The estimation of pseudo-differential operators utilising the coupled fractional Fourier transform and a certain inequality

Abstract. The symbol class $\Lambda(\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R})$ is discussed. The Pseudo-differential operators (p.d.o.) $A(x,y,D'_{x,y})$ and $\mathscr{A}(x,y,D'_{x,y})$ involving the coupled fractional Fourier transform (CFrFT) $\mathcal{F}_{\alpha_1,\alpha_2}$ associated with symbol classes are defined. Inequality and estimate of these operators are also obtained.

Keywords: Coupled fractional Fourier transform, Schwartz space, Sobolev type spaces, pseudo-differential operators.

1 Introduction and motivation

Firstly, Wiener developed the concept of the fractional Fourier transform (FrFT) in 1929 [1]. In 1980, Namias also explored the FrFT [2] as a means of determining the solutions to certain differential equations that sometimes arise in quantum physics. This transformation is crucial for resolving a number of issues in signal processing, optics, and quantum physics [2–11]. A variety of mathematical analytic fields have examined the FrFT, which is a generalisation of the Fourier transform. Fourier transform of a function $\phi \in L_1(\mathbb{R})$, represented by $\widehat{\phi}$, is described as

$$\widehat{\phi}(\eta) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\eta\zeta}\phi(\zeta)d\zeta$$

so that its inverse is given by

$$\phi(\zeta) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\eta\zeta}\widehat{\phi}(\eta)d\eta < \infty.$$

FrFT [4, 12–20] of a function of a function $\phi \in L_1(\mathbb{R})$ with parametre α, denoted by $(\mathcal{F}_\alpha\phi)(\eta) = \widehat{\phi}_\alpha(\eta)$ is given in $L_1(\mathbb{R})$ as follows:

$$(\mathcal{F}_\alpha\phi)(\eta) = \widehat{\phi}_\alpha(\eta) = \int_{\mathbb{R}} K_\alpha(\zeta,\eta)\phi(\zeta)d\zeta \tag{1}$$

where the kernel $K_\alpha(\zeta,\eta)$ is given by

$$K_\alpha(\zeta,\eta)=\begin{cases} C_\alpha e^{\frac{i(\zeta^2+\eta^2)\cot\alpha}{2}-i\zeta\eta\csc\alpha}, & \alpha\neq n\pi, n\in\mathbb{Z}\\ \frac{1}{\sqrt{2\pi}}e^{-i\zeta\eta}, & \alpha=\frac{\pi}{2}\\ \delta(\zeta-\eta), & \alpha=2n\pi\\ \delta(\zeta+\eta), & \alpha=(2n+1)\pi, \end{cases}$$

$C_\alpha=\sqrt{\frac{1-i\cot\alpha}{2\pi}}$.

The inverse of $(\mathcal{F}_\alpha\phi)(\eta)$ is as follows:

$$\phi(\zeta)=\int_{\mathbb{R}}\overline{K_\alpha(\zeta,\eta)}(\mathcal{F}_\alpha\varphi)(\eta)d\eta \tag{2}$$

$$\overline{K_\alpha(\zeta,\eta)}=\overline{C}_\alpha e^{\frac{-i(\zeta^2+\eta^2)\cot\alpha}{2}+i\zeta\eta\csc\alpha}$$

and $\overline{K_\alpha(\zeta,\eta)}=K_{-\alpha}(\zeta,\eta)$.

We consider that $\alpha=(\alpha_1,\alpha_2)$, $\mathbf{x}=(x,\eta)$, $\mathbf{y}=(y,\zeta)$,

$\mathcal{K}_\alpha(\mathbf{x},\mathbf{y})=\mathcal{K}_{\alpha_1}(x,\eta).\mathcal{K}_{\alpha_2}(y,\zeta)=\mathcal{K}_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)$, where $\mathcal{K}_{\alpha_1}(x,\eta)$ and $\mathcal{K}_{\alpha_2}(y,\zeta)$ explained as above.

The coupled fractional Fourier transform (CFrFT) [21–23] is explained as follows

$$[\mathcal{F}_\alpha\phi](\eta,\zeta)=[\mathcal{F}_{\alpha_1,\alpha_1}\phi](\eta,\zeta)=\int_{\mathbb{R}}\int_{\mathbb{R}}K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)\phi(x,y)dxdy. \tag{3}$$

The inverse of (3) is defined as follows

$$\phi(x,y)=\int_{\mathbb{R}}\int_{\mathbb{R}}\overline{K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)}[\mathcal{F}_{\alpha_1,\alpha_1}\phi](\eta,\zeta)d\eta d\zeta. \tag{4}$$

Definition 1. *A tempered distribution ϕ belongs to the Sobolev type spce $\mathcal{H}^s(\mathbb{R}\times\mathbb{R})$, and $s\in\mathbb{R}$ if a locally integrable function $(\mathcal{F}_{\alpha_1,\alpha_2}\phi)(\xi,\eta)$ over $\mathbb{R}\times\mathbb{R}$ such that*

$$\|\phi\|_s=\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\{(1+|\xi|^2)(1+|\eta|^2)\}^{\frac{s}{2}}(\mathcal{F}_{\alpha_1,\alpha_2}\phi)(\xi,\eta)\right|^2 d\eta d\xi\right)^{\frac{1}{2}}<\infty. \tag{5}$$

$\mathcal{H}^s(\mathbb{R}\times\mathbb{R})$, is a complete space.

