A Multifractional SABR Model for Pricing Variance and Volatility Swaps

Abstract

This paper presents a robust methodology for the valuation of options on variance swaps and volatility swaps. While existing literature has often focused on the pricing of the swaps themselves under stochastic volatility models, the valuation of options on these second-order derivative products remains a challenge, particularly within a framework that captures empirical market properties such as volatility clustering and long memory. Our approach addresses this problem by relying on the precise calculation of the optimal exercise prices (or strike rates) for variance and volatility swaps. To achieve this, we introduce and utilize a multifractional extension of the classic SABR model. This enhanced version of the model incorporates a deterministic and time-varying Hurst function, h(t), which allows for the modeling of a richer and more flexible volatility dynamics, capable of replicating the varying persistence phenomena observed in real markets. The proposed methodology is detailed comprehensively. We derive the theoretical formulas for the swap prices, which form the basis for calculating the optimal exercise prices. The valuation of options (call and put) on these swaps is then performed by employing advanced numerical techniques, adapted to the complexity of the multifractional process..

Key words: *SABR-fractional model, Wick-Itô, swaps, volatility option, variance option*

2020MSC Classification: 60G22, 60H20, 60J65

1 Introduction

The management of volatility risk has become a central challenge in quantitative finance. Since the advent of the CBOE Volatility Index (VIX) in 1993, market participants have gained access to a growing ecosystem of volatility-based derivatives. Instruments such as variance and volatility swaps, which are over-the-counter (OTC) agreements to exchange a fixed strike for the realized volatility of an asset, have seen their trading volumes surge as they provide a direct method for hedging volatility exposure.

A substantial body of literature has addressed the fair pricing of these swaps. The dominant approach has been to use stochastic volatility models that are based on standard Brownian motion, thereby assuming that volatility dynamics are **Markovian**. This research includes seminal work under the Heston model [6], [12], [4], the SABR model [5], [2] , and various extensions incorporating features like discrete sampling or jumps

[10], [13]. However, a key limitation of these models is their inability to capture the long-memory properties, or long-range dependence, that are well-documented features of financial asset volatility.

To bridge this gap between classic models and empirical reality, this paper proposes a novel pricing framework for variance and volatility swaps. We depart from the traditional Markovian assumption by employing a **fractional version of the SABR model**. This approach, driven by fractional Brownian motion, is specifically designed to model the long-range dependence inherent in volatile markets. Our work thus provides a more realistic valuation methodology tailored for today's complex and rapidly evolving financial environments.

1.1 Variance swap

A variance swap is a derivative contract in which two parties exchange payments based on changes in the price or volatility of the underlying asset. In trading practitioners use variance trades to speculate on future levels of volatility of an asset, spread traders use swaps to bet on the difference between realized volatility and implied volatility. Hedging traders use swaps to hedge volatility short positions. If the realized volatility is greater than the strike price, then the gains at maturity are positive.

A variance swap works like a simple vanilla swap, one of the two parties involved in a variance swap transaction will bet an amount based on the actual variance of the underlying asset's price changes. The other party will bet a fixed amount called the "strike" specified at the start of the contract so that the net present value (NPV) of the win is zero.

At the end of the contract, the net payment to the counterparty will be a theoretical amount multiplied by the difference between the variance and a fixed amount of volatility settled in cash. Due to margin requirements specified in the contract, some payouts may occur during the term of the contract if the value of the contract exceeds the agreed limits. The variance swaps in mathematical terms is the arithmetic mean of the squared differences from the mean value. The square root of the variance being the standard deviation, the payout of a variance swap will be greater than that of a volatility swap.

1.2 Volatility Swap

A volatility swap is a forward contract on realized future volatility. Although apparently very similar to the variance swap, the volatility swap is more commonly traded in practice. In 2006, the Financial Times published an article quoting a derivatives products trader saying this: "Variance is easier to hedge. Volatility can be a nightmare." Asset volatility is a good way to measure risk and uncertainty, and so such a swap will provide direct exposure to volatility, making it an attractive choice. The gain of the volatility swap is close to the variance swap and is defined as

2 Variance and volatility swaps under the multifractional version of the SABR

Our objective in this part is to determine a better strike price for variance and volatility swaps and options under a fractional version of the SABR model in a rapidly changing stochastic environment.

