# A novel wavelet-based approximation scheme for the approximate solution/source of Fredholm integral equation of the second kind with variable coefficient and the pseudo-logarithmic kernels

**Abstract.** This study introduces an efficient orthonormal polynomial wavelet-based approximation scheme for solving Fredholm integral equations of the second kind with logarithmic/pseudo-logarithmic singular kernels. The method exhibits high accuracy and computational efficiency and seems applicable simultaneously when the exact solution or the source term is unknown to be determined. The examples tested here validate the accuracy and effectiveness of the scheme proposed here.

**Keywords:** Fredholm integral equation; Pseudo-logarithmic singular kernel; Orthonormal polynomial wavelet basis; Sinh-Tanh quadrature formula

**MSC Classification:** 42C40 , 45B99 , 65D15 , 65R20

## 1. Introduction

The Fredholm integral equation of second-kind

$$a_0(x)u(x) + \int_a^b K(x,t)u(t)dt = f(x),\ x \in [a,b] \tag{1}$$

plays an important role in the theoretical investigation of various physical phenomena in applied sciences. Here, $a_0(x)$, $f(x)$ are given continuous functions, $K(x,t)$ is the kernel, and $u(x)$ is the unknown function to be determined. A particular challenging class of these equations involves kernels with logarithmic/pseudo-logarithmic singularities

$$K(x,t) = a_{VS}(x,t)\log|g(x-t)| + a_{FS}(x,t) + a_R(x,t), \tag{2}$$

where the coefficient functions of the singular part and the regular part of the kernel, $a_{VS}(x,t)$ and $a_R(x,t)$ respectively, are continuous functions, while the other function, $a_{FS}(x,t)$ may have fixed integrable singularity. The behavior of the argument $g(x)$ is assumed to be $g(x) \to x^\nu$ as $x \to 0$, $\nu \in \mathbb{R}$, arises in potential theory [1, 2], elasticity [3, 4], electromagnetic scattering [5, 6] and fluid dynamics [7]. Such kernels often appear in modeling stress intensity near crack tips, surface currents in scattering, and vortex sheet dynamics, among other phenomena.

Despite their importance, these equations pose significant mathematical and computational challenges due to the singular behavior of their kernels. Analytical approaches, such as the method of inversion [8, 9], regularization [3], and series expansions [10], have been developed to isolate and manage the singularity. In contrast, numerical techniques, for example, singular quadrature [11, 12], collocation methods [13], and hybrid schemes [14, 15], offer numerical solutions.

This article presents a computational scheme to solve the above-mentioned problems by using an orthonormal polynomial wavelet basis. The advantage of using an orthonormal polynomial wavelet basis is the availability of analytical expression of elements on the basis, including a variety of their algebraic properties, viz., recurrence relation, quadrature formula, etc., which minimizes the computational cost as well as the comprehensive understanding on the approximate solution simultaneously. Furthermore, a precise relation can be obtained between *a posteriori* error in the approximate solution and the wavelet coefficients of elements of the orthonormal polynomial wavelet basis. Another important aspect of this scheme is its applicability to obtain the analytical expression for the unknown source $(f(x))$ whenever an exact solution is given. This is a significant contribution of this work as it is the extension of an established Chebyshev polynomial-based approach of Shoukralla [16], initially developed for the integral equation of the first kind to that of the second kind, including variable coefficients as well.

Our manuscript is organized as follows. The mathematical prerequisites of an orthonormal polynomial wavelet basis involving the classical Legendre polynomial, in particular, have been presented in Sect. 2. The successive steps of transformation of the integral equation with the pseudo-logarithmic kernel to equations with the logarithmic kernel and their conversion to a system of linear simultaneous equations for the coefficients of the elements of wavelet basis in an approximation of the unknown solution have been discussed in Sect. 3. The scheme for evaluating an approximation to the unknown source $f(x)$ (whenever the exact solution is known) for Eq. (1) is also available there. The scheme developed in the previous section has been tested on a few problems and compared with the results now available in the literature in Sect. 4. Our results have been analyzed and concluded in Sect. 5.

