Viscosity solution and stochastic HJB equation to maximize the expectation of the utility function for the fractional SABR model

Abstract

This work addresses the fundamental problem of portfolio optimization in quantitative finance, which aims to maximize returns for a given level of risk. Building on previous research, this study analyzes a portfolio composed of a risky asset and a less-risky asset. The main contribution is to solve the optimization problem by using the Hamilton-Jacobi-Bellman (HJB) equation to approximate the solution of a fractional stochastic partial differential equation (FSPDE), which models the optimal investment strategy.
Keywords: *Optimal stochastic control; fractional stochastic Hamilton-Jacobi-Bellman equation, viscosity solution.*

2020 MSC Classification: 60G22, 60H20, 60J65

1 Introduction

In quantitative finance, determining an optimal investment strategy represents a fundamental challenge. The objective is to maximize a portfolio's return for a given level of risk. The model we study, inspired by [2], consists of a quantity λ_t^0 of a less-risky asset and a quantity λ_t^1 of a risky asset.

Our previous research [12] established that the optimization of this portfolio is governed by a fractional stochastic partial differential equation (FSPDE), identified as equation (42). Its solution was approximated via the Hamilton-Jacobi-Bellman (HJB) equation. The following developments are conducted under hypotheses H1 to H4, previously defined in [12].

$$(H_1): \quad \left|\frac{\partial p(t,x)}{\partial t} - \frac{\partial p(t,y)}{\partial t}\right| \le K_1 |x-y|$$

$$(H_2): \quad \left|\frac{\partial p(t,x)}{\partial x}\right| \le K_2$$

$$(H_3): \quad \left|\frac{\partial^2 p(t,x)}{\partial x^2}\right| \le K_3$$

$$(H_4): \quad \left|\frac{\partial^3 p(t,x)}{\partial x^3}\right| \le K_4$$

where K_1, K_2, K_3, K_4 are positive real constants.
Under the above hypotheses, we obtained relation (42) given by:

$$\frac{\varepsilon^2 \delta_t^1 p(t, F_{t,\Delta}^{\circ,H}) \left(\frac{\partial p(t, F_{t,\Delta}^{\circ,H})}{\partial F_{t,\Delta}^{\circ,H}} \right)^2}{\delta_t^0 + \delta_t^1 p(t, F_{t,\Delta}^{\circ,H})} =$$

$$\frac{\partial p(t, F_{t,\Delta}^{\circ,H})}{\partial t} + Ht^{2H-1}\sigma_t^2 (F_{t,\Delta}^{\circ,H})^2 \frac{\partial^2 p(t, F_{t,\Delta}^{\circ,H})}{\partial (F_{t,\Delta}^{\circ,H})^2} \sqrt{1-\rho^2} M(t) \frac{\partial p(t, F_{t,\Delta}^{\circ,H})}{\partial F_{t,\Delta}^{\circ,H}} \tag{1}$$

$$\varepsilon = \sqrt{1-\rho^2}\eta \Delta^{H-1/2} \sigma_t F_{t,\Delta}^{\circ,H}$$

and

$$M(t) = C\sqrt{2H} \left(H - \frac{1}{2} \right) \int_{-\infty}^{t} (t - s + \Delta)^{H-3/2} dW_s$$

In this paper, we seek to find an approximate solution to (1), which is a continuation of our work in [12].

2 Problem Formulation

Let the pair be $\theta_t(t) = (\lambda_t^0, \lambda_t^1)$ with $t \in \mathbb{R}_+$. The terminal wealth of the portfolio is given by the relation $R_{u,t} = u + \int_0^T \lambda_t^1 dF_t$ with

$$\mathbb{E}\left[U\left(R_{u,T}\right)\right] = \mathbb{E}\left[U\left(R_{u,t} = u + \int_0^T \lambda_t^1 dF_t\right)\right] \tag{2}$$

and

$$F_t = F_0 \exp\left\{\rho \int_0^t \sigma(x) dB(x) + \sqrt{1-\rho^2} \int_0^t \sigma(x) dZ(x) - \frac{1}{2}\left(1 - 2\rho^2\sqrt{1-\rho^2}\right) \int_0^t \sigma^2(x) dx\right\} \tag{3}$$

and $\sigma^H(t) = \alpha\sigma_0 e^{\gamma W_t^H}$, where B_t and Z_t are standard Brownian motions and W_t^H is a fractional Brownian Motion.