Definition 2. *The collection of all complex valued infinitely differentiable functions which are defined over $\mathbb{R}\times\mathbb{R}$, is denoted by $\mathscr{S}(\mathbb{R}\times\mathbb{R})$. Now $\phi(\xi,\eta)\in\mathscr{S}\mathbb{R}\times\mathbb{R}$ for every selection of $l_1,\ l_2,\ m_1,\ m_2\in\mathbb{N}_0$ for which*

$$\mathit{\Gamma}^{l_1,\ l_2}_{m_1,\ m_2}(\phi)=\sup_{(x,y)\in\mathbb{R}\times\mathbb{R}}\left|x^{l_1}y^{l_2}\frac{\partial^{m_1}}{\partial x^{m_1}}\frac{\partial^{m_2}}{\partial y^{m_2}}\phi(x,y)\right|<\infty. \tag{6}$$

The spce $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$ denotes the dual of $\mathscr{S}(\mathbb{R}\times\mathbb{R})$.

Theorem 1. *We have*

(i) $D^r_{x,y}K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)=\{i(\eta\csc\alpha_1+\zeta\csc\alpha_2)\}^r K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)$,

(ii) $\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(x,y) D^r_{x,y} K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta) dxdy$
$= \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(x,y,\eta,\zeta)(D'_{x,y})^r \varphi(x,y) dxdy$,
(iii) $\mathcal{F}_{\alpha_1,\alpha_2}\{(D'_{x,y})^r \varphi(x,y)\}(\eta,\zeta) = \{i(\eta \csc\alpha_1 + \zeta \csc\alpha_2)\}^r(\mathcal{F}_{\alpha_1,\alpha_2}\varphi(x,y))(\eta,\zeta)$, *for all* $r \in \mathbb{N}$, *where* $D_{x,y} = [\partial_x + \partial_y + i(x\cot\alpha_1 + y\cot\alpha_2)]$, $D'_{x,y} = -[\partial_x + \partial_y - i(x\cot\alpha_1 + y\cot\alpha_2)]$.

Proof. See [21].

2 Symbol Classes

The class Λ is the set of all functions $a(x,y,\xi,\zeta) \in \mathbb{C}^\infty(\mathbb{R}\times\mathbb{R}\times\mathbb{R}-\{0\}\times\mathbb{R}-\{0\})$ and for $t_1 > 0,\ t_2 > 0,\ a(x,y,t_1\xi,t_2\zeta) = a(x,y,\xi,\zeta)$ with

$$\lim_{(|x|,|y|)\to(\infty,\infty)} = a(\infty,\infty,\xi,\zeta) < \infty.$$

$a(\infty,\infty,\xi,\zeta)$ is also a $\mathbb{C}^\infty$-mapping.

Now we introduce $a'(x,y,\xi,\zeta) = a(x,y,\xi,\zeta) - a(\infty,\infty,\xi,\zeta)$, assume the estimates

$$(1+x^2+y^2)^p \left| \frac{\partial^k}{\partial x^k}\frac{\partial^l}{\partial y^l}\frac{\partial^m}{\partial \xi^m}\frac{\partial^n}{\partial \zeta^n} a'(x,y,\xi,\zeta)\right| \le \mathbb{C}_{p,k,l,m,n}, \tag{7}$$

here, p=1,2,3,....., and k, l, m, n are natural numbers.

> **Theorem 2.** *(i) We get* $|a(\infty,\infty,\xi,\zeta) - a(\infty,\infty,\delta,\eta)| \le \mathbb{C}\big((|\xi-\delta| + |\zeta-\eta|)/(|\xi|+|\zeta|+|\delta|+|\eta|)\big)$,
> *(ii) The estimates* $(1 + x^2csc^2\alpha_1 + y^2csc^2\alpha_2)^p|\mathcal{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)| \le \mathbb{M}_p$, $p = 1,\ 2,\ 3,\ 4,\ 5...;$

(iii) $(1 + x^2csc^2\alpha_1 + y^2csc^2\alpha_2)^p|\mathcal{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta) - \mathcal{F}_{\alpha_1,\alpha_2}(a')(x,y,\delta,\eta)| \le \mathbb{M}_p(|\xi-\delta| + |\zeta-\eta|)(|\xi|+|\zeta|+|\delta|+|\eta|)^{-1}$, $\forall\xi,\ \zeta,\ \delta,\ \eta \in \mathbb{R}-\{0\}$, $\forall x,y \in \mathbb{R}$, $p = 1,2,...$ *being*

$$\mathcal{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,x,y)a'(t,u,\xi,\zeta)dtdu,$$

are verified.

