The estimation of pseudo-differential operators utilising the coupled fractional Fourier transform and a certain inequality

Abstract. In this research work, the symbol class $\Lambda(\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$ is discussed. Then, we give the fundamental properties of this symbol class. Furthermore, the Pseudo-differential operators (p.d.o.) $A(x, y, D'_{x,y})$ and $\mathscr{A}(x, y, D'_{x,y})$ involving the coupled fractional Fourier transform (CFrFT) $\mathcal{F}_{\alpha_1,\alpha_2}$ associated with symbol classes are defined. Finally, inequality and estimate of these operators are also obtained.

Keywords: Coupled fractional Fourier transform, Schwartz space, Sobolev type spaces, pseudo-differential operators.

1 Introduction and motivation

There are many types of integral transformations used in applied mathematics, physics and engineering. Some of the most commonly used are Fourier, Fourier sine, Fourier cosine, Laplace, Mellin, Hilbert, Stieltjes and Hankel. Each of these transformations has unique properties and applications. Also for studies on these transformation (Ata, E., & Kıymaz, I. O.[1], Jafari [2], Ata, E., & Kıymaz, I.O.[3], Watugala, G. K.[4], • Ata, E., & Kıymaz, I.O. [5], Jumarie, G. [6]).
In the last 20 years, nonlinear phenomena in applied mathematics and physics have played a crucial role. There are many techniques to solve a large number of problems that have been written about (Alkan, A., Aktürk, T., & Bulut, H.[7], Alkan, A. [8], Aktürk, T., Alkan, A., Bulut, H., & Güllüoğlu, N. [9], Alkan, A., & Anaç, H. [10], Alkan, A., & Anaç, H.[11], Alkan, A.[12], Avit, Ö., & Anaç, H.[13], Avit, Ö., & Anaç, H.[14], Alkan, A., Kayalar, M., & Bulut, H.[15], Erol, A. S., Anaç, H.,& Olgun, A.[16]).

Firstly, Wiener developed the concept of the fractional Fourier transform (FrFT) in 1929 [17]. In 1980, Namias also explored the FrFT [18] as a means of determining the solutions to certain differential equations that sometimes arise in quantum physics. This transformation is crucial for resolving a number of issues in signal processing, optics, and quantum physics [18–27]. A variety of mathematical analytic fields have examined the FrFT, which is a generalisation of the Fourier transform. Fourier transform of a function $\phi \in L_1(\mathbb{R})$, represented by $\widehat{\phi}$, is described as

$$\widehat{\phi}(\eta) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\eta\zeta}\phi(\zeta)d\zeta$$

so that its inverse is given by

$$\phi(\zeta) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\eta\zeta}\widehat{\phi}(\eta)d\eta < \infty.$$

FrFT [20, 28–36] of a function of a function $\phi \in L_1(\mathbb{R})$ with parametre α, denoted by $(\mathcal{F}_\alpha\phi)(\eta) = \widehat{\phi}_\alpha(\eta)$ is given in $L_1(\mathbb{R})$ as follows:

$$(\mathcal{F}_\alpha\phi)(\eta) = \widehat{\phi}_\alpha(\eta) = \int_{\mathbb{R}} K_\alpha(\zeta,\eta)\phi(\zeta)d\zeta \tag{1}$$

where the kernel $K_\alpha(\zeta,\eta)$ is given by

$$K_\alpha(\zeta,\eta) = \begin{cases} C_\alpha e^{\frac{i(\zeta^2+\eta^2)\cot\alpha}{2} - i\zeta\eta\csc\alpha}, & \alpha \neq n\pi, n \in \mathbb{Z} \\ \frac{1}{\sqrt{2\pi}}e^{-i\zeta\eta}, & \alpha = \frac{\pi}{2} \\ \delta(\zeta-\eta), & \alpha = 2n\pi \\ \delta(\zeta+\eta), & \alpha = (2n+1)\pi, \end{cases}$$

$C_\alpha = \sqrt{\frac{1-i\cot\alpha}{2\pi}}$.
The inverse of $(\mathcal{F}_\alpha\phi)(\eta)$ is as follows:

$$\phi(\zeta) = \int_{\mathbb{R}} \overline{K_\alpha(\zeta,\eta)}(\mathcal{F}_\alpha\varphi)(\eta)d\eta \tag{2}$$

$$\overline{K_\alpha(\zeta,\eta)} = \overline{C}_\alpha e^{\frac{-i(\zeta^2+\eta^2)\cot\alpha}{2} + i\zeta\eta\csc\alpha}$$

and $\overline{K_\alpha(\zeta,\eta)} = K_{-\alpha}(\zeta,\eta)$.
We consider that $\alpha = (\alpha_1,\alpha_2)$, $\mathbf{x} = (x,\eta)$, $\mathbf{y} = (y,\zeta)$,
$\mathcal{K}_\alpha(\mathbf{x},\mathbf{y}) = \mathcal{K}_{\alpha_1}(x,\eta).\mathcal{K}_{\alpha_2}(y,\zeta) = \mathcal{K}_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)$, where $\mathcal{K}_{\alpha_1}(x,\eta)$ and $\mathcal{K}_{\alpha_2}(y,\zeta)$ explained as above.
The coupled fractional Fourier transform (CFrFT) [37–39] is explained as follows

$$[\mathcal{F}_\alpha\phi](\eta,\zeta) = [\mathcal{F}_{\alpha_1,\alpha_1}\phi](\eta,\zeta) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)\phi(x,y)dxdy. \tag{3}$$