## 2. The orthonormal Legendre polynomial wavelet basis

The scale functions and wavelets of the multiresolution analysis $V_{j_0} \bigoplus_{j=1}^{\infty} W_j$ of the space $L^2([a,b])$ are defined as the linear combination [17]

$$\phi_{j_0,i}(x) = \sum_{k=0}^{2^{j_0}} C^{\phi}_{j_0 ik} P_k(x), \qquad j_0 = 0,1,2,\cdots, \qquad i = 0,1,\cdots,2^{j_0} \tag{3}$$

$$\psi_{j,i}(x) = \sum_{k=2^j+1}^{2^{j+1}} C^{\psi}_{jik} P_k(x), \quad j = j_0, j_0+1, j_0+2, \cdots, \; i = 0,1,\cdots,2^j - 1, \tag{4}$$

of Legendre polynomials $P_k\left(\frac{2x-b-a}{b-a}\right)$, $x \in [a,b]$ [18] with coefficients

$$C^{\phi}_{j_0 ik} = \sqrt{\frac{2}{2^{j_0}+2}} \sin\left(\pi \frac{i+1}{2^{j_0}+2}\right) U_k(y_i^{(2^{j_0}+1)}) \sqrt{\frac{2k+1}{b-a}}, \tag{5}$$

$$C^{\psi}_{jik} = \sqrt{\frac{2}{2^j+1}}\sin\left(\pi\frac{i+1}{2^j+1}\right)U_k(y_i^{(2^j)})\sqrt{\frac{2k+1}{b-a}}, \tag{6}$$

and

$$y_i^n = -\cos\left(\frac{(i+1)\pi}{n+1}\right), \qquad i = 0,1,...,n-1 \tag{7}$$

are the zeroes of the $U_n(y)$. The polynomial $U_k(y)$ is given by [18]

$$U_k(y) = \frac{\sin\left[(k+1)\arccos(y)\right]}{\sin\left[\arccos(y)\right]} \tag{8}$$

in the interval $y \in [-1,1]$.

The functions $\phi_{j_0,i}(x)$ and $\psi_{j,i}(x)$ with bounds in the indices mentioned above span the approximation spaces $V_{j_0}$ and the detail spaces $W_j$ of the MRA $V_{j_0}\bigoplus_{j=1}^{\infty}W_j$. The elements of scaling and wavelet functions satisfy the orthonormal conditions

$$\begin{aligned}\langle\phi_{j,i},\phi_{j,l}\rangle_w &= \delta_{il},\\ \langle\psi_{j,i},\psi_{m,l}\rangle_w &= \delta_{il}\delta_{jm},\\ \langle\phi_{j,i},\psi_{m,l}\rangle_w &= 0 \qquad (m\geq j).\end{aligned} \tag{9}$$

The basis for the truncated MRA $V_{j_0}\bigoplus_{j=1}^{J-1}W_j$ up to resolution $J\in\mathbb{N}\cup\{0\}$ is

$$\mathcal{B}_{j_0J} = \{\boldsymbol{\phi}_{j_0},\boldsymbol{\psi}_{j_0},\boldsymbol{\psi}_{j_0+1},...,\boldsymbol{\psi}_{J-1}\} \tag{10}$$

with

$$\boldsymbol{\phi}_{j_0} = \left(\phi_{j_0,0},\ \phi_{j_0,1},\cdots,\phi_{j_0,2^{j_0}}\right) \tag{11}$$

$$\boldsymbol{\psi}_j = \left(\psi_{j,0},\ \phi_{j,1},\cdots,\phi_{j,2^j-1}\right),\ j=j_0,j_0+1,\cdots,J-1. \tag{12}$$

We now recast the basis $\boldsymbol{\phi}_{j_0}$ for the approximation space $V_{j_0}$ at the resolution $j_0$ and the basis $\boldsymbol{\psi}_j$ for the detail space $W_j$ at the resolution $j$ appearing in (10) in the form

$$\boldsymbol{\phi}_{j_0} = (P_0(x),P_1(x),\cdots,P_{2^{j_0}}(x))\cdot\mathcal{M}\phi_{j_0} \tag{13}$$

$$\boldsymbol{\psi}_j = (P_{2^j+1}(x),P_1(x),\cdots,P_{2^{j+1}}(x))\cdot\mathcal{M}\psi_j \tag{14}$$

involving the matrices

$$\mathcal{M}\phi_{j_0} = \begin{pmatrix} C^{\phi}_{j_000} & C^{\phi}_{j_010} & \cdots & C^{\phi}_{j_02^{j_0}0}\\ C^{\phi}_{j_001} & C^{\phi}_{j_011} & \cdots & C^{\phi}_{j_02^{j_0}1}\\ . & . & \cdots & .\\ . & . & \cdots & .\\ . & . & \cdots & .\\ C^{\phi}_{j_002^{j_0}} & C^{\phi}_{j_012^{j_0}} & \cdots & C^{\phi}_{j_02^{j_0}2^{j_0}}\end{pmatrix}_{(2^{j_0}+1)\times(2^{j_0}+1)}, \tag{15}$$