2.1 Variance swaps and volatility swaps

A variance swap is an over-the-counter contract that guarantees a buyer to exchange a fixed price for a realized difference at expiration.
In contrast, a volatility swap is equal to the variance swap except that the spreads relate to volatility instead of variance. Thus the gains at maturity ($T > 0$) of variance and volatility swaps are given by the equations:

$$S_{var}(T) = (RV - E_{var}) \times N_{var}$$

$$S_{vol}(T) = (\sqrt{RV} - E_{vol}) \times N_{vol}$$

with E_{var} and E_{vol} the strike prices of the variance and the volatility at time 0 and RV is the realized variance until expiration T and N_{var} and N_{vol} are the notional amounts of variance and volatility swaps.
We assume that the variance is realized in continuous time to define the realized variance RV until maturity by:

$$RV = \frac{1}{T}\int_0^T \sigma^2(t)dt \tag{1}$$

where $\sigma^2(t)$ denotes the volatility of the underlying asset at any time t.
The values of variance and volatility swaps at time t with maturity T are defined by

$$S_{var}(t) = \mathbb{E}\left[e^{-r(T-t)}(RV - E_{var}) \times N_{var}|\mathcal{F}_t\right]$$

$$S_{vol}(t) = \mathbb{E}\left[e^{-r(T-t)}(\sqrt{RV} - E_{vol}) \times N_{vol}|\mathcal{F}_t\right]$$

where $\mathbb{E}\left[\cdot|\mathcal{F}_t\right]$ denotes the conditional expectation with respect to the filtration $\mathcal{F}_t$. As the contract must be fair for both parties (condition of non-arbitrage), the strike prices E_{var} and E_{vol} of the variance and volatility swaps must obey:

$$E_{var} = \mathbb{E}\left[RV\right] \tag{2}$$

$$E_{vol} = \mathbb{E}\left[\sqrt{RV}\right] \tag{3}$$

2.2 The SABR-MSV model (Stochastic Alpha Beta Rho -Multifractional Stochastic Volatility)

Here we consider two multifractional versions of the SABR model called SABR-MSV1 and SABR-MSV2 described by the following Stochastic Differential Equation (SDE) systems:

Model 1 (SABR-MSV1) is as follows:
We consider a first multifractional version of the SABR model named SABR-MSV1

$$\begin{cases} dF_t = \sigma^H(t)F_t^\beta\left(\rho dW_t + \sqrt{1-\rho^2}dB_t\right) \\ \\ d\sigma(t) = \alpha\sigma(t)\left(d^\diamond B^h(t) + dW(t)\right) \end{cases}$$

with B_t^h a multifractional Brownian motion, W_t a standard Brownian motion. The differentiability $d^\diamond B_t^h$ is taken in the sense of Wick-Itô. Itô's theorem leads to the following result:

$$\sigma_t = \sigma_0 \exp \alpha \left(B_t^h + W_t - \alpha t \right) \tag{4}$$

Model 2 (SABR-MSV2) is as follows:
We consider another multifractional version of the SABR model called SABR-MSV2 described by the following Stochastic Differential Equation (SDE) system:

$$\begin{cases} dF_t = \sigma^H(t) F_t^\beta \left(\rho dW_t + \sqrt{1-\rho^2} dB_t \right) \\ \\ d \ln(\sigma_t) = \theta \left(\mu - \ln(\sigma_t) \right) dt + \alpha d^\diamond B_t^h + \beta d^\diamond W_t^H + \gamma dZ_t \end{cases}$$

with B_t^h a multifractional Brownian motion, W_t^H a fractional Brownian motion, Z_t is a standard Brownian motion. The differentiability $d^\diamond B_t^h$ and $d^\diamond W_t^H$ is taken in the sense of Wick-Itô. Itô's theorem leads to the following result:

$$\sigma_t = \exp \left(\sigma_0 e^{-\theta t} + \mu \left(1 - e^{-\theta} \right) + a \int_0^t e^{\theta(s-t)} d^\diamond B_s^h + b \int_0^t e^{\theta(s-t)} d^\diamond W_s^H + c \int_0^t e^{\theta(s-t)} dZ_s \right) \tag{5}$$

3 Valuation of variance and volatility swaps under the SABR-MSV model

We are now going to propose a pricing of variance and volatility swaps after giving some reminders about fractional and multifractional Brownian motion.