Let $(\Omega, \mathcal{A}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space, $S_{tr}(t,x)$ the set of admissible and self-financing strategies, and $U \subset S_{tr}(t,x)$. Let us consider the maximization of the following function:

$$\mathbb{E}\left[\int_0^T S_{tr}\left(t, X_t, a, \theta_t\right) dt + K\left(X_T, a, \theta_t\right)\right] \tag{4}$$

with $a \in \Omega$ and $\theta_t \in U$, $S_{tr} : [0,T] \times \mathbb{R} \times \Omega \times U \longrightarrow \mathbb{R}$ and $K : \mathbb{R} \times \Omega \times U \longrightarrow \mathbb{R}$, and where X_t is a process given by the following SDE:

$$\begin{cases} dX_t = \alpha\left(t, X_t, a, \theta_t\right) dt + \beta\left(t, X_t, a, \theta_t\right) dW_t^H \\ X_0 = x \end{cases} \tag{5}$$

where W_t^H is a fBm (fractional Brownian motion).
Let $\mathcal{L}(t,x) = \sup_{\theta_t \in U} \mathbb{E}\left[\int_0^T S_{tr}\left(t, X_t, a, \theta_t\right) ds + K\left(X_T, a, \theta_t\right)\right]$ be a solution to the problem (4). We

note that these types of problems have been studied by [8], [5], [4], [10],[3], [7]. Under certain conditions in [8],[3] and [6], the authors showed that $\mathcal{L}(t,x)$ satisfies the following HJB equation:

$$\frac{-\partial \mathcal{L}}{\partial t}(t,x) = \sup_{\theta_t \in U}\left[\frac{1}{2}\beta^2(t,x,\theta_t)\frac{\partial^2 \mathcal{L}}{\partial x^2}(t,x) + \alpha(t,x,\theta_t)\frac{\partial \mathcal{L}}{\partial x}(t,x) + S_{tr}(t,x,\theta_t)\right] \tag{6}$$

with the terminal condition $\mathcal{L}(x,T) = g(x)$ where $g : \mathbb{R} \longrightarrow \mathbb{R}$.
In [7], the authors showed that if the coefficients of (4) are stochastic and under certain regularity conditions, $\mathcal{L}(t,x)$ is a semimartingale given by:

$$d\mathcal{L}(t,x) = F(t,x)\,dt + G(t,x)\,dW_t \tag{7}$$

with $x \in \mathbb{R}$, where $F(t,x)$ and $G(t,x)$ are $\mathcal{F}_t$-adapted processes and $\mathcal{L}(t,x)$ satisfies the following stochastic HJB equation:

$$-d\mathcal{L}(t,x) = \sup_{\theta_t \in U}\left[\frac{1}{2}\sigma_t^2\frac{\partial^2 \mathcal{L}}{\partial x^2}(t,x) + \alpha_t\frac{\partial \mathcal{L}}{\partial x}(t,x) + \beta_t\frac{\partial G}{\partial x}(t,x) + S_{tr}(t,x,a,\theta_t)\right]dt - G(t,x)dW_t \tag{8}$$

with the terminal condition $\mathcal{L}(t,x) = g(x)$ where $g : \mathbb{R} \longrightarrow \mathbb{R}$.
The existence and uniqueness of a viscosity solution to (8) was demonstrated in [5], [11]. We consider a portfolio consisting of a less-risky asset that follows the dynamics F_t^0 and a risky asset that follows $F_{t,\Delta}^{\circ,H}$, given by the dynamics of the following fractional SABR model: $dF_t^0 = rF_t^0 dt$ and

$$\begin{cases} dF_t = \sigma_t^H F_t^v\left(\rho dB_t + \sqrt{1-\rho^2}dZ_t\right) \\ \sigma_t^H = \alpha\sigma_0 e^{\gamma W_t^H} \end{cases} \tag{9}$$

See [12] for more details.
We recall here the formulation for maximizing the expected utility of terminal wealth [11].