The inverse of (3) is defined as follows

$$\phi(x,y) = \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)}[\mathcal{F}_{\alpha_1,\alpha_1}\phi](\eta,\zeta)d\eta d\zeta. \tag{4}$$

Definition 1. *A tempered distribution ϕ belongs to the Sobolev type spce $\mathcal{H}^s(\mathbb{R}\times\mathbb{R})$, and $s \in \mathbb{R}$ if a locally integrable function $(\mathcal{F}_{\alpha_1,\alpha_2}\phi)(\xi,\eta)$ over $\mathbb{R}\times\mathbb{R}$ such that*

$$\|\phi\|_s = \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\{(1+|\xi|^2)(1+|\eta|^2)\}^{\frac{s}{2}}(\mathcal{F}_{\alpha_1,\alpha_2}\phi)(\xi,\eta)\right|^2 d\eta d\xi\right)^{\frac{1}{2}} < \infty. \tag{5}$$

$\mathcal{H}^s(\mathbb{R}\times\mathbb{R})$, is a complete space.

Definition 2. *The collection of all complex valued infinitely differentiable functions which are defined over* $\mathbb{R}\times\mathbb{R}$*, is denoted by* $\mathscr{S}(\mathbb{R}\times\mathbb{R})$*. Now* $\phi(\xi,\eta)\in\mathscr{S}\mathbb{R}\times\mathbb{R}$ *for every selection of* $l_1,\ l_2,\ m_1,\ m_2\in\mathbb{N}_0$ *for which*

$$\Gamma^{l_1,\ l_2}_{m_1,\ m_2}(\phi)=\sup_{(x,y)\in\mathbb{R}\times\mathbb{R}}\left|x^{l_1}y^{l_2}\frac{\partial^{m_1}}{\partial x^{m_1}}\frac{\partial^{m_2}}{\partial y^{m_2}}\phi(x,y)\right|<\infty. \tag{6}$$

The spce $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$ *denotes the dual of* $\mathscr{S}(\mathbb{R}\times\mathbb{R})$*.*

Theorem 1. *We have*
(i) $D^r_{x,y}K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)=\{i(\eta\csc\alpha_1+\zeta\csc\alpha_2)\}^rK_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)$*,*
(ii) $\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(x,y)D^r_{x,y}K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)dxdy$
$=\int_{\mathbb{R}}\int_{\mathbb{R}}K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)(D'_{x,y})^r\varphi(x,y)dxdy,$
(iii) $\mathcal{F}_{\alpha_1,\alpha_2}\{(D'_{x,y})^r\varphi(x,y)\}(\eta,\zeta)=\{i(\eta\csc\alpha_1+\zeta\csc\alpha_2)\}^r(\mathcal{F}_{\alpha_1,\alpha_2}\varphi(x,y))(\eta,\zeta)$*,*
for all $r\in\mathbb{N}$*, where* $D_{x,y}=[\partial_x+\partial_y+i(x\cot\alpha_1+y\cot\alpha_2)]$*,* $D'_{x,y}=-[\partial_x+\partial_y-i(x\cot\alpha_1+y\cot\alpha_2)]$*.*

Proof. Das, S. Mahato, K. and Zayed, A. I have proved the Theorem 1 in [37].

2 Symbol Classes

The class Λ is the set of all functions $a(x,y,\xi,\zeta)\in\mathbb{C}^\infty(\mathbb{R}\times\mathbb{R}\times\mathbb{R}-\{0\}\times\mathbb{R}-\{0\})$ and for $t_1>0,\ t_2>0,\ a(x,y,t_1\xi,t_2\zeta)=a(x,y,\xi,\zeta)$ with

$$\lim_{(|x|,|y|)\to(\infty,\infty)}=a(\infty,\infty,\xi,\zeta)<\infty.$$

$a(\infty,\infty,\xi,\zeta)$ is also a $\mathbb{C}^\infty$-mapping.

Now we introduce $a'(x,y,\xi,\zeta)=a(x,y,\xi,\zeta)-a(\infty,\infty,\xi,\zeta)$, assume the estimates

$$(1+x^2+y^2)^p\left|\frac{\partial^k}{\partial x^k}\frac{\partial^l}{\partial y^l}\frac{\partial^m}{\partial\xi^m}\frac{\partial^n}{\partial\zeta^n}a'(x,y,\xi,\zeta)\right|\le\mathbb{C}_{p,k,l,m,n}, \tag{7}$$

here, p=1,2,3,....., and k, l, m, n are natural numbers.

Theorem 2. *(i) We get* $|a(\infty,\infty,\xi,\zeta)-a(\infty,\infty,\delta,\eta)|\le\mathbb{C}\big((|\xi-\delta|+|\zeta-\eta|)/\big(|\xi|+|\zeta|+|\delta|+|\eta|\big)\big)$*,*
(ii) The estimates $(1+x^2csc^2\alpha_1+y^2csc^2\alpha_2)^p|\mathcal{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)|\le\mathbb{M}_p$*,* $p=1,\ 2,\ 3,\ 4,\ 5...;$

(iii) $(1+x^2csc^2\alpha_1+y^2csc^2\alpha_2)^p|\mathcal{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)-\mathcal{F}_{\alpha_1,\alpha_2}(a')(x,y,\delta,\eta)|\le\mathbb{M}_p(|\xi-\delta|+|\zeta-\eta|)(|\xi|+|\zeta|+|\delta|+|\eta|)^{-1},\quad\forall\xi,\ \zeta,\ \delta,\ \eta\in\mathbb{R}-\{0\},\ \forall x,y\in\mathbb{R},\ p=1,2,...$ *being*

$$\mathcal{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)=\int_{\mathbb{R}}\int_{\mathbb{R}}K_{\alpha_1,\alpha_2}(t,u,x,y)a'(t,u,\xi,\zeta)dtdu,$$

are verified.