Proof. (i) Similar proof of Theorem 1 (a)[24].
(ii) We get the equality

$$\begin{aligned} &(1 + x^2csc^2\alpha_1 + y^2csc^2\alpha_2)^p\mathcal{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,x,y)(I - D'_{x,y})^p a'(t,u,\xi,\zeta)dtdu, \\ &D'_{x,y} = -\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - i(xcot\alpha_1 + ycot\alpha_2)\right] \end{aligned} \tag{8}$$

and therefore is verified the estimate

$$\begin{aligned}
&\left|(1+x^2csc^2\alpha_1+y^2csc^2\alpha_2)^p\mathcal{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)\right|\\
&\leq C_{\alpha_1}C_{\alpha_2}\int_{\mathbb{R}}\int_{\mathbb{R}}(1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^q|(I-D'_{x,y})^p a'(t,u,\xi,\zeta)|\\
&\quad\times(1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^{-q}dtdu\\
&\leq C_{\alpha_1}C_{\alpha_2}C_1\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{(1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^q}dtdu=\mathbb{M}_p
\end{aligned}$$

for q sufficient large.
(iii) We get

$$\begin{aligned}
&(1+x^2csc^2\alpha_1+y^2csc^2\alpha_2)^p|\mathcal{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)-\mathcal{F}_{\alpha_1,\alpha_2}(a')(x,y,\delta,\eta)|\\
&=\int_{\mathbb{R}}\int_{\mathbb{R}}K_{\alpha_1,\alpha_2}(t,u,x,y)(1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^q(I-D'_{x,y})^p\\
&\times\left[a'(t,u,\xi,\zeta)-a'(t,u,\delta,\eta)\right](1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^{-q}dtdu.
\end{aligned}$$

Let us put now

$$b_{p,q}(t,u,\xi,\zeta)=(1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^q(I-D'_{x,y})^p a'(t,u,\xi,\zeta). \tag{9}$$

We obtain then the estimate

$$\begin{aligned}
&(1+x^2csc^2\alpha_1+y^2csc^2\alpha_2)^p\left|\mathcal{F}_{\alpha_1,\alpha_2}(a')(x,y,\xi,\zeta)-\mathcal{F}_{\alpha_1,\alpha_2}(a')(x,y,\delta,\eta)\right|\\
&\leq|C_{\alpha_1}C_{\alpha_2}|\int_{\mathbb{R}}\int_{\mathbb{R}}\left|b_{p,q}(t,u,\xi,\zeta)-b_{p,q}(t,u,\delta,\eta)\right|(1+t^2csc^2\alpha_1+u^2csc^2\alpha_2)^{-q}dtdu.
\end{aligned}$$

Consequently, the estimate

$$\left|b_{p,q}(t,u,\xi,\zeta)-b_{p,q}(t,u,\delta,\eta)\right|\leq D_{\alpha_1,\alpha_2}(|\xi-\delta|+|\zeta-\eta|)(|\xi|+|\zeta|+|\delta|+|\eta|)^{-1}. \tag{10}$$

It can be easily proved from (ii), (9) and (10).

In 1965, Kohn-Nirenberg and Hörmander [25] were the ones who first introduced the pseudo-differential calculus, and later authors expanded on it, primarily in a local context, to examine local regularity and local solvability of PDEs.

Pseudo-differential operators on $\mathbb{R}_+$ are standard or conventional generalizations of partial differential operators or ordinary differential operators and singular integrals.

3 P.D.O. $A(x,y,D'_{x,y})$ related to $\mathcal{F}_{\alpha_1,\alpha_2}$

Let us define, for any $\phi \in \mathcal{S}^2(\mathbb{R})$ and $x, y \in \mathbb{R}$, a mapping $(A(x,y,D'_{x,y})\phi)(x,y)$, by

$$(A(x,y,D'_{x,y})\phi)(x,y) = \int_{\mathbb{R}^2} K_{\alpha_1,\alpha_2}(t,u,x,y) G_{\alpha_1,\alpha_2}(t,u)dtdu, \tag{11}$$

where the mapping $G_{\alpha_1,\alpha_2}(t,u)$, given by

$$\begin{aligned} G_{\alpha_1,\alpha_2}(t,u) &= a(\infty,\infty,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u) \\ &\quad + \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{a'}_{\alpha_1,\alpha_2}(t-\xi, u-\eta, t, u)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)d\xi d\eta. \end{aligned}$$

Evidently, it has to be proved that $G_{\alpha_1,\alpha_2}(t,u)$ is the Coupled fractional Fourier transformable, in fact, we have $G_{\alpha_1,\alpha_2}(t,u) \in L_1(\mathbb{R}\times\mathbb{R})$ as clearly, it is enough to demonstrate that

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\widehat{a'}_{\alpha_1,\alpha_2}(t-\xi, u-\eta, t, u)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)\right| d\xi d\eta dt du < \infty;$$

we have in fact, $\forall p = 1, 2, 3, ...$

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\widehat{a'}_{\alpha_1,\alpha_2}(t-\xi, u-\eta, t, u)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)\right| d\xi d\eta \\ &\leq \mathbb{M}_p \int_{\mathbb{R}}\int_{\mathbb{R}} (1 + |t-\xi|^2 csc^2\alpha_1 + |u-\eta|^2 \csc^2\alpha_2)^{-p} |\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)| d\xi d\eta. \end{aligned}$$