Definition 2.1. *Let W_t^H be a fBm (fractional Brownian motion). W_t^H is a Gaussian process with continuous paths such that $W_0 = 0$ and $W_{t+1}^H - W_t^H$ is independent of $\sigma\left(W_s^H, s \leq t, s \in T\right)$. By definition, the fBm is neither a martingale nor a semimartingale.*

The authors in [1] showed that the fBm can be approximated by a semimartingale by inserting a shift. The benefit of this approximation is significant as it allows avoiding the arbitrage opportunities that arise from using the fBm.
Let B_t^H be a fractional Brownian motion. We approximate B_t^H by a semimartingale as:

$$\hat{Z}_{t,\Delta}^{\circ,H} = C\sqrt{2H}\int_{-\infty}^{t}\left[(t-s+\Delta)_+^{H-\frac{1}{2}} - (s)_+^{H-\frac{1}{2}}\right]dW_s \tag{10}$$

We have

$$\tilde{F}_{t,\Delta}^{\circ,H} = p\left(t, F_{t,\Delta}^{\circ,H}\right) \tag{11}$$

and

$$F_{t,\Delta}^{\circ,H} = \exp\left\{f(t) - g\left(t, Z_{t,\Delta}^{\circ,H}\right)\right\} \tag{12}$$

with

$$f(t,\rho) = -\frac{1}{2}\left(1 - 2\rho^2\sqrt{1-\rho^2}\right)\int_0^t \sigma_s^H dZ_s^{\circ,H} \tag{13}$$

and

$$g\left(t, Z_{t,\Delta}^{\circ,H}\right) = \rho \int_0^t \sigma_s^H dB_s + \sqrt{1-\rho^2} dZ_s^{\circ,H} \tag{14}$$

and

$$Z_t^{\circ,H} = \sqrt{2H} \int_0^t (t-s)^{H-\frac{1}{2}} dB_s \tag{15}$$

See [12] for more details.
Let $p : [0,T] \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a function of class $C^{1,2}$ for $t \in [0,T]$. Considering the equation

$$dF_t = \mu F_t dt + \sigma F_t dB_t^H, \tag{16}$$

we can rewrite it using equation (1) as

$$d\tilde{F}_{t,\Delta}^{\circ,H} = \mu(t)\tilde{F}_{t,\Delta}^{\circ,H} dt + \nu(t)\tilde{F}_{t,\Delta}^{\circ,H} dB_t \tag{17}$$

with

$$\mu(t) = \frac{1}{p\left(t, F_{t,\Delta}^{\circ,H}\right)} \times \left[\frac{\partial p\left(t, F_{t,\Delta}^{\circ,H}\right)}{\partial t} + Ht^{2H-1}\sigma_t^2 \left(F_{t,\Delta}^{\circ,H}\right)^2 \times \frac{\partial^2 p\left(t, F_{t,\Delta}^{\circ,H}\right)}{\partial \left(F_{t,\Delta}^{\circ,H}\right)^2} \sqrt{1-\rho^2} \times M(t) \frac{\partial p\left(t, F_{t,\Delta}^{\circ,H}\right)}{\partial F_{t,\Delta}^{\circ,H}}\right] \tag{18}$$

and

$$\nu(t) = \frac{1}{p\left(t, F_{t,\Delta}^{\circ,H}\right)} \left[\frac{\partial p\left(t, F_{t,\Delta}^{\circ,H}\right)}{\partial F_{t,\Delta}^{\circ,H}} \left(\sqrt{1-\rho^2}\eta\Delta^{H-\frac{1}{2}} \sigma_t^H F_{t,\Delta}^{\circ,H}\right)\right] \tag{19}$$

and

$$M(t) = C\sqrt{2H}\left(H - \frac{1}{2}\right) \int_{-\infty}^t (t-s+\Delta)^{H-3/2} dW_s \tag{20}$$

3 Maximization of the Utility of Terminal Wealth

Let λ_t^0 be the proportion of the less-risky asset and λ_t^1 the quantity of the risky asset. By definition, we have:

$$R_{t,\Delta} = \lambda_t^0 F_t^0 + \lambda_t^1 d\hat{F}_{t,\Delta}^{\circ,H} \tag{21}$$

and using the self-financing property, we can rewrite:

$$dR_{t,\Delta} = \lambda_t^0 dF_t + \lambda_t^1 d\hat{F}_{t,\Delta}^{\circ,H} \tag{22}$$

with

$$\int_0^T \left|\lambda_t^0\right| dt < +\infty \ a.s \ and \ \int_0^T \left(\lambda_t^1\right)^2 dt < +\infty \ a.s \tag{23}$$