Proof. (i) Similar proof of Theorem 1 (a)[40].
(ii) We get the equality

$$\begin{aligned}&(1+x^2csc^2\alpha_1+y^2csc^2\alpha_2)^p\mathcal{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)\\&=\int_{\mathbb{R}}\int_{\mathbb{R}}K_{\alpha_1,\alpha_2}(t,u,x,y)(I-D'_{x,y})^p a'(t,u,\xi,\zeta)dtdu,\\&D'_{x,y}=-\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}-i(xcot\alpha_1+ycot\alpha_2)\right]\end{aligned}\tag{8}$$

and therefore is verified the estimate

$$\begin{aligned}&\left|(1+x^2csc^2\alpha_1+y^2csc^2\alpha_2)^p\mathcal{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)\right|\\&\leq C_{\alpha_1}C_{\alpha_2}\int_{\mathbb{R}}\int_{\mathbb{R}}(1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^q|(I-D'_{x,y})^p a'(t,u,\xi,\zeta)|\\&\quad\times(1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^{-q}dtdu\\&\leq C_{\alpha_1}C_{\alpha_2}C_1\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{(1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^q}dtdu=\mathbb{M}_p\end{aligned}$$

for q sufficient large.
(iii) We get

$$\begin{aligned}&(1+x^2csc^2\alpha_1+y^2csc^2\alpha_2)^p|\mathcal{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)-\mathcal{F}_{\alpha_1,\alpha_2}(a')(x,y,\delta,\eta)|\\&=\int_{\mathbb{R}}\int_{\mathbb{R}}K_{\alpha_1,\alpha_2}(t,u,x,y)(1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^q(I-D'_{x,y})^p\\&\times\left[a'(t,u,\xi,\zeta)-a'(t,u,\delta,\eta)\right](1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^{-q}dtdu.\end{aligned}$$

Let us put now

$$b_{p,q}(t,u,\xi,\zeta)=(1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^q(I-D'_{x,y})^p a'(t,u,\xi,\zeta).\tag{9}$$

We obtain then the estimate

$$\begin{aligned}&(1+x^2csc^2\alpha_1+y^2csc^2\alpha_2)^p\left|\mathcal{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)-\mathcal{F}_{\alpha_1,\alpha_2}(a')(x,y,\delta,\eta)\right|\\&\leq|C_{\alpha_1}C_{\alpha_2}|\int_{\mathbb{R}}\int_{\mathbb{R}}\left|b_{p,q}(t,u,\xi,\zeta)-b_{p,q}(t,u,\delta,\eta)\right|(1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^{-q}dtdu.\end{aligned}$$

Consequently, the estimate

$$\left|b_{p,q}(t,u,\xi,\zeta)-b_{p,q}(t,u,\delta,\eta)\right|\leq D_{\alpha_1,\alpha_2}(|\xi-\delta|+|\zeta-\eta|)(|\xi|+|\zeta|+|\delta|+|\eta|)^{-1}.\tag{10}$$

It can be easily proved from (ii), (9) and (10).

In 1965, Kohn-Nirenberg and Hörmander [41] were the ones who first introduced the pseudo-differential calculus, and later authors expanded on it, primarily in a local context, to examine local regularity and local solvability of PDEs.

Pseudo-differential operators on $\mathbb{R}_+$ are standard or conventional generalizations of partial differential operators or ordinary differential operators and singular integrals.

3 P.D.O. $A(x, y, D'_{x,y})$ related to $\mathcal{F}_{\alpha_1,\alpha_2}$

Let us define, for any $\phi \in \mathcal{S}^2(\mathbb{R})$ and $x, y \in \mathbb{R}$, a mapping $(A(x, y, D'_{x,y})\phi)(x, y)$, by

$$(A(x, y, D'_{x,y})\phi)(x, y) = \int_{\mathbb{R}^2} K_{\alpha_1,\alpha_2}(t, u, x, y) G_{\alpha_1,\alpha_2}(t, u) dt du, \tag{11}$$

where the mapping $G_{\alpha_1,\alpha_2}(t, u)$, given by

$$G_{\alpha_1,\alpha_2}(t, u) = a(\infty, \infty, t, u)\widehat{\phi}_{\alpha_1,\alpha_2}(t, u) + \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{a'}_{\alpha_1,\alpha_2}(t - \xi, u - \eta, t, u)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi, \eta) d\xi d\eta.$$

Evidently, it has to be proved that $G_{\alpha_1,\alpha_2}(t, u)$ is the Coupled fractional Fourier transformable, in fact, we have $G_{\alpha_1,\alpha_2}(t, u) \in L_1(\mathbb{R} \times \mathbb{R})$ as clearly, it is enough to demonstrate that