This last expression is the convolution between $(1 + |t|^2 csc^2\alpha_1 + |u|^2 \csc^2\alpha_2)^{-p}$ and $|\widehat{\phi}_{\alpha_1,\alpha_2}(t,u)|$. When p is large enough, both are integrable.
We obtain

$$\int_{\mathbb{R}^4} \left|\widehat{a'}_{\alpha_1,\alpha_2}(t-\xi, u-\eta, t, u)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)\right| d\xi d\eta dt du < \infty.$$

Hence, $A\phi$ is bounded on $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$ and is also continuous. Hence

$$\begin{aligned} \left[\mathcal{F}_{\alpha_1,\alpha_2}(A)\phi\right](t,u) &= a(\infty,\infty,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u) \\ &\quad + \int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{a'}_{\alpha_1,\alpha_2}(t-\xi, u-\eta, t, u)\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)d\xi d\eta \end{aligned}$$

is verified in $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$.

4 The P.D.O. $\mathscr{A}$

We introduce an operator $\mathscr{A}$ of $\mathscr{S}(\mathbb{R}\times\mathbb{R})$ in $\mathscr{S}'(\mathbb{R}\times\mathbb{R})$ as follows:

$$[\mathscr{A}\phi](x,y) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_2}(t,u,x,y)\mathscr{H}_{\alpha_1,\alpha_2}(t,u)dtdu,$$

where, for $\phi \in \mathcal{S}$, the function $\mathscr{H}_{\alpha_1,\alpha_2}$ is introduced as

$$\begin{aligned} &\mathscr{H}_{\alpha_1,\alpha_2}(t,u) \qquad (12)\\ = &\, a(\infty,\infty,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(t,u) + \overline{C_{\alpha_1}C_{\alpha_2}} \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t\lambda_1-\lambda_1^2)cot\alpha_1+i(u\lambda_2-\lambda_2^2)cot\alpha_2} \\ &\times \widehat{a'}_{\alpha_1,\alpha_2}(t-\lambda_1,u-\lambda_2,t,u)\widehat{\phi}_{\alpha_1,\alpha_2}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2, \end{aligned}$$

$\forall \phi \in \mathscr{S}$ and $t \neq 0,\ u \neq 0 \in \mathbb{R}$.
Similary, we can prove that the mapping $\mathscr{A}$ is continuous, bounded.

Theorem 3. *For the symbol a, let* $A : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ *be a pseudo-differential operator such that* $a = \overline{a}$ *and* $a \geq \gamma$. *Then for every* $\epsilon > 0$, $\exists$ *a constant* $C'(\epsilon)$ *such that for* $\phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R})$

$$Re(A\phi,\phi)_{L^2(\mathbb{R}^2)} + C'(\epsilon)\|\phi\|^2_{-\frac{1}{2}} \geq (\gamma-\epsilon)\|\phi\|^2_{L^2(\mathbb{R}\times\mathbb{R})}$$

is verified.

Proof. It is obvious that we get the inequality

$$a(x,y,\xi,\eta) - \gamma + \epsilon \geq \epsilon$$

We assume $b(x,y,\xi,\eta) = (a(x,y,\xi,\eta)-\gamma+\epsilon)^{\frac{1}{2}},\ x,\ y \in \mathbb{R},\ |\xi| = 1$ *and* $|\eta| = 1$; for arbitrary $\xi \neq 0,\ \eta \neq 0 \in \mathbb{R}$, we put $b(.,.,\xi,\eta) = b(.,.,\frac{\xi}{|\xi|},\frac{\eta}{|\eta|})$. Thus, the homogeneous order of b is zero. Easily verify that

$$\left|\partial x^k \partial y^l \partial \xi^m \partial \eta^n b'(x,y,\xi,\eta)\right| \leq C_{p,k,l,m,n}(1+|x|^2+|y|^2)^{-p}.$$

It also holds for $a'(x,y,\xi,\eta)$
For the symbol $b(x,y,\xi,\eta)$, we assume the operators $B(x,y,D'_{x,y})$ and $\mathscr{B}(x,y,D'_{x,y})$.
We obtain
1) the order of $\mathscr{A} - (\gamma-\epsilon)I - \mathscr{B}.B$ is ≤ 0.
It implies that the order of $BB - b^2(x,y,D'_{x,y},D'_{x,y})$ is also ≤ 0.
2) Let $U : \mathscr{S}(\mathbb{R}\times\mathbb{R}) \to \mathscr{S}'(\mathbb{R}\times\mathbb{R})$ with the inequality $||U\phi||_s \leq C||\phi||_{s-1},\ s \in \mathbb{R}$.
Thus, we obtain

$$\mathrm{Re}(U\phi,\ \phi)_0 \geq -C'\|\phi\|^2_{-\frac{1}{2}};\ \forall \phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R}).$$

In fact, we obtain obviously the estimate

$$|\mathrm{Re}(U\phi,\ \phi)_0| \leq |(U\phi,\ \phi)_0| \leq \|U\phi\|_s\|\phi\|_{-s}$$

by Schwartz's inequality (Generalized)

$$|(\phi,\ \psi)_0| \leq \|\phi\|_s\|\psi\|_{-s},\ \forall \phi,\ \psi \in \mathscr{S}(\mathbb{R}\times\mathbb{R}).$$

