t + o(\Delta t)\,;\text{ if } j = i+1 \\ \mu_i \Delta t + o(\Delta t)\,;\text{ if } j = i-1 \\ 1 - (\lambda_i + \mu_i)\Delta t + o(\Delta t)\,;\text{ if } j = i \\ o(\Delta t)\,;\text{ if } |j-i| \geq 2 \end{cases} \end{aligned}$$

λ_i is the birth rate and μ_i is the death rate.

Proposition 6 *Let* $(X_t)_{t \in IR^+}$ *be the discret and homogeneous* "Birth and Death" *stochastique process with conditions* (1) *, so* $P(t) = (P_0(t), P_1(t), ...)^t = (P_i(t))_{i \geq 1}$ *is solution of the folowing linear differential equations systeme.*

$$\begin{cases} P_1'(t) = -(\lambda_1 + \mu_1) P_1(t) + \mu_2 P_2(t) \\ P_j'(t) = \lambda_{j-1} P_{j-1}(t) - (\lambda_j + \mu_j) P_j(t) + \mu_{j+1} P_{j+1}(t) \ \ \forall j \geq 2 \\ ... \end{cases}$$

and

$$P'(t) = AP(t)$$

called the backward equation.
with

$$A = \begin{pmatrix} -(\lambda_1 + \mu_1) & \mu_2 & 0 & \cdots & \cdots & & \cdots & \cdots & \cdots \\ \lambda_1 & \ddots & \ddots & \ddots & & & & & \\ 0 & \ddots & \ddots & \ddots & \ddots & & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \ddots & & \\ \vdots & & 0 & \lambda_{k-1} & -(\lambda_k + \mu_k) & & \mu_{k+1} & 0 & \\ \vdots & & & \ddots & \ddots & & \ddots & \ddots & \ddots \\ \vdots & & & & \ddots & & \ddots & \ddots & \ddots \end{pmatrix}$$

$$\& \; P(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \end{pmatrix}$$

Proof.

$$\begin{aligned} P\left(X_{t+\Delta t} = j\right) &= \sum_{i \in IN} P\left(X_{t+\Delta t} = j / X_t = i\right) P\left(X_t = i\right) \\ &= P\left(X_{t+\Delta t} = j / X_t = j-1\right) P\left(X_t = j-1\right) + \\ &\quad + P\left(X_{t+\Delta t} = j / X_t = j+1\right) P\left(X_t = j+1\right) + \\ &\quad + P\left(X_{t+\Delta t} = j / X_t = j\right) P\left(X_t = j\right) + \\ &\quad + \sum_{\substack{i \in IN \\ i \neq \pm j \\ i \neq j}} P\left(X_{t+\Delta t} = j / X_t = i\right) P\left(X_t = i\right) \\ &= \lambda_{j-1} \Delta t P\left(X_t = j-1\right) + \mu_{j+1} \Delta t P\left(X_t = j+1\right) + \\ &\quad + \left[1 - \left(\lambda_j + \mu_j\right) \Delta t\right] P\left(X_t = j\right) + o\left(\Delta t\right) \end{aligned}$$

$$\begin{aligned} P\left(X_{t+\Delta t} = j\right) - P\left(X_t = j\right) &= \lambda_{j-1} P\left(X_t = j-1\right) \Delta t + \\ &\quad + \mu_{j+1} P\left(X_t = j+1\right) \Delta t + \\ &\quad - \left(\lambda_j + \mu_j\right) P\left(X_t = j\right) \Delta t + o\left(\Delta t\right) \end{aligned}$$

$$\begin{aligned} \frac{P\left(X_{t+\Delta t} = j\right) - P\left(X_t = j\right)}{\Delta t} &= \lambda_{j-1} P\left(X_t = j-1\right) + \mu_{j+1} P\left(X_t = j+1\right) + \\ &\quad - \left(\lambda_j + \mu_j\right) P\left(X_t = j\right) + \\ &\quad + \frac{o\left(\Delta t\right)}{\Delta t} \end{aligned}$$