We have

$$\lambda_t^0 = \frac{R_{t,\Delta} - \lambda_t^1 \hat{F}_{t,\Delta}^{\circ,H}}{F_t^0} \tag{24}$$

(23) and (24) yield

$$dR_{t,\Delta} = \left[\lambda_t^1 \mu(t) \hat{F}_{t,\Delta}^{\circ,H} + r\left(F_{t,\Delta}^{\circ,H} - \lambda_t^1 \hat{F}_{t,\Delta}^{\circ,H}\right)\right] dt + \lambda_t^1 \sigma_t^H \hat{F}_{t,\Delta}^{\circ,H} dW_t. \tag{25}$$

Let

$$\psi_t = \lambda_t^1 \hat{F}_{t,\Delta}^{\circ,H} \tag{26}$$

be the amount of the risky asset in the portfolio.
(25) and (26) allow us to establish:

$$dR_{t,\Delta} = a_t\left(R_{t,\Delta}\right) dt + b_t\left(R_{t,\Delta}\right) dW_t^H \tag{27}$$

with

$$b_t\left(R_{t,\Delta}\right) = \psi_t \sigma_t^H. \tag{28}$$

$U(x)$ being a utility function, and using relation (5), we can define:
For all $t \in [0,T]$ and $u \in \mathbb{R}$, and let $\tau | 0 \leq t \leq \tau \leq T$ be a stopping time, the objective here is to maximize

$$\mathbb{E}\left[U\left(R_{t,\Delta}\right)\right] = \mathbb{E}\left[U\left(u + \int_0^T \lambda_t^1 \hat{F}_{t,\Delta}^{\circ,H}\right)\right] \tag{29}$$

Denoting the controlled wealth process by $R_{t,\psi}^{\tau,u}$, we have:

$$J(t,u,\psi) = \mathbb{E}\left[U\left(R_{t,\psi}^{\tau,u}\right) | \mathcal{F}_t\right] = \mathbb{E}\left[U\left(R_{t,\psi}^{\tau,u}\right)\right] \tag{30}$$

and

$$V_t(x) = v(t,u,\psi) = \sup_{\psi \in U} \mathbb{E}\left(R_{t,\psi}^{\tau,u}\right) = \sup_{\psi \in U} J(t,u,\psi) \tag{31}$$

The function V is concave and increasing such that $\dfrac{\partial^2 V}{\partial x^2}(t,x) < 0$ and $\dfrac{\partial V}{\partial x}(t,x) > 0$. The law of iterated expectations:

$$\begin{aligned} J(t,u,\psi) &= \mathbb{E}\left[\mathbb{E}\left[U\left(R_{t,\psi}^{\tau,u}\right) | \mathcal{F}_\tau\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[U\left(R_{t,\psi}^{\tau,u}\right) | X_{\tau,\psi}^{t,u} = u\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[U\left(R_t^{t,R_\tau^{t,u,\psi},\psi}\right)\right]\right] \end{aligned} \tag{32}$$

thus

$$J(t,u,\psi) = \mathbb{E}\left[J\left(\tau, R_\tau^{t,u,\psi}, \psi\right)\right] \tag{33}$$

4 Results

Lemma 4.1. *Let $t \in [0,T]$ and $\mu \in \mathbb{R}_+^*$. For any stopping time τ such that $0 \leq t \leq \tau \leq T$, we have:*

$$V(t,\mu,\psi) = \sup \lim_{t \in U} \mathbb{E}\left[V\left(\tau, R_\tau^{t,\mu,\psi_t}, \psi\right)\right] \tag{34}$$

where $(t,\mu) \in [0,T] \times \mathbb{R}_+^*$.