Hence

$$|\mathrm{Re}(U\phi,\ \phi)_0| \leq C_s\|\phi\|_{s-1}\|\phi\|_{-s},\ \forall \text{real } s,\ \phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R});$$

we take $s = \frac{1}{2}$ and obtain

$$|\mathrm{Re}(U\phi,\ \phi)_0| \leq C_{\frac{1}{2}} \|\phi\|^2_{-\frac{1}{2}}$$

therefore is

$$\mathrm{Re}(U\phi,\ \phi)_0 \geq -C_{\frac{1}{2}} \|\phi\|^2_{-\frac{1}{2}}.$$

By combining 1) and 2), we deduce that

$$\mathrm{Re}((A - (\gamma - \epsilon)I - \mathscr{B}.B)\phi,\ \phi)_0 \geq -C_{\frac{1}{2}} \|\phi\|^2_{-\frac{1}{2}},\ \forall \phi \in \mathscr{S}(\mathbb{R} \times \mathbb{R})$$

$$\mathrm{Re}(A\phi,\ \phi)_0 - (\gamma - \epsilon)\|\phi\|^2_0 - \mathrm{Re}(\mathscr{B}.B\phi,\ \phi)_0 \geq -C_{\frac{1}{2}} \|\phi\|^2_{-\frac{1}{2}};$$

$$\mathrm{Re}(A\phi,\ \phi)_0 - (\gamma - \epsilon)\|\phi\|^2_0 - \|B\phi\|^2_0 \geq -C_{\frac{1}{2}} \|\phi\|^2_{-\frac{1}{2}};$$

and therefore

$$\mathrm{Re}(A\phi,\ \phi)_0 + C_{\frac{1}{2}} \|\phi\|^2_{-\frac{1}{2}} \geq (\gamma - \epsilon)\|\phi\|^2_0,\ \forall \phi \in \mathscr{S}(\mathbb{R} \times \mathbb{R}).$$

Theorem 4. *For the symbol a, we consider the operator A. Let be* $\mathscr{K} = max\{|a(s,t,u,v)| : s,\ t \in \mathbb{R}$ *and* $|u| = |v| = 1\}$. *We have the inequality*

$$\|A(x,y,D'_{x,y})\phi\|_0 \leq (\mathscr{K} + \epsilon)\|\phi\|_0 + C_s\|\phi\|_{-1};$$

for $\phi \in \mathscr{S}(\mathbb{R} \times \mathbb{R})$, *is verified, where* C_s *as a constant.*

Proof. In fact, let be $b = a\overline{a} = |a|^2$; Consider $B(x,y,D'_{x,y})$ as the operator related with the symbol b; after that $\overline{\mathscr{A}}$ related to $\overline{a}$. We obtainthat the order of $B - \overline{\mathscr{A}}A$ is ≤ 0.
From Theorem 3 and 2), we get

$$\mathrm{Re}((B - \overline{\mathscr{A}}A)\phi,\ \phi)_{L^2(\mathbb{R}\times\mathbb{R})} \geq -C_{\frac{1}{2}} \|\phi\|^2_{-\frac{1}{2}},\ \forall \phi \in \mathscr{S}(\mathbb{R} \times \mathbb{R})$$

and therefore:

$$\begin{aligned} &\mathrm{Re}(B\phi,\ \phi)_{L^2(\mathbb{R}\times\mathbb{R})} - \mathrm{Re}(\mathscr{A}.A\phi,\ \phi)_{L^2(\mathbb{R}\times\mathbb{R})} \\ &= \mathrm{Re}(B\phi,\ \phi)_0 - \|\phi\|^2_0 \geq -C_{\frac{1}{2}} \|\phi\|^2_{-\frac{1}{2}},\ \forall \phi \in \mathscr{S}(\mathbb{R} \times \mathbb{R}) \end{aligned} \tag{13}$$

Let us consider now the symbol $\alpha = \mathscr{K}^2 - a\overline{a}$ which satisfies obviously the condition of Theorem 3.
Puting $\gamma = 0$ in Theorem 3, $\forall \epsilon' > 0,\ \exists\ C'(\epsilon')$, for $\phi \in \mathscr{S}(\mathbb{R} \times \mathbb{R})$, we get

$$\mathrm{Re}((\mathscr{K}^2 - B)\phi,\ \phi)_0 + C'(\epsilon')\|\phi\|^2_{-\frac{1}{2}} \geq -\epsilon'\|\phi\|^2_0 \tag{14}$$