and

$$\mathcal{M}\psi_j = \begin{pmatrix} C^{\psi}_{j\ 0\ 2^j+1} & C^{\psi}_{j\ 1\ 2^j+1} & \cdots & C^{\psi}_{j\ 2^j-1\ 2^j+1} \\ C^{\psi}_{j\ 0\ 2^j+2} & C^{\psi}_{j\ 1\ 2^j+2} & \cdots & C^{\psi}_{j\ 2^j-1\ 2^j+2} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ C^{\psi}_{j\ 0\ 2^{j+1}} & C^{\psi}_{j\ 1\ 2^{j+1}} & \cdots & C^{\psi}_{j\ 2^j-1\ 2^{j+1}} \end{pmatrix}_{2^j\times 2^j} . \tag{16}$$

Then the basis introduced in (10) can be recast into the compact form

$$\mathcal{B}_{j_0 J}(x) = \mathbf{P}_{j_0,J}(x).\mathcal{M}_{j_0,J} \tag{17}$$

with

$$\mathbf{P}_{j_0,J}(x) = (P_0(x), P_1(x), \cdots, P_{2^J}(x)) \tag{18}$$

and,

$$\mathcal{M}_{j_0,J} = \begin{pmatrix} \mathcal{M}\phi_{j_0} & 0 & 0 & 0 \ldots & 0 \\ 0 & \mathcal{M}\psi_{j_0} & 0 & 0 \ldots & 0 \\ 0 & 0 & \mathcal{M}\psi_{j_0+1} & 0 \ldots & 0 \\ 0 & 0 & 0 & \ldots\ldots & 0 \\ 0 & 0 & 0 & 0 \ldots & 0 \\ 0 & 0 & 0 & 0 \ldots & \mathcal{M}\psi_{J-1} \end{pmatrix}_{(2^J+1)\times(2^J+1)} . \tag{19}$$

## 2.1. Multiscale approximation of a function

The multiscale representation of a function $f \in L^2([a,b])$ is the orthogonal projection $P_{V_{j_0}\bigoplus_{j=1}^{J-1} W_j} : L^2([a,b]) \to V_{j_0}\bigoplus_{j=1}^{J} W_j$ defined by

$$P_{V_{j_0}\bigoplus_{j=j_0}^{J-1} W_j}[f](x) = f^{\text{Approx}}_{j_0 J}(x) = \sum_{k=0}^{2^{j_0}} c_{j_0,k}\phi_{j_0,k} + \sum_{j=1}^{J-1}\sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k} = \mathcal{B}_{j_0 J}\cdot \mathcal{CD}_{j_0,J}. \tag{20}$$

Here, $\mathcal{CD}_{j_0,J} = (\mathbf{c}_{j_0}, \mathbf{d}_{j_0}, \mathbf{d}_{j_0+1}, \cdots, \mathbf{d}_{J-1})$ with $\mathbf{c}_{j_0} = (c_{j_0,0}, c_{j_0,1}, \cdots, c_{j_0,2^{j_0}})$, and $\mathbf{d}_j = (d_{j,0}, d_{j,1}, \cdots, d_{j,2^j-1})$, $j = j_0, \cdots, J-1$. The orthonormality condition (9) may be exploited to obtain elements of the above vectors as

$$c_{j_0,k} = \langle f(x), \phi_{j_0,k}(x)\rangle = \int_a^b f(x)\phi_{j_0,k}(x)dx \tag{21}$$

and,

$$d_{j,k} = \langle f(x), \psi_{j,k}(x)\rangle = \int_a^b f(x)\psi_{j,k}(x)dx. \tag{22}$$

These formulae suggest that coefficients $c_{j_0,k}$, $d_{j,k}$ can be interpreted as the components in the projection of $f(x)$ along the scale function $\phi_{j_0,k}$ in the approximation space $V_{j_0}$ and the wavelet $\psi_{j,k}$ in the detail space $W_j$, respectively. One of the delicate properties of these coefficients is their direct role in the estimation of error due to the omission of detail spaces $W_j,\ j \geq J$.