Proof. Let $t, \tau \in [0, T]$ such that $t < \tau$. Then

$$V(t, \mu, \psi) = \sup_{\psi \in U} \mathbb{E}\left[\left(R_t^{t,\mu,\psi}\right)\right] = \sup_{\theta_t \in U} J(t, \mu, \psi) \tag{35}$$

$$V\left(\tau, R_t^{t,\mu,\psi}, \psi\right) = \sup_{\psi_t \in S_{tr}\left(\tau, R_t^{t,\mu,\psi}\right)} J\left(\tau, R_t^{t,\mu,\psi}\right) \tag{36}$$

Relation (33) allows us to write

$$J(t, \mu, \psi) \leq \mathbb{E}\left[V\left(\tau, R_\tau^{t,\mu,\psi}, \psi\right)\right] \tag{37}$$

and

$$\sup_{\psi_t \in U} J(t, \mu, \psi) \leq \sup_{\psi_t \in U} \mathbb{E}\left[V\left(\tau, R_\tau^{t,\mu,\psi}, \psi\right)\right] \tag{38}$$

$$V(t, \mu, \psi) \leq \sup_{\psi_t \in U} \mathbb{E}\left[V\left(\tau, R_\tau^{t,\mu,\psi}, \psi\right)\right] \tag{39}$$

For an $\varepsilon > 0$, there exists $\psi^\varepsilon \in S_{tr}\left(\tau, R_t^{\tau,\mu,\psi}\right)$ such that $V\left(\tau, R_\tau^{t,\mu,\psi}, \psi\right) - \varepsilon \leq J\left(\tau, R_t^{\tau,\mu,\psi}, \psi^\varepsilon\right)$. Using (31) and (33), we obtain

$$\begin{aligned} V(t, \mu, \psi) &\geq \mathbb{E}\left[J\left(\tau, R_\tau^{t,\mu,\psi} \psi^\varepsilon\right)\right] \\ &\geq \mathbb{E}\left[V\left(\tau, R_\tau^{t,\mu,\psi}, \psi^\varepsilon\right)\right] - \varepsilon. \end{aligned} \tag{40}$$

For a very small ε, we have:

$$V(t, \mu, \psi) \geq \sup_{\psi_t \in U} \mathbb{E}\left[V\left(\tau, R_\tau^{t,\mu,\psi}, \psi\right)\right] \tag{41}$$

Using (39) and (41), we obtain the final result

$$V(t, \mu, \psi) = \sup_{\psi_t \in U} \mathbb{E}\left[V\left(\tau, R_\tau^{t,\mu,\psi^\varepsilon}, \psi\right)\right] \tag{42}$$

where $(t, \mu) \in [0, T] \times \mathbb{R}$. □

Theorem 4.1. *Let $x \in \mathbb{R}_+^*$ be a positive real number. Then, for every $t \in [0, T]$, there exists a pair of $F_t -$ adapted processes $(V_t(R_{t,\Delta}(x)), G_t(R_{t,\Delta}(x)))$ which are differentiable in x and satisfy the following equation:*

$$\begin{cases} d\phi(t, R_{t,\Delta}) = -\left\{\dfrac{1}{2}\left(\psi_t^* \sigma_t^H\right)^2 \dfrac{\partial^2 V(t, R_{t,\Delta})}{\partial \left(R_{t,\Delta}\right)^2} + \left[\psi_t^*(\mu_t - r) + r R_{t,\Delta}\right] \times \dfrac{\partial V\left(t, R_{t,\Delta}\right)}{\partial R_{t,\Delta}} \right. \\ \qquad\qquad \left. + \psi_t^* \sigma_t^H \dfrac{\partial^*}{\partial W^H} \phi\left(t, R_{t,\Delta}\right)\right\} \\ V(T, x) = \ln x \\ \psi_t^* = -\dfrac{(\mu_t - r)\frac{\partial V\left(t, R_{t,\Delta}\right)}{\partial R_{t,\Delta}} + \sigma_t^H \frac{\partial G\left(t, R_{t,\Delta}(x)\right)}{\partial x}}{\sigma_{t,H}^2 \frac{\partial^2 V(t, R_{t,\Delta})}{\partial \left(R_{t,\Delta}\right)^2}} \end{cases} \tag{43}$$

