On Some relations with Square normal , class Q^* and n-normal operators

Abstract

The relationship between different classes of operators in Hilbert spaces is an important concern in operator theory. Many authors have extended the properties of one operator to another to enhance their applicability and enable their comparison on a common domain. For instance, every normal operator is square normal but the converse is not true. In this paper, we looked at two classes of operators in Hilbert spaces; Square normal and class Q^*, and their relations with 2-normal operators. We investigated the conditions under which a 2-normal operator is class Q^* and also showed that square normal operators and $3N$ operators are independent. Furthermore, square normal operators are not necessarily class Q^* operators. To achieve this, the relationship and independence among different classes of operators were applied.

Keywords: Normal operators, Square normal operators, independence of operators, class Q^*, n-normal, adjoint.

1 Introduction

Throughout this paper $B(H)$ denotes the algebra of all bounded linear operators on Hilbert space H. A linear operator T on a Hilbert space H is said to be bounded if there exist a constant $c > 0$ such that $\|Tx\| \leq c\|x\| \, \forall x \in H$. An operator T is called self-adjoint if $T = T^*$, invertible with inverse S if there exists $S \in B(H)$ such that $ST = I = TS$ where $I \in B(H)$ is the identity operator. An operator $T \in B(H)$ is called isometry if $\|Tx\| = \|x\| \; \forall x \in H$ or equivalently $T^*T = I$. An operator $T \in B(H)$ is called unitary if $TT^* = T^*T = I$. An operator $T \in B(H)$ is said to be normal if it commutes with its adjoint i.e $(T^*T = TT^*)$, equivalently $T^*T - TT^* = 0$. An operator $T \in B(H)$ is said to be n-power normal if $T^nT^* = T^*T^n$ for $n \in \mathbb{N}$. An operator T is class Q operator if for any $T \in Q$, $T^{*2}T^2 = (T^*T)^2$. $T \in B(H)$ is called a class Q^* if $T^{*2}T^2 = (TT^*)^2$. An operator $T \in B(H)$ is n power class Q if $T^{*2}T^{2n} = (T^*T^n)^2$. An operator $T \in B(H)$ is square normal if $T^2(T^*)^2 = (T^*)^2T^2$.

2 Methodology

Authors such as Jibril (2011) have studied the connections between various operators in Hilbert spaces. According to Jibril (2011), if T, a class of operators whose squares are 2-normal is an isometry, then T is unitary. Krutan*et al.*(2019) showed that every normal operator is skew n-normal. Such studies has enabled other authors to extend properties of one operator to another, identify independent operators and also unravel properties that can be attached to an operator to establish a connection with another. For instance, Panayappan and Sivamani (2012) proved that if T, a bounded linear operator, is an n-power class Q and has an inverse, then T is an n-normal operator. Mahmood (2016) established the relationship between normal and square normal operators and proved that every normal operator is a square normal operator but the converse is not true. Furthermore, Mahmood (2016) showed that T is a square normal operator if and only if T^2 is normal. Proposition 2.1 and Proposition 2.2 shows the relationship between normal and square normal operators.

Proposition 2.1: Manhood, (2016)

Let T be a normal operator. Then T is a square normal operator.

Proof

Since T is a normal operator,
Then,

$$TT^* = T^*T \tag{1}$$

Square both sides of equation (1), to get

$$(TT^*)^2 = (T^*T)^2 \tag{2}$$

Using the property of adjoint of operator in equation (2) to get equation (3)

$$T^2T^{*2} = T^{*2}T^2 \tag{3}$$

From equation (3), T is square normal. □

Proposition 2.2: Manhood, (2016)

T is square normal if and only if T^2 is normal.

Proof

Let T be a square normal operator and show that T^2 is normal.
Since T is square normal, then, by definition of square normal operators,

$$T^2(T^*)^2 = (T^*)^2T^2 \tag{4}$$

By the property of adjoint of an operator, equation (4) yields equation (5)

$$T^2(T^2)^* = (T^2)^*T^2 \tag{5}$$

This implies that T^2 is normal.
Conversely, suppose T^2 is normal and show T is square normal.
Since T^2 is normal, then, by definition of a normal operator, equation (6) is obtained.

$$T^2(T^2)^* = (T^2)^*T^2 \tag{6}$$

From the property of adjoint of an operator, $(T^2)^* = (T^*)^2$ and equation (6) yields

$$T^2(T^*)^2 = (T^*)^2T^2 \tag{7}$$

This implies that T is square normal. □

Remark 2.3
If T is a square normal such that $T^2 = 0$, then it is not necessarily that $T = 0$.

Consider $T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ acting on $\mathbb{R}^2$.