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\widehat{a'}_{\alpha_1,\alpha_2}(t - \xi, u - \eta, t, u)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi, \eta)\right| d\xi d\eta dt du < \infty;$$

we have in fact, $\forall p = 1, 2, 3, ...$

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\widehat{a'}_{\alpha_1,\alpha_2}(t - \xi, u - \eta, t, u)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi, \eta)\right| d\xi d\eta \\ &\leq \mathbb{M}_p \int_{\mathbb{R}}\int_{\mathbb{R}} (1 + |t - \xi|^2 csc^2\alpha_1 + |u - \eta|^2 \csc^2 \alpha_2)^{-p} |\widehat{\phi}_{\alpha_1,\alpha_2}(\xi, \eta)| d\xi d\eta.\end{aligned}$$

This last expression is the convolution between $(1 + |t|^2 csc^2\alpha_1 + |u|^2 \csc^2 \alpha_2)^{-p}$ and $|\widehat{\phi}_{\alpha_1,\alpha_2}(t, u)|$. When p is large enough, both are integrable.
We obtain

$$\int_{\mathbb{R}^4} \left|\widehat{a'}_{\alpha_1,\alpha_2}(t - \xi, u - \eta, t, u)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi, \eta)\right| d\xi d\eta dt du < \infty.$$

Hence, $A\phi$ is bounded on $\mathscr{S}'(\mathbb{R} \times \mathbb{R})$ and is also continuous. Hence

$$\left[\mathcal{F}_{\alpha_1,\alpha_2}(A)\phi\right](t, u) = a(\infty, \infty, t, u)\widehat{\phi}_{\alpha_1,\alpha_2}(t, u) + \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{a'}_{\alpha_1,\alpha_2}(t - \xi, u - \eta, t, u)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi, \eta) d\xi d\eta$$

is verified in $\mathscr{S}'(\mathbb{R} \times \mathbb{R})$.

4 The P.D.O. $\mathscr{A}$

We introduce an operator $\mathscr{A}$ of $\mathscr{S}(\mathbb{R}\times\mathbb{R})$ in $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$ as follows:

$$[\mathscr{A}\phi](x,y)=\int_{\mathbb{R}}\int_{\mathbb{R}}K_{\alpha_1,\alpha_2}(t,u,x,y)\mathscr{H}_{\alpha_1,\alpha_2}(t,u)dtdu,$$

where, for $\phi\in\mathcal{S}$, the function $\mathscr{H}_{\alpha_1,\alpha_2}$ is introduced as

$$\begin{aligned}&\mathscr{H}_{\alpha_1,\alpha_2}(t,u) \qquad (12)\\ &=a(\infty,\infty,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)+\overline{C_{\alpha_1}C_{\alpha_2}}\int_{\mathbb{R}}\int_{\mathbb{R}}e^{i(t\lambda_1-\lambda_1^2)cot\alpha_1+i(u\lambda_2-\lambda_2^2)cot\alpha_2}\\ &\times\widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2,\end{aligned}$$

$\forall\phi\in\mathscr{S}$ and $t\neq 0,\ u\neq 0\in\mathbb{R}$.
Similary, we can prove that the mapping $\mathscr{A}$ is continuous, bounded.

Theorem 3. *For the symbol a, let $A:L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)$ be a pseudo-differential operator such that $a=\overline{a}$ and $a\geq\gamma$. Then for every $\epsilon>0$, $\exists$ a constant $C'(\epsilon)$ such that for $\phi\in\mathscr{S}(\mathbb{R}\times\mathbb{R})$*

$$Re(A\phi,\phi)_{L^2(\mathbb{R}^2)}+C'(\epsilon)\|\phi\|^2_{-\frac{1}{2}}\geq(\gamma-\epsilon)\|\phi\|^2_{L^2(\mathbb{R}\times\mathbb{R})}$$

is verified.

Proof. It is obvious that we get the inequality

$$a(x,y,\xi,\eta)-\gamma+\epsilon\geq\epsilon$$

We assume $b(x,y,\xi,\eta)=(a(x,y,\xi,\eta)-\gamma+\epsilon)^{\frac{1}{2}},\ x,\ y\in\mathbb{R},\ |\xi|=1$ *and* $|\eta|=1$; for arbitrary $\xi\neq 0,\ \eta\neq 0\in\mathbb{R}$, we put $b(.,.,\xi,\eta)=b(.,.,\frac{\xi}{|\xi|},\frac{\eta}{|\eta|})$. Thus, the homogeneous order of b is zero. Easily verify that

$$\left|\partial x^k\partial y^l\partial\xi^m\partial\eta^n b'(x,y,\xi,\eta)\right|\leq C_{p,k,l,m,n}(1+|x|^2+|y|^2)^{-p}.$$

It also holds for $a'(x,y,\xi,\eta)$
For the symbol $b(x,y,\xi,\eta)$, we assume the operators $B(x,y,D'_{x,y})$ and $\mathscr{B}(x,y,D'_{x,y})$.
We obtain
1) the order of $\mathscr{A}-(\gamma-\epsilon)I-\mathscr{B}.B$ is ≤ 0.
It implies that the order of $BB-b^2(x,y,D'_{x,y},D'_{x,y})$ is also ≤ 0.
2) Let $U:\mathscr{S}(\mathbb{R}\times\mathbb{R})\to\mathscr{S}'(\mathbb{R}\times\mathbb{R})$ with the inequality $||U\phi||_s\leq C||\phi||_{s-1},\ s\in\mathbb{R}$.
Thus, we obtain

$$\mathrm{Re}(U\phi,\ \phi)_0\geq -C'\|\phi\|^2_{-\frac{1}{2}};\ \forall\phi\in\mathscr{S}(\mathbb{R}\times\mathbb{R}).$$

