Original Research Article

On Coefficient Inequalities of Certain Subclasses of Bi-Univalent Functions Defined Using q-Differential Operator Involving q-Poisson Distribution

ABSTRACT

In this research paper, a new subclass of bi-univalent and analytic functions on unit disc $\Delta = \{z \in \mathbb{C}: |z| < 1\}$ is defined . The zero-truncated Poisson distribution function and q-differential operator are used to define this subclass. Further, initial coefficient bounds $|a_2|$ and $|a_3|$ are estimated, for functions belonging to this subclass.

Keywords: Analytic function, q- differential operator, Bi-univalent function, Taylor-Maclaurin series expansion, coefficient bounds, zero-truncated Poisson differential operator.

1. INTRODUCTION

Let A be the class of analytic functions which are defined on the open unit disc $\Delta = \{z \in \mathbb{C}: |z| < 1\}$, expressed in the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \Delta). \tag{1}$$

Let S be the subset of A which contains functions univalent within Δ. The Koebe One-Quarter Theorem (Duren P.L.(1983)) plays a crucial role here, guaranteeing that the image of Δ under h in S contains a disc of radius $1/4$.

It is well known that, for each function $h \in S$, h^{-1} exists and it is defined as

$$h^{-1}(h(z)) = z, \quad z \in \Delta$$

and

$$h(h^{-1}(w)) = w, \quad |w| < r_0, r_0(h) \geq 1/4$$

where

$$g(w) = h^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{2}$$

If for $h \in S$, both h and h^{-1} are univalent in Δ then h is said to be bi-univalent function. The set of all such bi-univalent functions is denoted by Σ. The study of subclasses of bi-univalent functions and their coefficient bounds has been a compelling topic in mathematical research and this line of inquiry began with Lewin M. (1967) and he proved that $|a_2| < 1.51$. Subsequently, Netanyahu, E. (1969) established that $max|a_2| = 4/3$. Further, Brannan D.A. and Clunie J.G.(Eds.) (1980) conjectured that, if a function $h \in \sum \text{ then } |a_2| \leq \sqrt{2}$. Brannan D.A. and Taha T.S.(1986) introduced two interesting subclasses $S^*(\beta)$ (the class of starlike functions of order β) and $\mathcal{K}(\beta)$ (the class of convex functions of order β) $(0 \leq \beta < 1)$ in Δ (see Netanyahu E. (1969)). Later on, the class $S_{\Sigma}^*(\beta)$, that is the class of bi-starlike function of order β and the class $\mathcal{K}_{\Sigma}(\beta)$ that is the class of bi-convex functions of order β were introduced on Δ. Over time, numerous researchers have explored these classes, deriving initial coefficient bounds. Additionally, several related subclasses of bi-univalent functions have been introduced, with mathematicians obtaining coefficient bounds for these diverse subclasses (see Akgul A. and Altmkaya (2017), Patil A.B. and Naik U.H. (2017) and Khatu R.S., Naik U.H. and Patil A.B. (2022)).

For any real number s and positive real number q $(q \neq 1)$, the number $[s]$ (Kanas S. and Raducanu D. (2014)) is defined as

$$[s] = \frac{1-q^s}{1-q}, \quad [0] = 0. \tag{3}$$

Now the q-number shift factorial for any non-negative number n is defined as

$$[n]! = [n][n-1][n-2]\cdots[2][1], \quad ([0]! = 1). \tag{4}$$

It can be clearly observed that $\lim_{q\to 1}[n] = n$. We assume that q is a fixed number and $q \in (0,1)$.

The q-difference operator or the q-differential operator for $h \in \Delta$ is defined as

$$\partial_q h(z) = \frac{h(qz)-h(z)}{qz-z}, \quad z \in \Delta. \tag{5}$$

It can be easily observed that, for any natural number n and $z \in \Delta$,

$$\partial_q(z^n) = [n]z^{n-1},$$
$$\partial_q\{\sum_{n=1}^{\infty} a_n z^n\} = \sum_{n=1}^{\infty} [n]a_n z^{n-1}. \tag{6}$$

The Gegendauer polynomials $C_n^{\alpha}(x)$ for $n = 2,3,\cdots$ and $\alpha > -\frac{1}{2}$ which is defined as

$$C_0^{\alpha}(x) = 1;$$
$$C_1^{\alpha}(x) = 2\alpha x;$$
$$C_n^{\alpha}(x) = \frac{1}{n}[2x(\alpha + n - 1)C_{n-1}^{\alpha}(x) - (2\alpha + n - 2)C_{n-2}^{\alpha}(x)].$$

By setting $\alpha = \frac{1}{2}$ in above formula, we get Legendre polynomials $P_n(x)$ and for $\alpha = 1$, we get, Chebyshev polynomials of second kind $U_n(x)$.