Clearly, $T \neq 0$
But,

$$T^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Furthermore,

$$T^{*2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T^{*2}T^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T^2T^{*2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{Now , } T^{*2}T^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = T^2T^{*2}$$

Hence T is square normal.

As noted earlier, there exist square normal operators which are not normal.

In this case, T is not normal since

$$TT^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$T^*T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

□

The concept of independence of operators has been studied by authors such as Panayappan and Sivamani (2012). According to Panayappan and Sivamani (2012), the class of 2 power class Q need not be 3 power class Q and vice versa. Consider operators $T = \begin{bmatrix} i & 2 \\ 0 & -i \end{bmatrix}$ and $M = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ acting on a 2 dimensional complex Hilbert spaces.

We can show that $T \in 2$ power class Q but $T \notin 3$ power class Q. In addition, $M \in 3$ power class Q operators but $M \notin 2$-power class Q operators.

This shows that that 2-power class Q and 3-power class Q operators are independent.

Jibril, (2011) introduced (SN) operators, the class of operators whose squares are 2-normal. According to Jibril,(2011), this class (SN) is independent to the class of n-power normal where $n = 3$ introduced by Jibril (2008).

According to Panayappan and Sivamani,(2012), there is a relationship between n-normal and n-power class Q as stated in theorem 2.4 below.

Theorem 2.4 : Panayappan and Sivamani, (2012)
If $T \in B(H)$ is n-normal, then $T \in$ n power class Q

Proof
Since T is n-normal, then from the definition,

$$T^*T^n = T^nT^* \tag{8}$$

multiplying equation (8) by T^* from left to obtain

$$T^*T^*T^n = T^*T^nT^* \tag{9}$$

Multiplying equation (9) by T^n from right to get

$$T^*T^*T^nT^n = T^*T^nT^*T^n \tag{10}$$

This yields,

$$T^{*2}T^{2n} = (T^*T^n)^2 \tag{11}$$

Hence $T \in$ n power class Q. $\square$

Example 2.5 shows that an operator of 2 power class Q need not be 2-normal.

Example 2.5

Consider operator $T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ acting on a three-dimensional complex space.

Then $T^* = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$T^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$T^{*2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T^4 = T^2T^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T^{*2}T^4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T^*T^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(T^*T^2)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly, $T^{*2}T^4 = (T^*T^2)^2$. Hence T is a 2 power class Q operator.
However,

$$T^*T^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and}$$

$$T^2T^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Therefore T is not 2-normal.

Theorem 2.6: Wanjala and Nyongesa, (2021)

Let $T \in B(H)$ be a class Q^* operator, Then T is a square normal operator.

Proof

Since $T \in Q^*$, then

$$\begin{aligned} T^{*2}T^2 &= (TT^*)^2 \\ &= TT^*TT^* \text{ (but T is normal)} \\ &= TTT^*T^* \\ &= T^2T^{*2} \end{aligned}$$

Now $T^{*2}T^2 = T^2T^{*2}$. Hence T is square normal.

3 Results and Discussion

Theorem 3.1 gives the relationship between 2- normal and square normal operators.

Theorem 3.1
Let $T \in B(H)$ be a 2-normal operator. If T is normal, then $T \in B(H)$ is a square normal operator.

Proof
Since T is a 2-normal operator, from the definition,

$$T^*T^2 = T^2T^* \tag{12}$$

Multiply equation (12) by T^* from the left to obtain

$$T^*T^*T^2 = T^*T^2T^* \tag{13}$$

$$\begin{aligned} T^*T^*T^2 = (T^*)^2T^2 &= T^*TTT^* \\ &= TT^*TT^* (\text{using normality of } T) \\ &= TTT^*T^* (\text{using normality of } T) \\ &= T^2(T^*)^2 \end{aligned}$$

Hence T is square normal.

Theorem 3.2
Square normal and class $3N$ operators are independent.

Example 3.3 and Example 3.4 gives an example of a square normal operator which is not a $3N$ operator and $3N$ operator which is not a square normal respectively.

Example 3.3
Let T be given by $T = \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix}$ where $i \in \mathbb{C}$, then $T^* = \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix}$

$$T^2 = \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix} \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T^{*2} = \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix} \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T^2T^{*2} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T^{*2}T^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$T^2T^{*2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = T^{*2}T^2$ and hence square normal.