In fact, we obtain obviously the estimate

$$|\mathrm{Re}(U\phi,\ \phi)_0|\leq|(U\phi,\ \phi)_0|\leq\|U\phi\|_s\|\phi\|_{-s}$$

by Schwartz's inequality (Generalized)

$$|(\phi,\ \psi)_0| \leq \|\phi\|_s \|\psi\|_{-s},\ \forall \phi,\ \psi \in \mathscr{S}(\mathbb{R}\times\mathbb{R}).$$

Hence

$$|\mathrm{Re}(U\phi,\ \phi)_0| \leq C_s \|\phi\|_{s-1}\|\phi\|_{-s},\ \forall \text{real } s,\ \phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R});$$

we take $s = \frac{1}{2}$ and obtain

$$|\mathrm{Re}(U\phi,\ \phi)_0| \leq C_{\frac{1}{2}} \|\phi\|^2_{-\frac{1}{2}}$$

therefore is

$$\mathrm{Re}(U\phi,\ \phi)_0 \geq -C_{\frac{1}{2}} \|\phi\|^2_{-\frac{1}{2}}.$$

By combining 1) and 2), we deduce that

$$\mathrm{Re}((A - (\gamma - \epsilon)I - \mathscr{B}.B)\phi,\ \phi)_0 \geq -C_{\frac{1}{2}} \|\phi\|^2_{-\frac{1}{2}},\ \forall \phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R})$$

$$\mathrm{Re}(A\phi,\ \phi)_0 - (\gamma-\epsilon)\|\phi\|_0^2 - \mathrm{Re}(\mathscr{B}.B\phi,\ \phi)_0 \geq -C_{\frac{1}{2}} \|\phi\|^2_{-\frac{1}{2}};$$

$$\mathrm{Re}(A\phi,\ \phi)_0 - (\gamma-\epsilon)\|\phi\|_0^2 - \|B\phi\|_0^2 \geq -C_{\frac{1}{2}} \|\phi\|^2_{-\frac{1}{2}};$$

and therefore

$$\mathrm{Re}(A\phi,\ \phi)_0 + C_{\frac{1}{2}} \|\phi\|^2_{-\frac{1}{2}} \geq (\gamma-\epsilon)\|\phi\|_0^2,\ \forall \phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R}).$$

Theorem 4. *For the symbol a, we consider the operator A. Let be* $\mathscr{K} = max\{|a(s,t,u,v)| : s,\ t \in \mathbb{R}$ *and* $|u| = |v| = 1\}$. *We have the inequality*

$$\|A(x,y,D'_{x,y})\phi\|_0 \leq (\mathscr{K}+\epsilon)\|\phi\|_0 + C_s\|\phi\|_{-1};$$

for $\phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R})$, *is verified, where* C_s *as a constant.*

Proof. In fact, let be $b = a\overline{a} = |a|^2$; Consider $B(x,y,D'_{x,y})$ as the operator related with the symbol b; after that $\overline{\mathscr{A}}$ related to $\overline{a}$. We obtainthat the order of $B - \overline{\mathscr{A}}A$ is ≤ 0.
From Theorem 3 and 2), we get

$$\mathrm{Re}((B - \overline{\mathscr{A}}A)\phi,\ \phi)_{L^2(\mathbb{R}\times\mathbb{R})} \geq -C_{\frac{1}{2}} \|\phi\|^2_{-\frac{1}{2}},\ \forall \phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R})$$

and therefore:

$$\begin{aligned} &\mathrm{Re}(B\phi,\ \phi)_{L^2(\mathbb{R}\times\mathbb{R})} - \mathrm{Re}(\mathscr{A}.A\phi,\ \phi)_{L^2(\mathbb{R}\times\mathbb{R})} \\ &= \mathrm{Re}(B\phi,\ \phi)_0 - \|\phi\|_0^2 \geq -C_{\frac{1}{2}} \|\phi\|^2_{-\frac{1}{2}},\ \forall \phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R}) \end{aligned} \tag{13}$$

Let us consider now the symbol $\alpha = \mathscr{K}^2 - a\overline{a}$ which satisfies obviously the condition of Theorem 3.
Puting $\gamma = 0$ in Theorem 3, $\forall \epsilon' > 0,\ \exists\ C'(\epsilon')$, for $\phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R})$, we get

$$\mathrm{Re}((\mathscr{K}^2 - B)\phi,\ \phi)_0 + C'(\epsilon')\|\phi\|^2_{-\frac{1}{2}} \geq -\epsilon'\|\phi\|_0^2 \tag{14}$$