The generating function of Gegenbauer polynomials is

$$H_{\alpha}(x,z) = \frac{1}{(1-2xz+z^2)^{\alpha}}, \quad z \in \Delta, -1 \leq x \leq 1. \tag{7}$$

For any fixed $x \in [-1,1]$, $H_{\alpha}(x,z)$ is an analytic in Δ; therefore it can be expressed as follows:

$$H_{\alpha}(x,z) = \sum_{n=0}^{\infty} C_n^{\alpha}(x)z^n, \quad z \in \Delta. \tag{8}$$

The probability density function of a discrete random variable X with parameter mean $m > 0$ and which follows a zero-truncated Poissson distribution can be written as

$$\mathcal{P}_m(X = s) = \frac{m^s}{(e^m-1)s!}, \quad s = 1,2,3,\cdots.$$

Ferus Yousef F., Amourah A., Frasin B.A. and Bulboaca T. (2022) introduced the novel power series using above poisson distribution as

$$\mathbb{P}(m,z) := z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(e^m-1)(n-1)!} z^n, \quad z \in \Delta, \tag{9}$$

where $m \in \mathbb{R}^+$ and $\Delta = \{z \in \mathbb{C}: |z| < 1\}$.

Now, we generalize above series by using q-number shift factorial as

$$\mathbb{P}_q(m,z) := z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(e^m-1)[n-1]!} z^n, \quad z \in \Delta, \tag{10}$$

where $m \in \mathbb{R}^+$, $[n]! = [n][n-1]\cdots[2][1] \quad ([0]! = 1)$

and $[t] = \frac{1-q^t}{1-q}, [0] = 0.$

It can be easily observed that, $\lim_{q\to 1}[n] = n$.

Now, we define the linear operator $\chi_{m,q}: A \to A$ as

$$\chi_{m,q}h(z) = \mathbb{P}_q(m,z) * h(z), \quad z \in \Delta, \tag{11}$$

where the symbol "$*$" denotes the Hadamard product of the two series. By putting the series of two operators and using the definition of Hadamard product, we get

$$\chi_{m,q}g(z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}a_n z^n}{(e^m-1)[n-1]!} z^n, \quad z \in \Delta. \tag{12}$$

2. The class $\mathcal{B}_{\Sigma,q}^{\mu}(\lambda, x, \alpha, \delta)$

The Class $\mathcal{B}_{\Sigma,q}^{\mu}(\lambda, x, \alpha, \delta)$ Recently, several researchers have investigated the relationship between Gegenbauer polynomials and other orthogonal polynomials (see Amourah A. (2019), Amourah A., Al Amoush A.G. and Al-Kaseasbeh M. (2021), Altinkaya S. and Yalcin S. (2017), Amourah A., Frasin B.A. and Abdeljawad T. (2021), Amourah A., Frasin B.A., Ahmad M. and Yousef F.(2022), Amourah A. (2020)). However, limited work has been conducted on the connection between Gegenbauer polynomials and bi-univalent functions. Motivated by the work of Ahmad I., Ali Shah S.G., HUssain S., Darus M. and Ahmad B. (2022) on the application of Gegenbauer polynomials and utilizing the operator defined in equation (12), we propose a new class of bi-univalent functions.

Definition 2.1: *Let* $\delta \geq 0, \mu \geq 0, \lambda \geq 1, \frac{1}{2} < x \leq 1, m > 0, q \in \mathbb{R}^+(q \neq 1)$ *and* $\alpha \in \mathbb{R}(\alpha \neq 0)$. *Then,* $h \in \Sigma$ *given by (1) is in the class* $\mathcal{B}_{\Sigma,q}^{\mu}(\lambda, x, \alpha, \delta)$ *if it satisfies*

$$(1-\lambda)\left(\frac{\chi_{m,q}h(z)}{z}\right)^{\mu} + \lambda\partial_q(\chi_{m,q}h(z))\left(\frac{\chi_{m,q}h(z)}{z}\right)^{\mu-1} + \xi\delta\partial_q^2(\chi_{m,q}h(z)) \prec H_\alpha(x,z)$$

and

$$(1-\lambda)\left(\frac{\chi_{m,q}g(w)}{w}\right)^{\mu} + \lambda\partial_q(\chi_{m,q}g(w))\left(\frac{\chi_{m,q}g(w)}{w}\right)^{\mu-1} + \xi\delta\partial_q^2(\chi_{m,q}g(w)) \prec H_\alpha(x,w)$$

where $g(w)$ is given by (2) and $\xi = \frac{2\lambda+\mu}{2\lambda+1}$.