3.1 Elements of calculations on fractional and multifractional Brownian motion

Definition 3.1. *(Fractional Brownian motion)*
Let H be a constant belonging to the interval $]0,1[$ and I an interval on $\mathbb{R}$. A fractional Brownian motion (mBf) on the interval I of Hurst index H is a Gaussian process centered on I whose covariance, denoted Cov_H is given by:
For any real pair $(t,s) \in I^2$, we have

$$Cov_H(t,s) = \frac{d_H}{2} \left(|t|^{2H} + |s|^{2H} - |t-s|^{2H} \right) \tag{6}$$

where d_H is a positive constant,
and

$$\mathbb{E} \left[\left(B_t^H - B_s^H \right)^2 \right] = Z_H |t-s|^{2H}$$

An approximation of the fractional Brownian motion is given in [?]

$$B_t^H = \int_0^t \mathbf{1}_{0 \le x \le t \le T}(t,X) K_H(t,x) W(dX) \tag{7}$$

with

- if $0 < H < \dfrac{1}{2}$ so

$$K_H(t,X) = \alpha_H \left[\left(\frac{t}{X}\right)^{H-1/2} - \left(H - \frac{1}{2}\right) X^{1/2-H} \int_X^t (Y-X)^{H-1/2} Y^{H-1/2} dY \right] \tag{8}$$

- if $\dfrac{1}{2} < H < 1$ so

$$K_H(t,X) = \left(\frac{H\left(2H-1\right)}{\beta\left(2-2H, H-\dfrac{1}{2}\right)} \right)^{1/2} X^{1/2-H} \int_X^t (Y-X)^{H-3/2} Y^{H-1/2} dY \tag{9}$$

where for all $H \in (0,1)$, $\alpha_H = \left(\dfrac{2H}{\left(1-2H\right)\beta\left(1-2H, H+\dfrac{1}{2}\right)} \right)^{\frac{1}{2}}$

3.2 Variance swap formula under the SABR-MSV1 model

Using 3, we get:

$$E_{var} = \mathbb{E}[RV] = \mathbb{E}\left[\frac{1}{T}\int_0^T \sigma^2(t)dt\right]$$

$$= \mathbb{E}\left[\frac{1}{T}\int_0^T \sigma_0^2 e^{2\left(\alpha B_t^h + \alpha W_t - \alpha^2 t\right)} dt\right]$$

$$= \frac{\sigma_0^2}{T}\mathbb{E}\left[\int_0^T e^{2\left(\alpha B_t^h + \alpha W_t - \alpha^2 t\right)} dt\right]$$

L'intégrale stochastique par rapport au mouvement brownien multifractionnaire est définie de la façon suivante comme dans [8] par

Definition 3.2. *(Intégrale de Wick- Itô)*
Let $Y : \mathbb{R} \longrightarrow (S)^*$ *be a process such that* $t \mapsto Y(t) \diamond W^h(t)$ *is* $(S)^* -$ *integrable on* $\mathbb{R}$. *We say that the process* Y *is* $dB^h -$ *integrable or integrable on* $\mathbb{R}$ *with respect to the multifractional Brownian motion (mBm). The integral of* Y *with respect to* B^h *is defined by:*

$$\int_{\mathbb{R}} Y(s)dB^h(s) := \int_{\mathbb{R}} Y(s) \diamond W^h(s)ds \tag{10}$$

For any interval I *of* $\mathbb{R}$, *we define* $\int_I Y(s)dB^h(s) := \int_{\mathbb{R}} \mathbb{1}_I(s)Y(s)dB^h(s)$

The following theorems are generalizations of Itô's theorem for fractional and multifractional Brownian motion.

Theorem 3.1. *(2.5 dans [1]) (Fractional and multifractional Itô formula)*
Let $f(t,x)$ be a continuously differentiable function with respect to t and twice with respect to x and suppose that the random variables $f(t,B^H(t))$, $\int_0^t \frac{\partial f}{\partial t}(f(t,B^H(t)))dt$, $\int_0^t \frac{\partial^2 f}{\partial x^2}(f(t,B^H(t)))dt$ are in $L^2(u)$ so

$$f(T,B^H(t)) = f(0,0) + \int_0^T \frac{\partial f}{\partial t}(f(t,B^H(t)))dt + \int_0^T \frac{\partial f}{\partial x}(f(t,B^H(t)))dB^H(t)$$

$$+H\int_0^T t^{2H-1}\frac{\partial^2 f}{\partial x^2}(f(t,B^H(t)))dt$$

For $H = \frac{1}{2}$ we recover the classical Itô formula of the standard Brownian motion.

For the multifractional case the formula becomes:

$$f(T,B^H(t)) = f(0,0) + \int_0^T \frac{\partial f}{\partial t}(f(t,B^H(t)))dt + \int_0^T \frac{\partial f}{\partial x}(f(t,B^H(t)))dB^H(t)$$

$$+\frac{1}{2}\int_0^T 2t^{2h-1}(h'(t)t\ln t + h(t))\frac{\partial^2 f}{\partial x^2}(f(t,B^H(t)))dt$$

The following theorem is an approximation of the Wick-Itô multifractional integral.