Proof. According to the lemma, we have:

$$V_t(x) = \sup_{\psi_t \in U} \mathbb{E}\left[V_\tau\left(R_\tau^{t,\mu,\psi}, \psi\right)\right] = \sup_{\psi_t \in U} \mathbb{E}\left[V_{t+r}\left(R_{\tau+r}^{t,\mu,\psi}, \psi\right)\right] \tag{44}$$

and according to (7), we have:

$$dV_t(u) = F_t(u)du + G_t(u)dW_u^H \tag{45}$$

using (45) and the Itô-Kunita lemma [17] applied to $V_{t+r}(x)$, and relation (27), we obtain:

$$V(t,u) = \int_0^t \tilde{F}(t,u)dt + \int_0^t \tilde{G}(t,u)dW_u^H \tag{46}$$

and

$$R_{t,\Delta}(x) = \int_0^t a_t\left(R_{t,\Delta}(x)\right)dx + \int_0^t b_t\left(R_{t,\Delta}(x)dW_x^H\right) \tag{47}$$

with $a_t\left(R_{t,\Delta}\right) = \psi_t(\mu_t - r) + rF_{t,\Delta}$ and $b_t\left(R_{t,\Delta}\right) = \psi_t\sigma_t^H$.

We have

$$\begin{aligned} V\left(t, R_{t,\Delta}\right) = \int_0^t &\left[\frac{\partial V\left(t, R_{t,\Delta}\right)(x)}{\partial t} + \tilde{F}\left(t, R_{t,\Delta}\right)(x) + a_t\left(R_{t,\Delta}\right)\frac{\partial V\left(t, R_{t,\Delta}(x)\right)}{\partial x}\right. \\ &\left.+\frac{1}{2}b_t^2\left(R_{t,\Delta}(x)\right)^2\frac{\partial^2 V\left(t, R_{t,\Delta}(x)\right)}{\partial x^2}\right]dt + \int_0^t\left[\tilde{G}\left(t, R_{t,\Delta}(x)\right) + b_t\left(t, R_{t,\Delta}(x)\right)\frac{\partial V\left(t, R_{t,\Delta}(x)\right)}{\partial x}\right]dW_t^H \end{aligned} \tag{48}$$

$$\begin{aligned} V_{t+r}\left(R_{t,\Delta}(x)\right) = &\int_t^{t+r}\left[\frac{\partial V\left(t, R_{t,\Delta}(x)\right)}{\partial t} + \tilde{F}\left(t, R_{t,\Delta}(x)\right) + a_t R_{t,\Delta}(x)\frac{\partial V\left(t, R_{t,\Delta}(x)\right)}{\partial x}\right. \\ &\left.\frac{1}{2}b_t^2\left(R_{t,\Delta}(x)\right)^2\frac{\partial^2 V\left(t, R_{t,\Delta}(x)\right)}{\partial x^2}\right]dt \\ &+\int_t^{t+r}\left[\tilde{G}\left(t, R_{t,\Delta}(x)\right) + b_t\left(R_{t,\Delta}(x)\right)\frac{\partial V\left(t, R_{t,\Delta}(x)\right)}{\partial x}\right]dW_t^H \end{aligned} \tag{49}$$

By substituting the expression for $V\left(t+r, x\right)$ into (44), we obtain:

$$\begin{aligned} V_t(x) &= \sup_{\psi_t \in U}\mathbb{E}\left[V_{t+r}\left(R_{t,\Delta}(x), \psi\right)\right] \\ &= V_t(x) + \sup_{\psi_t \in U}\left[\int_t^{t+r}\frac{\partial V\left(t, R_{t,\Delta}(x)\right)}{\partial t} + \tilde{F}\left(t, R_{t,\Delta}(x)\right) + a_t\left(R_{t,\Delta}(x)\right)\frac{\partial V\left(t, R_{t,\Delta}(x)\right)}{\partial x}\right. \\ &\left.+\frac{1}{2}b_t^2\left(R_{t,\Delta}(x)\right)^2\frac{\partial^2 V\left(t, R_{t,\Delta}(x)\right)}{\partial x^2}\right]dt \end{aligned} \tag{50}$$

We have

$$\tilde{F}(t,x) = -\sup_{\psi_t \in U}\left(\frac{\partial V\left(t, R_{t,\Delta}(x)\right)}{\partial t} + a_t R_{t,\Delta}(x)\frac{\partial V\left(t, R_{t,\Delta}(x)\right)}{\partial x} + \frac{1}{2}b_t^2 R_{t,\Delta}^2(x)\frac{\partial^2 V\left(t, R_{t,\Delta}(x)\right)}{\partial x^2}\right) \tag{51}$$