By adding (13) and (14), we arrive at the inequality

$$\mathscr{K}^2\|\phi\|^2_0 - \|A\phi\|^2_0 + C'(\epsilon')\|\phi\|^2_{-\frac{1}{2}} \geq -C_{\frac{1}{2}} \|\phi\|^2_{-\frac{1}{2}} - \epsilon'\|\phi\|^2_0$$

$$\|A\phi\|_0^2 - (\mathscr{K}^2 + \epsilon')\|\phi\|_0^2 \leq C_1(\epsilon')\|\phi\|^2_{-\frac{1}{2}}$$

$$\|A\phi\|_0^2 \leq (\mathscr{K}^2 + \epsilon')\|\phi\|_0^2 + C_1(\epsilon')\|\phi\|^2_{-\frac{1}{2}},\ \forall\phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R}),\ \forall\epsilon' > 0$$

and applying $\sqrt{f+g} \leq \sqrt{f} + \sqrt{g},\ f,\ g > 0$, we obtain for constant $C_1(\epsilon') > 0$;

$$\|A\phi\|_0 \leq (\mathscr{K} + \sqrt{\epsilon'})\|\phi\|_0 + C_2(\epsilon')\|\phi\|_{-\frac{1}{2}}. \tag{15}$$

On the other hand, $\epsilon'' > 0,\ \exists\gamma(\epsilon'');\ \|\phi\|_{-\frac{1}{2}} \leq \epsilon''\|\phi\|_0 + \gamma(\epsilon'')\|\phi\|_{-1}$ whence we obtain, from (15), the estimate

$$\|A\phi\|_0 \leq (\mathscr{K} + \sqrt{\epsilon'})\|\phi\|_0 + C_2(\epsilon')\epsilon"\|\phi\|_0 + \gamma(\epsilon")C_2(\epsilon'')\|\phi\|_{-1}.$$

Taking ϵ' with $\sqrt{\epsilon'} < \frac{\epsilon}{2}$; and $C_2(\epsilon')\epsilon'' < \frac{\epsilon}{2}$; after that we obtain

$$\|A\phi\|_0 \leq (\mathscr{K} + \epsilon)\|\phi\|_0 + \gamma'(\epsilon)\|\phi\|_{-1},\ \forall\epsilon > 0,\ \forall\phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R}),$$

where $\gamma(\epsilon'')C_2(\epsilon'') = \gamma'(\epsilon)$ is a constant.

Theorem 5. *Let a be a symbol: $\mathscr{K} = max\{|a| : x,\ y \in \mathbb{R}$ and $|\xi| = |\eta| = 1\}$ and A the associated operator and the collection of operators with order ≤ 0 is denoted by $\mathcal{G}_0$.*
Then we have

$$\inf\{\|A(x, y, D'_{x,y}) + T\| : T \in \mathcal{G}_0\} \leq \mathscr{K}.$$

Proof. Actually, we need to prove that if $\forall\epsilon > 0$, then there exists a zero order operator U_ϵ as follows:

$$\|(A + U_\epsilon)\phi\| \leq (\mathscr{K} + \epsilon)\|\phi\|_0, \quad \forall\phi \in L^2(\mathbb{R}\times\mathbb{R}).$$

We construct the operator U_ϵ by using a function $\psi \in C^\infty(\mathbb{R}\times\mathbb{R}),\ \psi_{R_1,R_2}(\xi,\eta)$ depends on the parameter $R_1 > 0$ and $R_2 > 0$, such that $0 \leq \psi_{R_1,R_2}(\xi,\eta) \leq 1,\ \psi_{R_1,R_2}(\xi,\eta) = 1$ for $|\xi| < 2R_1,\ |\eta| < 2R_2,\ \psi_{R_1,R_2}(\xi,\eta) = 0$ for $|\xi| \geq 2R_1,\ |\eta| \geq 2R_2$.
The operator $U_{R_1,R_2} = -A\psi_{R_1,R_2}(D'_{x,y})$ is of order ≤ 0; in fact, we have for every $\psi \in \mathscr{S}(\mathbb{R}\times\mathbb{R})$, the estimates

$$\begin{aligned}
&\|U_{R_1,R_2}\phi\|_s \\
&= \|A\psi_{R_1,R_2}(D'_{x,y})\phi\|_s \\
&\leq C_{s,\alpha_1,\alpha_2}\|\psi_{R_1,R_2}(D'_{x,y})\phi\|_s \\
&= C_{s,\alpha_1,\alpha_2}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\xi|^2)^s(1+|\eta|^2)^s\psi^2_{R_1,R_2}(\xi,\eta)|\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)|^2 d\xi d\eta\right)^{\frac{1}{2}} \\
&\leq C_{s,\alpha_1,\alpha_2}\left(\int_{|\xi|<2R_1}\int_{|\eta|<2R_2}(1+|\xi|^2)^s(1+|\eta|^2)^s|\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)|^2 d\xi d\eta\right)^{\frac{1}{2}} \\
&= C_{s,\alpha_1,\alpha_2} \\
&\times\left(\int_{|\xi|<2R_1}\int_{|\eta|<2R_2}(1+|\xi|^2)^{s-1}(1+|\eta|^2)^{s-1}|\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)|^2(1+|\xi|^2)(1+|\eta|^2)d\xi d\eta\right)^{\frac{1}{2}} \\
&\leq (1+4R_1^2)(1+4R_2^2)C_{s,\alpha_1,\alpha_2}\|\phi\|_{s-1}.
\end{aligned}$$