Now by taking particular values of parameters q, μ and δ, we get some interesting subclasses of Σ, few of them are given as follows.

Remarks

1. For $\mu = 1$ and $q = 1$, the class $\mathcal{B}_{\Sigma,q}^{\mu}(\lambda, x, \alpha, \delta)$ reduces to the class $\zeta_{\Sigma}(x, \alpha, \delta, \lambda)$ introduced and studied by Yousef F., Amourah A., Frasin B.A. and Bulboaca T. (2022).

2. For $\delta = 0, \mu = 1$ and $q = 1$, the class $\mathcal{B}_{\Sigma,q}^{\mu}(\lambda, x, \alpha, \delta)$ reduces to the class $\mathcal{B}_{\Sigma,0}^{1}(\lambda, x, \alpha, 0) = \mathcal{B}_{\Sigma}(\lambda, x, \alpha)$ and it is defined as follows:

A function $h \in \Sigma$ is in the class $\mathcal{B}_{\Sigma}(\lambda, x, \alpha)$ if it satisfies the following subordinations.

$$(1-\lambda)\frac{\chi_m h(z)}{z} + \lambda(\chi_m h(z))' \prec H_\alpha(x,z)$$

and

$$(1-\lambda)\frac{\chi_m g(w)}{w} + \lambda(\chi_m g(w))' \prec H_\alpha(x,w)$$

where $\alpha \in (0,\infty), \lambda \in [0,\infty]$ and $x \in \left(\frac{1}{2}, 1\right]$..

3. For $\lambda = 1, \delta = 0, \mu = 1$ and $q = 1$, the class $\mathcal{B}_{\Sigma,q}^{\mu}(\lambda, x, \alpha, \delta)$ reduces to the class $\mathcal{B}_{\Sigma,0}^{1}(1, x, \alpha, 0) = \mathcal{B}_{\Sigma}(x, \alpha)$ and it is defined as follows:

A function $h \in \Sigma$ is said to be in the class $\mathcal{B}_{\Sigma}(x, \alpha)$ if it satisfies the following subordinations.

$$(\chi_m h(z))' \prec H_\alpha(x,z)$$

and

$$(\chi_m g(w))' \prec H_\alpha(x,w)$$

where $\alpha \in (0,\infty)$ and $x \in \left(\frac{1}{2}, 1\right]$.

We use the following lemma to obtain our result.

Lemma 2.1 (Nehari Z.(1952))**:** *Assume that* $f(z) = \sum_{n=1}^{\infty} b_n z^n$, $z \in \Delta$ *is an analytic function in* Δ *such that* $|f(z)| < 1$ *for all* $z \in \Delta$. *Then,* $|b_1| \leq 1$, $|b_n| \leq 1 - |b_1|^2, n \in \mathbb{N} - \{1\}$.

3. Coefficient Bounds for the Function Class $\mathcal{B}_{\Sigma,q}^{\mu}(\lambda, x, \alpha, \delta)$

In this section, we determine bounds for initial coefficients for the function class $\mathcal{B}_{\Sigma,q}^{\mu}(\lambda, x, \alpha, \delta)$.

Theorem 3.1: *If $h \in \sum$ given by (1) and it is from the class $\mathcal{B}_{\Sigma,q}^{\mu}(\lambda, x, \alpha, \delta)$, then*

$$|a_2| \leq \frac{2\sqrt{2|\alpha|x}}{\sqrt{|\beta_1|}} \tag{13}$$

and

$$|a_3| \leq \frac{4|\alpha|x}{|\beta_1|} + \frac{2|\alpha|x(e^m-1)(1+q)}{m^2|\Omega|} \tag{14}$$

where

$$\Omega = \mu + \lambda q + \lambda q^2 + \xi\delta + 2q\xi\delta + 2q^2\xi\delta + q^3\xi\delta,$$

$$\psi = \mu + \lambda q + \xi\delta + \xi\delta q$$

and

$$\beta_1 = \frac{m^2(\mu-1)(\mu+2q\lambda)}{(e^m-1)^2} + \frac{2\Omega m^2}{(e^m-1)(1+q)} - \frac{m^2\psi^2[2(1+\alpha)x^2-1]}{2\alpha x^2(e^m-1)^2}.$$

Proof. Let $h \in \mathcal{B}_{\Sigma,q}^{\mu}(\lambda, x, \alpha, \delta)$.