By adding (13) and (14), we arrive at the inequality

$$\mathscr{K}^2\|\phi\|_0^2 - \|A\phi\|_0^2 + C'(\epsilon')\|\phi\|^2_{-\frac{1}{2}} \geq -C_{\frac{1}{2}}\|\phi\|^2_{-\frac{1}{2}} - \epsilon'\|\phi\|_0^2$$

$$\|A\phi\|_0^2 - (\mathscr{K}^2 + \epsilon')\|\phi\|_0^2 \leq C_1(\epsilon')\|\phi\|^2_{-\frac{1}{2}}$$

$$\|A\phi\|_0^2 \leq (\mathscr{K}^2 + \epsilon')\|\phi\|_0^2 + C_1(\epsilon')\|\phi\|^2_{-\frac{1}{2}},\ \forall\phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R}),\ \forall\epsilon' > 0$$

and applying $\sqrt{f+g} \leq \sqrt{f} + \sqrt{g},\ f,\ g > 0$, we obtain for constant $C_1(\epsilon') > 0$;

$$\|A\phi\|_0 \leq (\mathscr{K} + \sqrt{\epsilon'})\|\phi\|_0 + C_2(\epsilon')\|\phi\|_{-\frac{1}{2}}. \tag{15}$$

On the other hand, $\epsilon'' > 0,\ \exists\gamma(\epsilon'');\ \|\phi\|_{-\frac{1}{2}} \leq \epsilon''\|\phi\|_0 + \gamma(\epsilon'')\|\phi\|_{-1}$ whence we obtain, from (15), the estimate

$$\|A\phi\|_0 \leq (\mathscr{K} + \sqrt{\epsilon'})\|\phi\|_0 + C_2(\epsilon')\epsilon"\|\phi\|_0 + \gamma(\epsilon")C_2(\epsilon'')\|\phi\|_{-1}.$$

Taking ϵ' with $\sqrt{\epsilon'} < \frac{\epsilon}{2}$; and $C_2(\epsilon')\epsilon'' < \frac{\epsilon}{2}$; after that we obtain

$$\|A\phi\|_0 \leq (\mathscr{K} + \epsilon)\|\phi\|_0 + \gamma'(\epsilon)\|\phi\|_{-1},\ \forall\epsilon > 0,\ \forall\phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R}),$$

where $\gamma(\epsilon'')C_2(\epsilon'') = \gamma'(\epsilon)$ is a constant.

Theorem 5. *Let a be a symbol:* $\mathscr{K} = max\{|a| : x,\ y \in \mathbb{R}$ *and* $|\xi| = |\eta| = 1\}$ *and A the associated operator and the collection of operators with order* ≤ 0 *is denoted by* $\mathcal{G}_0$.
Then we have

$$\inf\{\|A(x,y,D'_{x,y}) + T\| : T \in \mathcal{G}_0\} \leq \mathscr{K}.$$

Proof. Actually, we need to prove that if $\forall\epsilon > 0$, then there exists a zero order operator U_ϵ as follows:

$$\|(A + U_\epsilon)\phi\| \leq (\mathscr{K} + \epsilon)\|\phi\|_0, \quad \forall\phi \in L^2(\mathbb{R}\times\mathbb{R}).$$

We construct the operator U_ϵ by using a function $\psi \in C^\infty(\mathbb{R}\times\mathbb{R}),\ \psi_{R_1,R_2}(\xi,\eta)$ depends on the parameter $R_1 > 0$ and $R_2 > 0$, such that $0 \leq \psi_{R_1,R_2}(\xi,\eta) \leq 1,\ \psi_{R_1,R_2}(\xi,\eta) = 1$ for $|\xi| < 2R_1,\ |\eta| < 2R_2,\ \psi_{R_1,R_2}(\xi,\eta) = 0$ for $|\xi| \geq 2R_1,\ |\eta| \geq 2R_2$.
The operator $U_{R_1,R_2} = -A\psi_{R_1,R_2}(D'_{x,y})$ is of order ≤ 0; in fact, we have for

every $\psi \in \mathscr{S}(\mathbb{R}\times\mathbb{R})$, the estimates

$$\begin{aligned}
&\|U_{R_1,R_2}\phi\|_s \\
&= \|A\psi_{R_1,R_2}(D'_{x,y})\phi\|_s \\
&\leq C_{s,\alpha_1,\alpha_2}\|\psi_{R_1,R_2}(D'_{x,y})\phi\|_s \\
&= C_{s,\alpha_1,\alpha_2}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\xi|^2)^s(1+|\eta|^2)^s\psi^2_{R_1,R_2}(\xi,\eta)|\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)|^2 d\xi d\eta\right)^{\frac{1}{2}} \\
&\leq C_{s,\alpha_1,\alpha_2}\left(\int_{|\xi|<2R_1}\int_{|\eta|<2R_2}(1+|\xi|^2)^s(1+|\eta|^2)^s|\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)|^2 d\xi d\eta\right)^{\frac{1}{2}} \\
&= C_{s,\alpha_1,\alpha_2} \\
&\times\left(\int_{|\xi|<2R_1}\int_{|\eta|<2R_2}(1+|\xi|^2)^{s-1}(1+|\eta|^2)^{s-1}|\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)|^2(1+|\xi|^2)(1+|\eta|^2)d\xi d\eta\right)^{\frac{1}{2}} \\
&\leq (1+4R_1^2)(1+4R_2^2)C_{s,\alpha_1,\alpha_2}\|\phi\|_{s-1}.
\end{aligned}$$

By using here above Theorem, we get

$$\begin{aligned}
&\|(A-A\psi_{R_1,R_2}(D'_{x,y}))\phi\|_0 \\
&= \|A(I-\psi_{R_1,R_2}(D'_{x,y}))\phi\|_0 \\
&\leq (\mathscr{K}+\epsilon)\|(I-\psi_{R_1,R_2}(D'_{x,y}))\phi\|_0 + C_{s,\alpha_1,\alpha_2}\|(I-\psi_{R_1,R_2}(D'_{x,y}))\phi\|_{-1}.
\end{aligned}$$