However,

$$T^3 = T^2T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix} = \begin{bmatrix} -i & 0 \\ -i & i \end{bmatrix}$$

$$T^3T^* = \begin{bmatrix} -i & 0 \\ -i & i \end{bmatrix} \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}$$

$$T^*T^3 = \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix} \neq \begin{bmatrix} -i & 0 \\ -i & i \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$$

Hence $T \notin 3N$

Example 3.4

Let $T = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ be an operator acting on complex Hilbert space.

$$T^* = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$T^2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$T^3 = T^2T = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T^3T^* = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$T^*T^3 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$T^3T^* = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} = T^*T^3$$

Hence $T \in 3N$

Now,

$$T^{*2} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$T^2T^{*2} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$T^{*2}T^2 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$T^2T^{*2} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \neq T^{*2}T^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Hence T is not a square normal operator.

Theorem 3.5

Let $T \in Q^*$, then T is a normal operator.

Proof

Since $T \in Q^*$, then, from the definition,
$T^{*2}T^2 = (TT^*)^2$
$(T^*T)^2 = (TT^*)^2$
$T^*T = TT^*$ which is normal.

Theorem 3.6 states the relationship between class Q^* operators and 2-normal operators.

Theorem 3.6
Let T be a normal operator. If T is a 2-normal operator, then $T \in Q^*$.
Proof
Since T is a 2-normal operator, then,

$$T^*T^2 = T^2T^* \tag{14}$$

Multiply equation (14) by T^* from left to obtain,

$$T^*T^*T^2 = T^*T^2T^* \tag{15}$$

Equation (15) yields

$$T^{*2}T^2 = T^*TTT^* \tag{16}$$

Since T is normal, the right side of equation (16) becomes

$$T^{*2}T^2 = TT^*TT^* \tag{17}$$

Equation (17) becomes

$$\begin{aligned} T^{*2}T^2 &= TT^*TT^* \\ &= (TT^*)^2 \end{aligned} \tag{18}$$

Hence $T \in Q^*$.

Example 3.7 shows that if the normality of the 2-normal operator is ignored, then,T is not a class Q^* operator.

Example 3.7
Consider the operator $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Now, $T^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

T is 2-normal since,

$$T^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T^2T^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T^*T^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Here $T^2T^* = T^*T^2$.
On normality of T, observe that,

$$TT^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and,}$$

$$T^*T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore T is not normal.
Consequently, T is not a class Q^* operator since,

$$T^{*2} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T^{*2}T^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$(TT^*)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Clearly, $T^{*2}T^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq (TT^*)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Remarks 3.8
Square normal operators are not necessarily class Q^* operators. Example 3.8 shows the existence of a square normal operator which is not class Q^*.

Example 3.9

Let $T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ be an operator in $B(H)$.

$$T^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$T^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$T^{*2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$T^2T^{*2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and

$T^{*2}T^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Since $T^2T^{*2} = T^{*2}T^2$, T is square normal.

Now,

$TT^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$(TT^*)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$T^{*2}T^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Clearly,$(TT^*)^2 \neq T^{*2}T^2$. Hence $T \notin Q^*$.

4 Conclusion

This research has focused on investigating the connection among square normal operators, class Q^* and n-normal operators in the Hilbert space. We have shown that square normal operators are not necessarily class Q^* operators. However, operators that are both 2-normal and normal are both square normal and class Q^*. In this case, the normality of the operator can not be ignored. Lastly, we have also shown that square normal operators and $3N$ operators are independent.

References

[1] Alzuraiqi, A., & Patel, A. (2010). On n-normal operators. *General mathematics notes*, 1(3):61–73.

[2] Jibril, A. (2008). On n-Power Normal Operators. *The Journal for Science and Engineering*, 33:247–251.

[3] Jibril, A. (2010). On operators for which $T^{*2}T^2 = (T^*T)^2$. *International Mathematical Forum*, 5(4):2255–2262.

[4] Jibril, A. (2011). On operators whose squares are 2-normal. *Journal of Mathematical Science:Advances and Applications*, 8:61–72.

[5] Krutan, R., Detrital.S, Alit,I. and Flamure,S.(2019).Skew n-Binormal Operators. *International Journal of Mathematical analysis*, 13(11):513–517

[6] Mahmood, K. (2016). On square normal operators. *British Journal of Mathematics and Computer Science*, 19(3):1–7.

[7] Muneo, C., Ji Eun, L., Kotaro, T., & Uchiyama, A. (2018). Remarks on n-normal operators. *Filomat*, 32:5441

[8] Panayappan, S. and Sivamani, N. A. (2012). On n-powerclass Q operators. *Journal of Math Analysis*, 6:1513–1518

[9] Panayappan, S., Sivamani, N., & Jibril, A. (2012). On n-binormal operators. *General Mathematical Notes*, 10(1):1–8.

[10] Wanjala, V. and Nyongesa, M. (2021). On class (Q^*). *International Journal of Mathematics And its Application*, 9(2):227–230