Theorem 3.2. *(theorem (2.25) dans [**?**])*
Let $h : \mathbb{R} \longrightarrow (0,1)$ be a deterministic function belonging to C^1 and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a deterministic measurable function belonging to $L^1_{loc}(\mathbb{R})$. Let $Z_t := (Z_t)_{t\in\mathbb{R}}$ be the process defined by $Z_t := \int_0^t f(t)d \diamond B^h(t)$. Then Z is a is $(S)^* -$ process which verifies the following equality in $(S)^*$

$$\int_0^t f(t)d^{\diamond}B^h(t) = \sum_{k=0}^{+\infty}\left(\int_0^t f(t)\frac{d}{dt}\left[g_{ek}(t,h(t))\right]dt\right)\langle\cdot,e_k\rangle \tag{11}$$

Moreover Z is a Gaussian process if and only if

$\sum_{k=0}^{+\infty}\left(\int_0^t f(t)\frac{d}{dt}\left[g_{ek}(t,h(t))\right]dt\right)^2 < \infty$ *for all $t \in \mathbb{R}$. In this case we have for all $t \in \mathbb{R}$,*

$$Z_t = \int_0^t f(t)d^{\diamond}B^h(t) \overset{L}{\sim} \mathcal{N}\left(0, \sum_{k=0}^{+\infty}\left(\int_0^t f(t)\frac{d}{dt}\left[g_{ek}(t,h(t))\right]dt\right)^2\right) \tag{12}$$

In particular, Z is a Gaussian process when $f \in C^1(\mathbb{R},\mathbb{R})$.
*However if $f \in C(\mathbb{R})$, by applying Itô's theorem with respect to mBm ([**?**] theorem (6.9)), we obtain the following integration by parts formula*

$$\int_0^t f(t)d^{\diamond}B^h(t) \overset{(L^2)}{=} f(t)B^h(t) - \int_0^t f'(t)dB^h(s)dt \tag{13}$$

and

$$\mathbb{E}\left[Z_t^2\right] = f(t)^2 t^{2h(t)} + \int_0^t\int_0^t f'(t)f'(u)R_h(t,u)dsdu - 2f(t)\int_0^t f'(t)R(t,t)dt \tag{14}$$

We state the following proposition which will help us in the computation of $\mathbb{E}[RV]$

Proposition 3.1. *Let* $F(T) := \int_0^T e^{2\alpha\zeta(t)}dt$ *with* $\zeta(t) = B^h(t) + W(t) - \alpha t$ *and* α *a positive constant. The random variable* $F(T)$ *can be expressed by the multifractional integral Wick-Itô as follows:*

$$F(T) = g(T)e^{2\alpha\zeta(T)} - \frac{1}{2\alpha^2} - \int_0^T g(t)e^{2\alpha\zeta(t)}dB^h(t) - 2\alpha^2 \int_0^T 2t^{2h'(t)-1}(h'(t)t\ln t + h(t))g(t)e^{2\alpha\zeta(t)}dt \tag{15}$$

with $g(x) = Ke^{-2\alpha x} + \frac{1}{2\alpha^2}$, $K \in \mathbb{R}$ *et* $h(x)$ *is a Hurst functional parameter belonging in* $(0,1)$.

Proof. Let's put $F(T) = \int_0^T \exp\left(2\alpha B_t^h + 2\alpha W_t - \alpha^2 t\right)$.

Let $f(t,x) = g(t)\exp\left(2\alpha x + 2W - \alpha^2 t\right)$ for all $g \in C^1(\mathbb{R})$ et $f \in C^{1,2}(\mathbb{R}_+, \mathbb{R})$ and we assume that the random variables $f\left(t, B_t^h, W_t\right)$, $\int_0^t \frac{\partial f}{t}\left(t, B_t^h\right)dt$ and $\int_0^t \frac{\partial^2 f}{t^2}\left(t, B_t^h\right)dt$ are in $L^2(u)$

According to Itô's multifractional theorem (3.1) we have

$$f\left(T, B_t^h\right) = f\left(0, B_0^h\right) + \int_0^T e^{2\alpha\zeta(t)}\left(g'(t) - 2\alpha^2 g(t)\right)d + \int_0^T 2\alpha g(t)e^{2\alpha\zeta(t)}dB_t^h$$

$$+2\alpha^2 \int_0^T 2t^{2h_t-1}\left(h'(t)t\ln sh(t)\right)g(t)e^{2\alpha\zeta(t)}dt$$

with $g(t) = ke^{-2\alpha t} + \frac{1}{2\alpha^2}$ solution of the differential equation $g'(t) - 2\alpha^2 g(t) = 1$. Which allows us to write

$$F(T) = g(T)e^{2\alpha\zeta(T)} - \frac{1}{2\alpha^2} - \int_0^T g(t)e^{2\alpha\zeta(t)}dB^h(t) - 2\alpha^2 \int_0^T 2t^{2h(t)-1}\left(h'(t)t\ln t + h(t)\right)g(t)e^{2\alpha\zeta(t)}$$