### 2.2. Estimates of *a posteriori* error

The error in the projection $f_{j_0 J}^{Approx}$ of $f \in L^2([a,b])$ into the MRA $V_{j_0} \bigoplus_{j=j_0}^{\infty} W_j$ may be recast (by using orthonormality condition provided in (9)) to

$$E_{L^2} = \sqrt{||f(x) - f_{j_0 J}^{\text{Approx}}(x)||_{L^2}} \simeq \sqrt{\sum_{j=J}^{\infty} ||\boldsymbol{\psi}_j \cdot \mathbf{d}_j||_{L^2}} = \sqrt{\sum_{j=J}^{\infty} \mathbf{d}_j \cdot \mathbf{d}_j}. \tag{23}$$

For $f \in L^2([a,b])$ with $\sup_{x\in[a,b]} | \frac{f^{(n)}(x)}{n!} | << 1$, the ratio $r = \sup_{j\in\{j_0,\cdots,J-1\}} \frac{\mathbf{d}_{j+1}\cdot\mathbf{d}_{j+1}}{\mathbf{d}_j\cdot\mathbf{d}_j} < 1$. In such cases, *a posteriori* error can be estimated as

$$E_{L^2\ J}^{apost} \simeq \frac{1}{1-r}\sqrt{\mathbf{d}_{J-1} \cdot \mathbf{d}_{J-1}}. \tag{24}$$

## 3. Approximation of unknown solution and source

**Proposition 1.** *[15] The equation*

$$a_0(x)u(x) + IL[a\,u](x) + IR[b\,u](x) = f(x) \tag{25}$$

*with*

$$IL[a\,u](x) = \int_{\Omega} a(x,t)u(t)\log|x-t|\,dt, \tag{26a}$$

$$IR[b\,u](x) = \int_{\Omega} b(x,t)u(t)dt, \tag{26b}$$

*where $a(x,t)$ and $b(x,t)$ are satisfying the condition stated earlier (after (2)) and $a_0(x) \neq 0,\ x \in \Omega$ has unique solution in $L^2(\Omega)$ if $-1 \notin$ spectrum of $ILR[\cdot] := \frac{1}{a_0(x)}(IL[\cdot] + IR[\cdot]) : L^2(\Omega) \to L^2(\Omega)$ and $f \in L^2(\Omega)$.*

### 3.1. Equation for the coefficients of approximate solution of Eq. (1) when source (f(x)) are given

For the Fredholm integral equation (1) of the second kind with pseudo-logarithmic kernel (2) and $g(x) \sim x$ as $x \to 0$, the variable singular term may be split into

$$\log|g(x-t)| = \log\left|\frac{g(x-t)}{x-t}\right| + \log|x-t|, \tag{27}$$

so that Eq. (1) can be recast as

$$a_0(x)u(x) + IL_{[a,b]}[a_{VS}u](x) + IR_{[a,b]}[b_R u](x) = f(x). \tag{28}$$

Here,

$$IL_{[a,b]}[F](x) = \int_a^b F(x,t)\log|x-t|\,dt, \tag{29a}$$

$$IR_{[a,b]}[F](x) = \int_a^b F(x,t)dt, \tag{29b}$$

and

$$b_R(x,t) = a_{VS}(x,t)\log\left|\frac{g(x-t)}{x-t}\right| + a_{FS}(x,t) + a_R(x,t). \tag{29c}$$

For the evaluation of singular integral $IL_{[a,b]}[a_{VS}u](x)$ in (28) , we adopt the change of variable

$$x = x(\xi) = \frac{1}{2}\left\{(b-a)\xi + b + a\right\}, \quad t = t(\tau) = \frac{1}{2}\left\{(b-a)\tau + b + a\right\}, \quad \xi,\tau \in [-1,1] \tag{30a,}$$

to change the domain $[a,b]$ to $[-1,1]$ and get,

$$\begin{aligned} IL_{[a,b]}[a_{VS}u](x) &= \frac{b-a}{2}\int_{-1}^{1}\bar{a}_{VS}(\xi,\tau)\bar{u}(\tau)\log|\xi-\tau|\,d\tau + \frac{b-a}{2}\log\frac{b-a}{2}\int_{-1}^{1}\bar{a}_{VS}(\xi,\tau)\bar{u}(\tau)d\tau \\ &= \frac{b-a}{2}IL_{[-1,1]}[\bar{a}_{VS}\bar{u}]\left(\frac{2x-b-a}{b-a}\right) + \log\frac{b-a}{2}\int_a^b a_{VS}(x,t)u(t)dt. \end{aligned} \tag{31}$$

Here the symbol $\bar{F}$ is defined as $\bar{F}(\xi) = F(x(\xi)) = F\left(\frac{1}{2}\left\{(b-a)\xi + b + a\right\}\right)$.