By using here above Theorem, we get

$$\begin{aligned} &\|(A - A\psi_{R_1,R_2}(D'_{x,y}))\phi\|_0 \\ &= \|A(I - \psi_{R_1,R_2}(D'_{x,y}))\phi\|_0 \\ &\leq (\mathscr{K} + \epsilon)\|(I - \psi_{R_1,R_2}(D'_{x,y}))\phi\|_0 + C_{s,\alpha_1,\alpha_2}\|(I - \psi_{R_1,R_2}(D'_{x,y}))\phi\|_{-1}. \end{aligned}$$

Remark that we have

$$\begin{aligned} &\|(I - \psi_{R_1,R_2}(D'_{x,y}))\phi\|_0 \\ &= \left(\int_{R_1}\int_{R_2}(1 - \psi_{R_1,R_2}(\xi,\eta))^2|\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)|^2 d\xi d\eta\right)^{\frac{1}{2}} \leq \|\phi\|_0,\ \phi \in\in \mathscr{S}(\mathbb{R}\times\mathbb{R}) \end{aligned}$$

and also that

$$\begin{aligned} &\|(I - \psi_{R_1,R_2}(D'_{x,y}))\phi\|_{-1} \\ &= \left(\int_{\mathbb{R}}\int_{\mathbb{R}}(1 - \psi_{R_1,R_2}(\xi,\eta))^2|\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)|^2(1+|\xi|^2)^{-1}(1+|\eta|^2)^{-1} d\xi d\eta\right)^{\frac{1}{2}} \\ &\leq \left(\int_{|\xi|\geq R_1}\int_{|\eta|\geq R_2}|\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)|^2(1+|\xi|^2)^{-1}(1+|\eta|^2)^{-1} d\xi d\eta\right)^{\frac{1}{2}} \\ &\leq \left((1+|R_1|^2)^{-1}(1+|R_2|^2)^{-1}\int_{|\xi|\geq R_1}\int_{|\eta|\geq R_2}|\widehat{\phi}_{\alpha_1,\alpha_2}(\xi,\eta)|^2 d\xi d\eta\right)^{\frac{1}{2}} \\ &= (1+|R_1|^2)^{-\frac{1}{2}}(1+|R_2|^2)^{-\frac{1}{2}}\|\phi\|_0. \end{aligned}$$

Whence we get

$$\|(A + U_{R_1,R_2})\phi\|_0 \leq (\mathscr{K} + \epsilon)\|\phi\|_0 + C_{s,\alpha_1,\alpha_2}(1+|R_1|^2)^{-\frac{1}{2}}(1+|R_2|^2)^{-\frac{1}{2}}\|\phi\|_0.$$

We choose R_1^ϵ and R_2^ϵ such that $C_{s,\alpha_1,\alpha_2}(1+|R_1^\epsilon|^2)^{-\frac{1}{2}}(1+|R_2^\epsilon|^2)^{-\frac{1}{2}} < \epsilon$; hence we get finally

$$\|(A + U_{R_1,R_2})\phi\|_0 \leq (\mathscr{K} + 2\epsilon)\|\phi\|_0$$

and this proves Theorem 5.

Theorem 6. *If $a(x,y,\xi,\eta)$ is a symbol for $x,\ y,\ \xi \neq 0,\ \eta \neq 0 \in \mathbb{R}$, Ω is an open set, and $\mathscr{K}_\Omega = max\{|a(x,y,\xi,\eta)| : x,\ y \in \Omega$ and $|\xi| = |\eta| = 1\}$. Next, for any $\epsilon > 0$, there exists a constant C_{s,α_1,α_2} and*

$$\|A(x,y,D'_{x,y})\phi\|_0 \leq (\mathscr{K}_\Omega + \epsilon)\|\phi\|_0 + C_{s,\alpha_1,\alpha_2}\|\phi\|_{-\frac{1}{2}},\ \forall\phi \in C_0^\infty(\overline{\Omega})$$

be verified.

We deduce this Theorem from Theorem 4 by means of some additional reasonings. We have following

Lemma 1. *For, $\forall\epsilon > 0$ there is an open set $\overline{\Omega} \subset \Omega_\epsilon$ such that the relation $\mathscr{K}_{\Omega_\epsilon} \leq \mathscr{K}_\Omega + \epsilon$ is verified.*

Proof. In fact, we have, for every $x_\circ,\ y_\circ \in \mathbb{R},\ |a(x,y,\xi,\eta) - a(x_0,y_0,\xi,\eta)| \le \epsilon$ if $-\delta'_\epsilon < x - x_\circ < \delta'_\epsilon,\ \ -\delta''_\epsilon < y - y_\circ < \delta''_\epsilon$ and $\xi \neq 0,\ \eta \neq 0 \in \mathbb{R}$; here $\delta'_\epsilon,\ \delta''_\epsilon$ do not dependent on $x_\circ$ and $y_\circ$ respectively.
Let us take

$$\Omega_\epsilon = \Omega$$

$S(x_\circ, \delta'_\epsilon) = \{x : |x - x_\circ| \le \delta'_\epsilon\}$ and $S(y_\circ, \delta''_\epsilon) = \{y : |y - y_\circ| \le \delta''_\epsilon\}$. Therefore, if $t,\ u \in \Omega_\epsilon$, we have $t \in \Omega\ or\ t \in S(x^*, \delta'_\epsilon)$ and $u \in \Omega$ or $u \in S(y^*, \delta'')$ for certain $x^*,\ y^* \in \partial\Omega$. In the first case, we have $|a(x,y,\xi,\eta)| \le \max\{|a(x,y,\xi,\eta)| : |\xi| = |\eta| = 1,\ x,\ y \in \overline{\Omega}\} = \mathscr{K}_\Omega$.
In 2^{nd} case, we have

$$|a(t,u,\xi,\eta)| \le |a(t,u,\xi,\eta) - a(x^*,y^*,\xi,\eta)| \le \epsilon + \mathscr{K}_\Omega.$$