Remark that we have

$$\begin{aligned}
&\|(I-\psi_{R_1,R_2}(D'_{x,y}))\phi\|_0 \\
&= \left(\int_{R_1}\int_{R_2}(1-\psi_{R_1,R_2}(\xi,\eta))^2|\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)|^2 d\xi d\eta\right)^{\frac{1}{2}} \leq \|\phi\|_0,\ \phi\in\in\mathscr{S}(\mathbb{R}\times\mathbb{R})
\end{aligned}$$

and also that

$$\begin{aligned}
&\|(I-\psi_{R_1,R_2}(D'_{x,y}))\phi\|_{-1} \\
&= \left(\int_{\mathbb{R}}\int_{\mathbb{R}}(1-\psi_{R_1,R_2}(\xi,\eta))^2|\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)|^2(1+|\xi|^2)^{-1}(1+|\eta|^2)^{-1}d\xi d\eta\right)^{\frac{1}{2}} \\
&\leq \left(\int_{|\xi|\geq R_1}\int_{|\eta|\geq R_2}|\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)|^2(1+|\xi|^2)^{-1}(1+|\eta|^2)^{-1}d\xi d\eta\right)^{\frac{1}{2}} \\
&\leq \left((1+|R_1|^2)^{-1}(1+|R_2|^2)^{-1}\int_{|\xi|\geq R_1}\int_{|\eta|\geq R_2}|\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)|^2 d\xi d\eta\right)^{\frac{1}{2}} \\
&= (1+|R_1|^2)^{-\frac{1}{2}}(1+|R_2|^2)^{-\frac{1}{2}}\|\phi\|_0.
\end{aligned}$$

Whence we get

$$\|(A+U_{R_1,R_2})\phi\|_0 \leq (\mathscr{K}+\epsilon)\|\phi\|_0 + C_{s,\alpha_1,\alpha_2}(1+|R_1|^2)^{-\frac{1}{2}}(1+|R_2|^2)^{-\frac{1}{2}}\|\phi\|_0.$$

We choose R_1^ϵ and R_2^ϵ such that $C_{s,\alpha_1,\alpha_2}(1+|R_1^\epsilon|^2)^{-\frac{1}{2}}(1+|R_2^\epsilon|^2)^{-\frac{1}{2}} < \epsilon$; hence we get finally

$$\|(A+U_{R_1,R_2})\phi\|_0 \le (\mathscr{K}+2\epsilon)\|\phi\|_0$$

and this proves Theorem 5.

Theorem 6. *If $a(x,y,\xi,\eta)$ is a symbol for $x,\ y,\ \xi \neq 0,\ \eta \neq 0 \in \mathbb{R},\ \Omega$ is an open set, and $\mathscr{K}_\Omega = max\{|a(x,y,\xi,\eta)| : x,\ y \in \Omega$ and $|\xi| = |\eta| = 1\}$. Next, for any $\epsilon > 0$, there exists a constant C_{s,α_1,α_2} and*

$$\|A(x,y,D'_{x,y})\phi\|_0 \le (\mathscr{K}_\Omega+\epsilon)\|\phi\|_0 + C_{s,\alpha_1,\alpha_2}\|\phi\|_{-\frac{1}{2}},\ \forall\phi \in C_0^\infty(\overline{\Omega})$$

be verified.

We deduce this Theorem from Theorem 4 by means of some additional reasonings. We have following

Lemma 1. *For, $\forall\epsilon > 0$ there is an open set $\overline{\Omega} \subset \Omega_\epsilon$ such that the relation $\mathscr{K}_{\Omega_\epsilon} \le \mathscr{K}_\Omega + \epsilon$ is verified.*

Proof. In fact, we have, for every $x_\circ,\ y_\circ \in \mathbb{R},\ |a(x,y,\xi,\eta) - a(x_0,y_0,\xi,\eta)| \le \epsilon$ if $-\delta'_\epsilon < x - x_\circ < \delta'_\epsilon,\ -\delta''_\epsilon < y - y_\circ < \delta''_\epsilon$ and $\xi \neq 0,\ \eta \neq 0 \in \mathbb{R}$; here $\delta'_\epsilon,\ \delta''_\epsilon$ do not dependent on $x_\circ$ and $y_\circ$ respectively.
Let us take

$$\Omega_\epsilon = \Omega$$

$S(x_\circ,\delta'_\epsilon) = \{x : |x - x_\circ| \le \delta'_\epsilon\}$ and $S(y_\circ,\delta''_\epsilon) = \{y : |y - y_\circ| \le \delta''_\epsilon\}$. Therefore, if $t,\ u \in \Omega_\epsilon$, we have $t \in \Omega$ *or* $t \in S(x^*,\delta'_\epsilon)$ and $u \in \Omega$ or $u \in S(y^*,\delta'')$ for certain $x^*,\ y^* \in \partial\Omega$. In the first case, we have $|a(x,y,\xi,\eta)| \le \max\{|a(x,y,\xi,\eta)| : |\xi| = |\eta| = 1,\ x,\ y \in \overline{\Omega}\} = \mathscr{K}_\Omega$.
In 2^{nd} case, we have

$$|a(t,u,\xi,\eta)| \le |a(t,u,\xi,\eta) - a(x^*,y^*,\xi,\eta)| \le \epsilon + \mathscr{K}_\Omega.$$