Then, by definition of the class $\mathcal{B}_{\Sigma,q}^{\mu}(\lambda, x, \alpha, \delta)$,

$$(1-\lambda)\left(\frac{\chi_{m,q}h(z)}{z}\right)^{\mu} + \lambda\partial_q(\chi_{m,q}h(z))\left(\frac{\chi_{m,q}h(z)}{z}\right)^{\mu-1} + \xi\delta\partial_q^2(\chi_{m,q}h(z)) = H_\alpha(x, u(z)), \quad z \in \Delta \tag{15}$$

and

$$(1-\lambda)\left(\frac{\chi_{m,q}g(w)}{w}\right)^{\mu} + \lambda\partial_q(\chi_{m,q}g(w))\left(\frac{\chi_{m,q}g(w)}{w}\right)^{\mu-1} + \xi\delta\partial_q^2(\chi_{m,q}g(w)) = H_\alpha(x, v(w)), \quad w \in \Delta \tag{16}$$

From eqaulities (15) and (16), we obtain

$$(1-\lambda)\left(\frac{\chi_{m,q}h(z)}{z}\right)^{\mu} + \lambda\partial_q(\chi_{m,q}h(z))\left(\frac{\chi_{m,q}h(z)}{z}\right)^{\mu-1} + \xi\delta\partial_q^2(\chi_{m,q}h(z)) = 1 + C_1^\alpha(x)c_1 z + [C_1^\alpha(x)c_2 + C_2^\alpha(x)c_1^2]z^2 + \cdots, \quad z \in \Delta \tag{17}$$

and

$$(1-\lambda)\left(\frac{\chi_{m,q}g(w)}{w}\right)^{\mu} + \lambda\partial_q(\chi_{m,q}g(w))\left(\frac{\chi_{m,q}g(w)}{w}\right)^{\mu-1} + \xi\delta\partial_q^2(\chi_{m,q}g(w)) = 1 + C_1^\alpha(x)d_1 w + [C_1^\alpha(x)d_2 + C_2^\alpha(x)d_1^2]w^2 + \cdots, \quad w \in \Delta \tag{18}$$

where

$$u(z) = \sum_{j=1}^{\infty} c_j z^j, \quad z \in \Delta \quad and \quad v(w) = \sum_{j=1}^{\infty} d_j w^j, \quad w \in \Delta. \tag{19}$$

By applying Lemma (2.1), and using expression of $u(z)$ and $v(w)$ mentioned in (19), we get

$$\left|c_j\right| \leq 1 \quad and \quad \left|d_j\right| \leq 1 \quad \text{for all } j \in \mathbb{N}. \tag{20}$$

Thus, by equating the coefficients of like powers of z and w in (17) and (18), we get

$$\frac{m(\mu+\lambda q+\xi\delta+\xi\delta q)}{(e^m-1)} a_2 = C_1^\alpha(x)c_1, \tag{21}$$

$$\frac{m^2\Omega}{(e^m-1)(1+q)} a_3 + \frac{m^2(\mu-1)(\mu+2q\lambda)}{2(e^m-1)^2} a_2^2 = C_1^\alpha(x)c_2 + C_2^\alpha(x)c_1^2, \tag{22}$$

$$\frac{-m(\mu+\lambda q+\xi\delta+\xi\delta q)}{(e^m-1)} a_2 = C_1^\alpha(x)d_1 \tag{23}$$

and

$$\left[\frac{2\Omega}{(e^m-1)(1+q)} + \frac{(\mu+2q\lambda)(\mu-1)}{2(e^m-1)^2}\right] m^2 a_2^2 - \frac{m^2\Omega}{(e^m-1)(1+q)} a_3 = C_1^\alpha(x)d_2 + C_2^\alpha(x)d_1^2. \tag{24}$$

From (21) and (23), we get

$$c_1 = -d_1 \tag{25}$$

and

$$\frac{2m^2(\mu+\lambda q+\xi\delta+\xi\delta q)^2}{(e^m-1)^2}a_2^2 = [C_1^\alpha(x)]^2(c_1^2+d_1^2). \tag{26}$$

If we add (22) and (24), we get

$$\frac{m^2(\mu-1)(\mu+2q\lambda)}{(e^m-1)^2}a_2^2 + \frac{2m^2\Omega}{(e^m-1)(1+q)}a_2^2 = C_1^\alpha(x)(c_2+d_2) + C_2^\alpha(x)(c_1^2+d_1^2). \tag{27}$$

Substituting the value of $c_1^2+d_1^2$ from (26) in (27), we get

$$\left[\frac{m^2(\mu-1)(\mu+2q\lambda)}{(e^m-1)^2} + \frac{2m^2\Omega}{(e^m-1)(1+q)} - \frac{2m^2\psi^2C_2^\alpha(x)}{[C_1^\alpha(x)]^2(e^m-1)^2}\right]a_2^2 = C_1^\alpha(x)(c_2+d_2). \tag{28}$$

Now using the value of $C_1^\alpha(x)$ and inequalities obtained from (20) in (28), we find that (13) holds.