□

From the previous proposition we can state the following theorem

Theorem 3.3. *Under the SABR multifractional model, the strike price* E_{var} *of variance swaps with maturity* T *is given by the following formula*

$$\frac{\sigma_0^2}{T}\left(g(T)e^{2\alpha\zeta(T)} - \frac{1}{2\alpha^2} - 2\alpha^2\mathbb{E}\left[\int_0^T p(t)g(t)e^{2\alpha\zeta(t)}dt\right] - \mathbb{E}\left[\int_0^T g(t)e^{2\alpha\zeta(t)}dt\right]\right) \tag{16}$$

with $p(t) = t^{h'(t)-1}(h'(t)t\ln t + h(t))$

Proof. Using the proposition 3.1, we get

$$E_{var} = \mathbb{E}\left[RV\right] = \frac{\sigma_0^2}{T}\mathbb{E}\left[\int_0^T e^{2\alpha\zeta(t)}dt\right]$$

$$= \frac{\sigma_0}{T}\mathbb{E}\left[g(T)e^{2\alpha\zeta(T)} - \frac{1}{2\alpha^2} - \int_0^T g(t)e^{2\alpha\zeta(t)}dt - 2\alpha^2\int_0^T 2p(t)g(t)e^{2\alpha\zeta(t)}dt\right]$$

$$= \frac{\sigma_0^2}{T}\left(g(T)e^{2\alpha\zeta(T)} - \frac{1}{2\alpha^2} - 2\alpha^2\mathbb{E}\left[\int_0^T p(t)g(t)e^{2\alpha\zeta(t)}dt\right] - \mathbb{E}\left[\int_0^T g(t)e^{2\alpha\zeta(t)}dt\right]\right)$$

because

$$\mathbb{E}\left[\int_0^T g(t)e^{2\alpha\zeta(t)}dB^h(t)\right] = 0$$

with $g(s)$, $\zeta(s)$, $p(t)$ are defined above and $h = h(t)$ is the Hurst functional parameter for the mBm. □

Once the strike price of the variance swaps has been calculated, we can now calculate that of the volatility by an approximation. For this we use the following approximation (see [3], [9], [7] and [11] for more details).

The formula that allows the price of volatility swaps is given by the following theorem

Theorem 3.4. *Under the multi-fractional SABR model, the strike price E_{vol} of volatility swaps with maturity T is given by*

$$E_{vol} = \frac{9}{8}\left(E_{var}\right)^{1/2} - \frac{\sigma_0^2}{8T^2}\left(E_{var}\right)^{-3/2}\left(\frac{1}{4\alpha^4}\psi(T)\left(1 - \frac{1}{\alpha^2} - 6\alpha^2\mathbb{E}\left[\phi(T)\right]\right) + \mathbb{E}\int_0^T \psi^2(t)dt\right.$$

$$\left. + 4\mathbb{E}[\phi(T)] + \mathbb{E}\left[\phi^2(T)\right]\right)$$

with $p(t) = t^{h'(t)-1}(h'(t)t\ln t + h(t))$, $\phi(T) = \int_0^T p(t)g(t)dt$ et $\psi(T) = g(T)e^{2\alpha\zeta(T)}$

Proof. We have by definition $E_{vol} = \mathbb{E}[\sqrt{RV}]$. The approximation in [3] and [11] show that $\mathbb{E}[\sqrt{X}] \simeq \sqrt{\mathbb{E}[X]} - \frac{Var[X]}{8\left(\mathbb{E}[X]\right)^{3/2}}$

Using this approximation, we can write

$$E_{vol} = \sqrt{\mathbb{E}_{var}} - \frac{Var[RV]}{8\left(\mathbb{E}_{var}\right)^{3/2}} \tag{17}$$

let's calculate $Var[RV]$

$$By\ definition\ Var[RV] = \mathbb{E}[RV^2] - \mathbb{E}[RV]^2$$

$$Recall\ that\ \mathbb{E}[RV] = E_{\text{var}}\ so$$

$$\mathbb{E}[RV^2] = \mathbb{E}\left[\left(\frac{1}{T}\int_0^T \sigma^2(t)dt\right)^2\right]\ with\ \sigma(t) = \sigma_0 e^{\alpha B^h(t) + \alpha W(t) - \alpha^2 t}$$