Hence, for every $t,\ u \in \Omega_\epsilon,\ \xi,\ \eta \in \mathbb{R} - \{0\}$, we get $|a(t,u,\xi,\eta)| \le \epsilon + \mathscr{K}_\Omega$.
Proof of the Theorem: Given $\epsilon > 0$, and $\phi \in C_\circ^\infty(\overline{\Omega})$ we construct Ω_ϵ given in the Lemma. $\exists$, a mapping $\chi_\epsilon(x,y) \in C_\circ^\infty$ as follows:
$\chi_\epsilon(x,y) = \begin{cases} 1, \text{ on } \sup\phi \\ 0, x,\ y \in \Omega_\epsilon \end{cases}$
i.e. $0 \le \chi_\epsilon \le 1$.
Obviously $\chi_\epsilon(.,.)$ is a symbol, and $\gamma_\epsilon = \chi_\epsilon a$ is another symbol.
Furthermore, $\gamma_\epsilon(x,y,\xi,\eta) = 0$ if $x,\ y \in \Omega_\epsilon^c$; hence, we have
$\max\{|\gamma_\epsilon(x,y,\xi,\eta)| :\ x,\ y \in \mathbb{R},\ |\xi| = |\eta| = 1\} \le \max\{|a(x,y,\xi,\eta)| :\ x,\ y \in \mathbb{R},\ |\xi| = |\eta| = 1\} = \mathscr{K}_\Omega \le \mathscr{K}_{\Omega_\epsilon} + \epsilon$.
We define $\Gamma_\epsilon(x,y,D'_{x,y})$ the operator by using γ_ϵ.
We obtain

$$\Gamma_\epsilon = A(\chi_\epsilon).$$

In fact,

$$\begin{aligned}&\mathcal{F}_{\alpha_1,\alpha_2}[\Gamma_\epsilon(x,y,D'_{x,y})\phi](\xi,\eta)\\&=\int_{\mathbb{R}}\int_{\mathbb{R}}K_{\alpha_1,\alpha_1}(x,y,\xi,\eta)(\chi_\epsilon(x,y)a(x,y,\xi,\eta))\phi(x,y)dxdy\\&=\mathcal{F}_{\alpha_1,\alpha_2}[A(x,y,D'_{x,y})(\chi_\epsilon\phi)](\xi,\eta),\ \forall\phi\in\mathscr{S}(\mathbb{R}\times\mathbb{R}),\ \forall\xi\neq0,\ \eta\neq0\in\mathbb{R}.\end{aligned}$$

Hence, we get

$$\Gamma_\epsilon\phi=A(\chi_\epsilon\phi),\ \forall\phi\in\mathscr{S}(\mathbb{R}^2).$$

Thus getting the decomposition

$$\phi=\chi_\epsilon\phi+(1-\chi_\epsilon)\phi$$

and

$$\begin{aligned}A\phi&=A(\chi_\epsilon\phi)+A((1-\chi_\epsilon)\phi)\\&=\Gamma_\epsilon\phi+A((1-\chi_\epsilon)\phi),\end{aligned}$$

as it is $1-\chi_\epsilon(x,y)=0$ on $\sup\phi$, then it is $(1-\chi_\epsilon)\phi=0$ on $\mathbb{R}^2$, and therefore

$$A\phi=\Gamma_\epsilon\phi.$$

Hence, applying Theorem 4, we get

$$\begin{aligned}&\|A\phi\|_0\\&=\|\Gamma_\epsilon(x,y,D'_{x,y})\phi\|_0\\&\leq\left(\max\{|\gamma_\epsilon(x,y,\xi,\eta)|:\ x,\ y\in\mathbb{R}\text{ and }|\xi|=|\eta|=1\}+\epsilon\right)\|\phi\|_0+C_\epsilon\|\phi\|_{-\frac{1}{2}}\\&\leq(K_\Omega+2\epsilon)\|\phi\|_0+C_\epsilon\|\phi\|_{-\frac{1}{2}}\end{aligned}$$

and this proves the Theorem 6.

Theorem 7. *Let $A(x,y,D'_{x,y})$ be operator related with the symbol $a(x,y,\xi,\eta)$ and $\mathcal{G}_0$ be the set of zero order operators. Then we obtain*

$$\inf_{T\in\mathcal{G}_0}\|A+T\|\geq\mathscr{K}.$$

Proof. Combining with Theorem 5 we deduce equality

$$\inf_{T\in\mathcal{G}_0}\|A+T\|=\mathscr{K},$$

which achieves the result of Theorem 7.

5 Conclusion and Discussion

The manuscript is scientifically accurate. The manuscript addresses important theoretical aspects of pseudo-differential operators involving the coupled fractional Fourier transform, focusing on the estimation and a certain inequality. This topic is relevant for functional analysis, operator theory, and applications in mathematical physics and signal processing. The rigorous treatment and results provide new insights that can benefit researchers working with coupled fractional Fourier analysis and pseudo-differential operators. The topic enriches the literature by extending .The estimation of pseudo-differential operators with some inequalities in fractional Fourier spaces.

There are potential directions for future research of my manuscript by using many types of integral transformations (Ata, E., & Kıymaz, I. O.[1], Jafari [2], Ata, E., & Kıymaz, I.O.[3], Watugala, G. K.[4], • Ata, E., & Kıymaz, I.O. [5], Jumarie, G. [6]).

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