Now, we subtract (0.24) from (0.22), we get

$$\frac{2m^2\Omega}{(e^m-1)(1+q)}(a_3-a_2^2) = C_1^\alpha(x)(c_2-d_2). \tag{29}$$

Now, using the value of $C_1^\alpha(x)$ and substituting the value of a_2^2 obtained from (28) in (29), we get

$$a_3 = \frac{2\alpha x(c_2+d_2)}{\beta_1} + \frac{\alpha x(e^m-1)(1+q)(c_2-d_2)}{m^2\Omega}. \tag{30}$$

Now, by using inequalities obtained from (20) in (30), we find that (14) holds and it completes the proof.

4.COROLLARIES AND CONSEQUENCES

For particular values of q, μ, δ and λ, in Theorem (3.1), we get many well known results and corollaries and few of them are as follows.

By setting $\mu = 1$ and $q = 1$ in Theorem (3.1), we get the result obtained by Yousef F., Amourah A., Frasin B.A. and Bulboaca T. (2022).

Corollary 4.1: *If a function* $h \in \mathcal{B}_{\Sigma,1}^1(\lambda, x, \alpha, \delta)$ *then*

$$|a_2| \le \frac{2\alpha x(e^m-1)\sqrt{2x}}{m\sqrt{|[2\alpha(1+2\lambda+6\delta)(e^m-1)-2(1+\alpha)(1+\lambda+2\delta)^2]x^2+(1+\lambda+2\delta)^2|}}$$

and

$$|a_3| \le \frac{4\alpha^2x^2(e^m-1)^2}{m^2(1+2\delta+\lambda)^2} + \frac{4\alpha x(e^m-1)}{m^2(1+6\delta+2\lambda)}.$$

By setting $q = 1, \delta = 0$ and $\mu = 1$ in Theorem (3.1), we get the following result obtained by Yousef F., Amourah A., Frasin B.A. and Bulboaca T. (2022).

Corollary 4.2: *If a function* $h \in \mathcal{B}_\Sigma(\lambda, x, \alpha)$ *then*

$$|a_2| \le \frac{2\alpha x(e^m-1)\sqrt{2x}}{m\sqrt{|[2\alpha(1+2\lambda)(e^m-1)-2(1+\alpha)(1+\lambda)^2]x^2+(1+\lambda)^2|}}$$

and

$$|a_3| \le \frac{4\alpha^2x^2(e^m-1)^2}{m^2(1+\lambda)^2} + \frac{4\alpha x(e^m-1)}{m^2(1+2\lambda)}.$$

By setting $\delta = 0, \lambda = 1, \mu = 1$ and $q = 1$ in Theorem (3.1), we get the following result obtained by Yousef F., Amourah A., Frasin B.A. and Bulboaca T. (2022).

Corollary 4.3: *If a function* $h \in \mathcal{B}_\Sigma(x, \alpha)$ *then*

$$|a_2| \le \frac{2\alpha x(e^m-1)\sqrt{2x}}{m\sqrt{|[6\alpha(e^m-1)-8(1+\alpha)]x^2+4|}}$$

and

$$|a_3| \le \frac{\alpha^2x^2(e^m-1)^2}{m^2} + \frac{4\alpha x(e^m-1)}{3m^2}.$$

5. CONCLUSION

In this paper, new q-differential operator is introduced using q-Poisson distribution and a new class $\mathcal{B}_{\Sigma,q}^{\mu}(\lambda, x, \alpha, \delta)$ of analytic and bi-univalent function is defined. Later on, we obtained the coefficient bounds for $|a_2|$ and $|a_3|$. We observed that many interesting results and well-known findings are corollaries of our result. The study of coefficient estimation of a subclass of bi-univalent functions $\mathcal{B}_{\Sigma,q}^{\mu}(\lambda, x, \alpha, \delta)$ can be further extended to the Fekete–Szegö inequality.

DISCLAIMER (ARTIFICIAL INTELLIGENCE)

Authors hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during writing or editing of this manuscript.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.

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