According to proposition 3.1 the integral $\int_0^T e^{2\alpha(B_t^h)+W_t-\alpha^2 t}dt$ can be approximated by

$$\int_0^T e^{2\alpha(B_t^h)+W_t-\alpha^2 t}dt = g(T)e^{2\alpha\zeta(T)} - \frac{1}{2\alpha^2} - \int_0^T g(t)e^{2\alpha\zeta(t)}dB^h(t)$$

$$- 2\alpha^2 \int_0^T 2t^{2h'(t)-1}\left(h'(t)t\ln t + h(t)\right) g(t)e^{2\alpha\zeta(t)}dt$$

$$\left(\int_0^T e^{2\alpha(B_t^h)+W_t-\alpha^2 t}dt\right)^2 \simeq \int_0^T g^2(t)e^{4\alpha\zeta(t)}ds + \frac{1}{4\alpha^4} + g^2(T)e^{4\alpha\zeta(T)}$$

$$+ \left(\int_0^T 2t^{h'(t)-1}\left(h'(t)t\ln t + h(t)\right) g(t)e^{2\alpha\zeta(t)}dt\right)^2 - \frac{1}{\alpha^2}g(T)e^{2\alpha\zeta(T)}$$

$$- 6\alpha^2 g(T)e^{2\alpha\zeta(T)} \int_0^T 2t^{h'(t)-1}\left(h'(t)t\ln t + h(t)\right) g(t)e^{2\alpha\zeta(t)}dt$$

$$+ 4\int_0^T 2t^{h'(t)-1}\left(h'(t)t\ln t + h(t)\right) g(t)e^{2\alpha\zeta(t)}dt$$

$$- 2g(T)e^{2\alpha\zeta(T)} \int_0^T g(t)e^{2\alpha\zeta(t)}dB^h(t) + \frac{1}{\alpha^2}\int_0^T g(t)e^{2\alpha\zeta(t)}dB^h(t)$$

$$+ 2\left(\int_0^T g(t)e^{2\alpha\zeta(t)}dB^h(t)\right) \times \left(2\alpha^2 \int_0^T 2t^{h'(t)-1}\left(h'(t)t\ln t + h(t)\right) g(t)e^{2\alpha\zeta(t)}dt\right)$$

So

$$\mathbb{E}[RV^2] = \frac{\sigma_0^2}{T^2}\left(\frac{1}{4\alpha^4} + g^2(T)e^{4\alpha\zeta(T)} - \frac{1}{\alpha^2}g(T)e^{2\alpha\zeta(T)} - 6\alpha^2 g(T)e^{2\alpha\zeta(T)}\mathbb{E}\left[\int_0^T p(t)g(t)e^{2\alpha\zeta(t)}dt\right]\right.$$

$$\left. + \mathbb{E}\left[\int_0^T g^2(t)e^{4\alpha\zeta(t)}dt\right] + 4\mathbb{E}\left[\int_0^T p(t)g(t)e^{2\alpha\zeta(t)}dt\right] + \mathbb{E}\left[\left(\int_0^T 2p(t)g(t)e^{2\alpha\zeta(t)}dt\right)^2\right]\right)$$

because

$$\mathbb{E}\left[g(T)e^{2\alpha\zeta(T)} \int_0^T g(t)e^{2\alpha\zeta(t)}dB^h(t) + \frac{1}{\alpha^2}\int_0^T g(t)e^{2\alpha\zeta(t)}dB^h(t)\right] = 0$$

$$\mathbb{E}\left[\left(\int_0^T g(t)e^{2\alpha\zeta(t)}dB^h(t)\right) \times \left(2\alpha^2 \int_0^T 2t^{h'(t)-1}\left(h'(t)t\ln t + h(t)\right) g(t)e^{2\alpha\zeta(t)}dt\right)\right] = 0$$

replacing this result in 17, we get

$$E_{vol} = \frac{9}{8}\left(E_{var}\right)^{1/2} - \frac{\sigma_0^2}{8T^2}\left(E_{var}\right)^{-3/2}\left(\frac{1}{4\alpha^4}\psi(T)\left(1 - \frac{1}{\alpha^2} - 6\alpha^2\mathbb{E}\left[\phi(T)\right]\right) + \mathbb{E}\int_0^T \psi^2(t)dt\right.$$

$$+4\mathbb{E}[\phi(T)] + \mathbb{E}\left[\phi^2(T)\right]$$

□

3.3 Approximation of variance and volatility options

Having evaluated the strike prices of variance and volatility swaps, we can now evaluate the prices of variance and volatility options.
By definition the price of a variance option at expiration is given by

$$Call_{var} = Payoff_{var} = \max\left(RV - E_{var}, 0\right) \tag{18}$$

and the price of a volatility option is given by

$$Call_{vol} = Payoff_{vol} = \max\left(\sqrt{RV} - E_{vol}, 0\right) \tag{19}$$

An approximation of the variance and volatility options is in the following proposition