By replacing $\tilde{F}(t,x)$, a_t, and b_t with their expressions in (45), we obtain:

$$dV(t,R_{t,\Delta}) = -\sup_{\psi_t\in U}\left(\frac{1}{2}\left(\psi_t\sigma_t^H\right)^2\frac{\partial^2 V(t,R_{t,\Delta})}{\partial\left(R_{t,\Delta}\right)^2} + \left[\psi_t(\mu_t - r) + rR_{t,\Delta}\right]\times\frac{\partial V\left(t,R_{t,\Delta}\right)}{\partial R_{t,\Delta}}\right.$$
$$\left.+\psi_t\sigma_t^H\frac{\partial G\left(t,R_{t,\Delta}(x)\right)}{\partial x}\right)dx + G\left(t,R_{t,\Delta}(x)\right)dW_x^H \quad (52)$$

The optimal strategy ψ_t^* is reached when the term inside the supremum is maximized. By setting

$$M = \frac{1}{2}\left(\psi_t\sigma_t^H\right)^2\frac{\partial^2 V(t,R_{t,\Delta})}{\partial\left(R_{t,\Delta}\right)^2} + \left[\psi_t(\mu_t - r) + rR_{t,\Delta}\right]\times\frac{\partial V\left(t,R_{t,\Delta}\right)}{\partial R_{t,\Delta}} + \psi_t\sigma_t^H\frac{\partial G\left(t,R_{t,\Delta}(x)\right)}{\partial x} \quad (53)$$

$$\frac{\partial M}{\partial\psi} = \frac{\partial M}{\partial\psi_t} = \psi_t\sigma_{t,H}^2\frac{\partial^2 V(t,R_{t,\Delta})}{\partial\left(R_{t,\Delta}\right)^2} + (\mu_t - r)\frac{\partial V\left(t,R_{t,\Delta}\right)}{\partial R_{t,\Delta}} + \sigma_t^H\frac{\partial G\left(t,R_{t,\Delta}(x)\right)}{\partial x} \quad (54)$$

and

$$\frac{\partial^2 M}{\partial\psi^2} = \sigma_{t,H}^2\frac{\partial^2 V(t,R_{t,\Delta})}{\partial\left(R_{t,\Delta}\right)^2} \quad (55)$$

and since $\frac{\partial M}{\partial\psi}$ must be zero for the solution to be optimal, we can derive

$$\psi_t^* = -\frac{(\mu_t - r)\frac{\partial V\left(t,R_{t,\Delta}\right)}{\partial R_{t,\Delta}} + \sigma_t^H\frac{\partial G\left(t,R_{t,\Delta}(x)\right)}{\partial x}}{\sigma_{t,H}^2\frac{\partial^2 V(t,R_{t,\Delta})}{\partial\left(R_{t,\Delta}\right)^2}} \quad (56)$$

Using 45 and setting $F_t(\phi) = \frac{\partial^*}{\partial t}\phi(t,x)$ and $G_t(\phi) = \frac{\partial^*}{\partial W^H}\phi(t,x)$ (see [5], [9]), we can formulate

$$\begin{cases} d\phi(t,R_{t,\Delta}) = & -\left\{\frac{1}{2}\left(\psi_t^*\sigma_t^H\right)^2\frac{\partial^2 V(t,R_{t,\Delta})}{\partial\left(R_{t,\Delta}\right)^2} + \left[\psi_t^*(\mu_t - r) + rR_{t,\Delta}\right]\times\frac{\partial V\left(t,R_{t,\Delta}\right)}{\partial R_{t,\Delta}}\right. \\ & \left.+\psi_t^*\sigma_t^H\frac{\partial^*}{\partial W^H}\phi\left(t,R_{t,\Delta}\right)\right\} \\ V(T,x) = \ln x & \end{cases} \quad (57)$$

□

5 Conclusion

In this paper, we propose a viscosity solution using the HJB equation to maximize the expected value of the utility function in the case of the fractional SABR model. We intend to perform a sensitivity and complexity analysis of the resulting result and, if possible, perform numerical simulations.

References

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