Hence, for every $t,\ u \in \Omega_\epsilon,\ \xi,\ \eta \in \mathbb{R} - \{0\}$, we get $|a(t,u,\xi,\eta)| \le \epsilon + \mathscr{K}_\Omega$.
Proof of the Theorem: Given $\epsilon > 0$, and $\phi \in C^\infty_\circ(\overline{\Omega})$ we construct Ω_ϵ given in the Lemma. $\exists$, a mapping $\chi_\epsilon(x,y) \in C^\infty_\circ$ as follows:
$\chi_\epsilon(x,y) = \begin{cases} 1, \text{ on } \sup\phi \\ 0, x,\ y \in \Omega_\epsilon \end{cases}$
i.e. $0 \le \chi_\epsilon \le 1$.
Obviously $\chi_\epsilon(.,.)$ is a symbol, and $\gamma_\epsilon = \chi_\epsilon a$ is another symbol.
Furthermore, $\gamma_\epsilon(x,y,\xi,\eta) = 0$ if $x,\ y \in \Omega^c_\epsilon$; hence, we have
$\max\{|\gamma_\epsilon(x,y,\xi,\eta)| :\ x,\ y \in \mathbb{R},\ |\xi| = |\eta| = 1\} \le \max\{|a(x,y,\xi,\eta)| :\ x,\ y \in \mathbb{R},\ |\xi| = |\eta| = 1\} = \mathscr{K}_\Omega \le \mathscr{K}_{\Omega_\epsilon} + \epsilon$.
We define $\Gamma_\epsilon(x,y,D'_{x,y})$ the operator by using γ_ϵ.
We obtain

$$\Gamma_\epsilon = A(\chi_\epsilon).$$

In fact,

$$\begin{aligned}
&\mathcal{F}_{\alpha_1,\alpha_2}[\Gamma_\epsilon(x,y,D'_{x,y})\phi](\xi,\eta) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} K_{\alpha_1,\alpha_1}(x,y,\xi,\eta)(\chi_\epsilon(x,y)a(x,y,\xi,\eta))\phi(x,y)dxdy \\
&= \mathcal{F}_{\alpha_1,\alpha_2}[A(x,y,D'_{x,y})(\chi_\epsilon\phi)](\xi,\eta),\ \forall\phi \in \mathscr{S}(\mathbb{R}\times\mathbb{R}),\ \forall\xi \neq 0,\ \eta \neq 0 \in \mathbb{R}.
\end{aligned}$$

Hence, we get

$$\Gamma_\epsilon\phi = A(\chi_\epsilon\phi),\ \forall\phi \in \mathscr{S}(\mathbb{R}^2).$$

Thus getting the decomposition

$$\phi = \chi_\epsilon\phi + (1 - \chi_\epsilon)\phi$$

and

$$\begin{aligned}
A\phi &= A(\chi_\epsilon\phi) + A((1 - \chi_\epsilon)\phi) \\
&= \Gamma_\epsilon\phi + A((1 - \chi_\epsilon)\phi),
\end{aligned}$$

as it is $1 - \chi_\epsilon(x,y) = 0$ on $\sup\phi$, then it is $(1 - \chi_\epsilon)\phi = 0$ on $\mathbb{R}^2$, and therefore

$$A\phi = \Gamma_\epsilon\phi.$$

Hence, applying Theorem 4, we get

$$\begin{aligned}
&\|A\phi\|_0 \\
&= \|\mathit{\Gamma}_\epsilon(x, y, D'_{x,y})\phi\|_0 \\
&\leq \left(\max\{|\gamma_\epsilon(x, y, \xi, \eta)| :\ x,\ y \in \mathbb{R} \text{ and } |\xi| = |\eta| = 1\} + \epsilon \right) \|\phi\|_0 + C_\epsilon \|\phi\|_{-\frac{1}{2}} \\
&\leq (K_\Omega + 2\epsilon)\|\phi\|_0 + C_\epsilon \|\phi\|_{-\frac{1}{2}}
\end{aligned}$$

and this proves the Theorem 6.

Theorem 7. *Let $A(x, y, D'_{x,y})$ be operator related with the symbol $a(x, y, \xi, \eta)$ and $\mathcal{G}_0$ be the set of zero order operators. Then we obtain*

$$\inf_{T \in \mathcal{G}_0} \|A + T\| \geq \mathscr{K}.$$

Proof. Combining with Theorem 5 we deduce equality

$$\inf_{T \in \mathcal{G}_0} \|A + T\| = \mathscr{K},$$

which achieves the result of Theorem 7.

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