Proposition 3.2. *Under the multifractional SABR model, the prices of the variance and volatility options are given by*

$$Call_{var} = \max\left(\frac{\sigma_0^2}{T}\left(\int_0^T g(t)e^{2\alpha\zeta(t)}dB^h(t) + 2\mathbb{E}\left[\int_o^T p(t)g(t)e^{2\alpha\zeta(t)}dt\right] + \mathbb{E}\left[\int_0^T g(t)e^{2\alpha\zeta(t)}dt\right]\right.\right.$$

$$\left.\left. - \int_0^T g(t)e^{2\alpha\zeta(t)}dt - 2\alpha^2\int_0^T p(t)g(t)e^{2\alpha\zeta(t)}\right), 0\right)$$

$$Call_{vol} = \max\left(\frac{\sigma_0}{\sqrt{T}} \times \sqrt{\psi(T) - \frac{1}{2\alpha^2}\int_0^T \psi(t)dt - 2\alpha^2\phi(T)\int_0^T \psi(t)dB^h(t)} - \frac{9}{8}\sqrt{E_{var}}\right.$$

$$-\frac{\sigma_0^2}{8T^2}\left(E_{var}\right)^{-3/2}\left(\frac{1}{4\alpha^4}\psi(T)\left(1 - \frac{1}{\alpha^2} - 6\alpha^2\mathbb{E}\left[\phi(T)\right]\right)\right) + \mathbb{E}\left[\int_0^T \psi^2(t)dt\right]$$

$$\left.+\mathbb{E}\left[\phi(T)\right] + \mathbb{E}\left[\phi^2(T)\right]; 0\right)$$

with $\phi(T) = \int_0^T p(t)g(t)e^{2\alpha\zeta(t)dt}$, $\psi(T) = g(T)e^{2\alpha\zeta(T)}$, $g(t) = Ke^{-2\alpha t} + \frac{1}{\alpha^2}$,
$\zeta(t) = \alpha B^h(t) + \alpha W(t) - \alpha^2 t$, *and* $h = h(t)$ *is a Hurst parameter for mBm.*

Proof. This is a direct application of the proposition 3.1 □

3.4 Variance swap formula under the SABR-MSV2 model

Using 3, we get:

$$E_{var} = \mathbb{E}[RV] = \mathbb{E}\left[\frac{1}{T}\int_0^t \sigma^2(t)dt\right]$$

$$= \mathbb{E}\left[\frac{1}{T}\int_0^T \exp 2\left(\sigma_0 e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right) + a\int_0^t e^{\theta(s-t)}d^{\diamond}B_s^h + b\int_0^t e^{\theta(s-t)}d^{\diamond}W_s^H + c\int_0^t e^{\theta(s-t)}dZ_s\right)\right]$$

By using the Taylor expansion of the function e^x to order 1, and after some elementary calculations, we obtain the following formula:

$$\begin{aligned}
E_{var} = \mathbb{E}\Bigg[&\frac{2}{\theta}\sigma_0\left(1-e^{-\theta T}\right)+2\mu\left(T-\frac{1}{\theta}\left(1-e^{-\theta T}\right)\right)+2ae^{-\theta t}\int_0^T e^{\theta s}B^h(s)ds+2be^{-\theta t}\int_0^T e^{\theta s}W^H(s)ds \\
&-2a\theta e^{-\theta t}\int_0^T\left(\int_0^t e^{\theta s}B^h(s)ds\right)dt-2b\theta e^{-\theta t}\int_0^T\left(\int_0^t e^{\theta s}W^H(s)ds\right)dt \\
&+2a\int_0^T\left(\int_0^t e^{\theta(s-t)}W^H(s)dZ(s)\right)dt\Bigg] \\
&=\frac{2}{\theta}\sigma_0\left(1-e^{-\theta T}\right)(1-\mu)+2\mu T+2\mathbb{E}\Bigg[ae^{-\theta t}\int_0^T e^{\theta s}B^h(s)ds+be^{-\theta t}\int_0^T e^{\theta s}W^H(s)ds \\
&-a\theta e^{-\theta t}\int_0^T\left(\int_0^u e^{\theta s}B^h(s)ds\right)du-b\theta e^{-\theta t}\int_0^T\left(\int_0^u e^{\theta s}W^H(s)ds\right)du \\
&+a\int_0^T\left(\int_0^u e^{\theta(s-t)}dZ(s)\right)du\Bigg]
\end{aligned}$$