$P_j'(t) = \lambda_{j-1} P_{j-1}(t) - \left(\lambda_j + \mu_j\right) P_j(t) + \mu_{j+1} P_{j+1}(t), \forall j \geq 2$
(When $\Delta t \longrightarrow 0$)

... ■

let β be an eigenvalue of the matrix A and $x = (x_k)_{k \in IN^*}$ and an associated right eigenvector, so $Ax = \beta x$

$$\begin{cases} -\left(\lambda_1 + \mu_1\right) x_1 + \mu_2 x_2 = \beta x_1 \\ \cdots \\ \lambda_{k-1} x_{k-1} - \left(\lambda_k + \mu_k\right) x_k + \mu_{k+1} x_{k+1} = \beta x_k \quad 2 \leq k \\ \cdots \end{cases}$$

$$\mu_{k+1} x_{k+1} - \left(\lambda_k + \mu_k + \beta\right) x_k + \lambda_{k-1} x_{k-1} = 0, \quad \forall 1 \leq k$$

with $x_0 = 0$ or $\lambda_0 = 0$.

3.2.1 The case of constant birth and death rates: $\lambda_k = \lambda$ and $\mu_k = \mu$

The associated right eigenvectors ($Ax = \beta x$, $x = (x_k)_{k \in IN^*}$) will therefore verify the following system

$$(E_k): \qquad \mu x_{k+1} - (\lambda + \mu + \beta)\, x_k + \lambda x_{k-1} = 0, \qquad \forall 1 \leq k$$

with $x_0 = 0$ and $x_1 \in IK$.

If $\mu \neq 0$, (E_k) is a second-order linear recurrence sequence with constant coefficients.

$(E_k) \Longleftrightarrow x_{k+1} = \left(\frac{\lambda+\mu+\beta}{\mu}\right) x_k - \frac{\lambda}{\mu} x_{k-1}, \quad \forall 1 \leq k$

Applying the results of the last subsection concerning second-order linear recurrence sequences with constant coefficients with $a = \frac{\lambda+\mu+\beta}{\mu}$ and $b = -\frac{\lambda}{\mu}$, we will have

$x_n = u_n = (x_1 - a x_0) \sum\limits_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k} + x_0 \sum\limits_{k=0}^{n} \alpha_1^k \alpha_2^{n-k} \Longrightarrow \forall n \geq 1$

$$x_n = x_1 \sum_{k=0}^{n-1} \alpha_1^k \alpha_2^{n-1-k}$$

such α_1 and α_2 are solutions of $\mu\alpha^2 - (\lambda + \mu + \beta)\, \alpha + \lambda = 0$ (and $x_0 = 0$).

Let $\Delta = (\lambda + \mu + \beta)^2 - 4\lambda\mu$

1st case: If $(\lambda + \mu + \beta)^2 = 4\lambda\mu$, so $\alpha_1 = \alpha_2 = \frac{\lambda+\mu+\beta}{2\mu} = \pm\sqrt{\frac{\lambda}{\mu}}$, and $\forall n \geq 1$

$$x_n = n \left(\frac{\lambda + \mu + \beta}{2\mu}\right)^{n-1} x_1$$

2st case: If $(\lambda + \mu + \beta)^2 \neq 4\lambda\mu$, so $\alpha_1 \neq \alpha_2$, and $\forall n \geq 1$

$$x_n = \left(\frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}\right) x_1$$

such $\alpha_1 = \frac{(\lambda+\mu+\beta)+\Delta^{\frac{1}{2}}}{2\mu} \in IK$ and $\alpha_2 = \frac{(\lambda+\mu+\beta)-\Delta^{\frac{1}{2}}}{2\mu} \in IK$ ($IK = IR$ or $\mathbb{C}$).

$$\frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} \in IR$$

since, $\alpha_1 = \frac{(\lambda+\mu+\beta)+\Delta^{\frac{1}{2}}}{2\mu} \in \mathbb{C} \backslash IR \Longrightarrow \alpha_2 = \frac{(\lambda+\mu+\beta)-\Delta^{\frac{1}{2}}}{2\mu} = \overline{\alpha_1}$ (as $(\lambda, \mu, \beta) \in IR^3$)

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