$$\begin{aligned}
E_{var} = &\frac{2}{\theta}\sigma_0\left(1-e^{-\theta T}\right)(1-\mu)+2\mu T+2e^{-\theta t}\mathbb{E}\Bigg[a\int_0^T e^{\theta s}B^h(s)ds+b\int_0^T e^{\theta s}W^H(s)ds \\
&-a\theta\int_0^T\left(\int_0^u e^{\theta s}B^h(s)ds\right)du-b\theta\int_0^T\left(\int_0^u e^{\theta s}W^H(s)ds\right)du+a\int_0^T\left(\int_0^u e^{\theta s}Z(s)ds\right)du\Bigg]
\end{aligned}$$

3.5 Volatility swap formula under the SABR-MSV2 model

Still using the following approximation

$$E_{vol}=\sqrt{E_{var}}-\frac{Var(RV)}{8(E_{var})^{3/2}}$$

we can approximate E_{vol} as follows

$$var(RV)=\mathbb{E}(RV^2)-\mathbb{E}(RV)^2 \quad avec \quad RV^2=\frac{1}{T^2}\left(\int_0^T\sigma^2(s)ds\right)^2 \quad et \ \mathbb{E}(RV)^2=E_{var}$$

The approximation of $\mathbb{E}(RV^2)$ uses the same techniques as in the model 1.

4 Conclusion

In this article we have proposed a multifractional SABR model to calculate the strike prices of variance and volatility swaps using three Hurst functionals. The objective is to better price variance swap options and volatility swap options in situations of high volatility or low volatility. This allows investors to better hedge volatility risks. We plan for future work to conduct numerical simulations and an analysis of the sensitivity and complexity of the method.

References

[1] **Bender, Christian**. An itô formula for generalized functionals of a fractional brownian motion with arbitrary hurst parameter. *Stochastic Processes and their Applications*, 104:81–106, 03 2003.

[2] **Bossoney, Simon**. Volatility swap under the sabr model. *arXiv preprint arXiv:1303.6090*, 2013.

[3] **Dufresne, Daniel**. The log-normal approximation in financial and other computations. *Advances in applied probability*, 36(3):747–773, 2004.

[4] **Elliott, Robert J and Lian, Guang-Hua**. Pricing variance and volatility swaps in a stochastic volatility model with regime switching: discrete observations case. *Quantitative Finance*, 13(5):687–698, 2013.

[5] **Hagan, Patrick S and Kumar, Deep and Lesniewski, Andrew S and Woodward, Diana E**. Managing smile risk. *The Best of Wilmott*, 1:249–296, 2002.

[6] **Heston, Steven L**. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The review of financial studies*, 6(2):327–343, 1993.

[7] **Kim, See-Woo and Kim, Jeong-Hoon**. Volatility and variance swaps and options in the fractional sabr model. *The European Journal of Finance*, 26(17):1725–1745, 2020.

[8] **Lebovits, Joachim and Levy Vehel, Jacques**. Stochastic calculus with respect to multifractional brownian motion. *Stochastics An International Journal of Probability and Stochastic Processes*, 86, 03 2011.

[9] **Lin, Sha and He, Xin-Jiang**. Analytically pricing variance and volatility swaps with stochastic volatility, stochastic equilibrium level and regime switching. *Expert Systems with Applications*, 217:119592, 2023.

[10] **Pun, Chi Seng and Chung, Shing Fung and Wong, Hoi Ying**. Variance swap with mean reversion, multifactor stochastic volatility and jumps. *European Journal of Operational Research*, 245(2):571–580, 2015.

[11] **Schürger, Klaus**. Laplace transforms and suprema of stochastic processes. *Advances in finance and stochastics: essays in honour of Dieter Sondermann*, pages 285–294, 2002.

[12] **Swishchuk, Anatoliy**. Modeling of variance and volatility swaps for financial markets with stochastic volatilities. *WILMOTT magazine*, 2:64–72, 2004.

[13] **Zheng, Wendong and Kwok, Yue Kuen**. Closed form pricing formulas for discretely sampled generalized variance swaps. *Mathematical Finance*, 24(4):855–881, 2014.