Use of the formula (31) into Eq. (28) gives

$$a_0(x)u(x) + \frac{b-a}{2}IL_{[-1,1]}[\bar{a}_S\bar{u}]\left(\frac{2x-b-a}{b-a}\right) + IR_{[a,b]}[Gu](x) = f(x), \tag{32a}$$

where

$$G(x,t) = b_R(x,t) + a_{VS}(x,t)\log\frac{b-a}{2}. \tag{32b}$$

For the evaluation of the integral $IL_{[-1,1]}[\bar{F}]\,(\xi)$ defined in (29a), we use $N-$point quadrature formula with variable weight [19]

$$IS_{[-1,1]}[\bar{F}](\xi) = \int_{-1}^{1}\bar{F}(\xi,\tau)\log|\xi-\tau|d\tau \simeq \sum_{k=1}^{N}\omega_k^N\Omega_k^N(\xi)\bar{F}(\xi,\tau_k), \quad \xi\in[-1,1] \tag{33a}$$

where $\tau_k$ and $\omega_k$ are the nodes and weights of $N-$point Gauss-Legendre quadrature formula and variable weight factor $\Omega_k^N(\xi)$ accommodating the presence of logarithmic singular factor in the integral is given by

$$\begin{aligned} \Omega_k^N(\xi) &= (P_0(\tau_k) - P_1(\tau_k))\left(Q_0(\xi) + \frac{1}{4}\log(1-\xi)^2\right) + \sum_{l=1}^{N-2}(P_{l-1}(\tau_k) - P_{l+1}(\tau_k))\left(Q_l(\xi) + \frac{1}{4}\log(1-\xi)^2\right) \\ &+ P_{N-2}(\tau_k)\left(Q_{N-1}(\xi) + \frac{1}{4}\log(1-\xi)^2\right) + P_{N-1}(\tau_k)\left(Q_N(\xi) + \frac{1}{4}\log(1-\xi)^2\right). \end{aligned} \tag{33b}$$

Here, $P_k(\xi)$ and $Q_k(\xi)$ are Legendre polynomials of the first kind and Legendre function respectively.

Double Exponential (DE) or $\sinh-\tanh$ quadrature formula [20, 21]

$$IR_{[a,b]}[Gu](x) \simeq QDE_{jQ}[Gu](x) = \frac{1}{2^{jQ}}\sum_{k=-2^{jQ+2}}^{2^{jQ+2}} G\left(x,\theta\left(\frac{k}{2^{jQ}}\right)\right)u\left(\theta\left(\frac{k}{2^{jQ}}\right)\right)\theta'\left(\frac{k}{2^{jQ}}\right) \tag{34}$$

have been used for evaluation of the integral $IR_{[a,b]}[Gu](x)$ in Eq. (32a), due to its efficiency for obtaining highly accurate numerical value of definite (even improper) integrals (involving integrand with finite number of fixed integrable singularity). Use of (33a,b) for the integral $IL_{[-1,1]}[\bar{a}_{VS}\bar{u}]\left(\frac{2x-b-a}{b-a}\right)$ followed by replacement of $(\xi,\tau)$ in term of $(x,t)$ and substitution of expression of $IR_{[a,b]}[Gu](x)$ from (34) into Eq. (32a), gives

$$\begin{aligned} a_0(x)u(x) \;&+ \sum_{k=1}^{N} \bar{\omega}_k^N \Omega_k^N \left(\frac{2x-b-a}{b-a}\right) a_{VS}(x,t_k)u(t_k) \\ &+ \frac{1}{2^{jQ}} \sum_{k=-2^{jQ+2}}^{2^{jQ+2}} G\left(x, \theta\left(\frac{k}{2^{jQ}}\right)\right) u\left(\theta\left(\frac{k}{2^{jQ}}\right)\right) \theta'\left(\frac{k}{2^{jQ}}\right) \simeq f(x), \end{aligned} \tag{35}$$

where $t_k = t(\tau_k)$ and $\bar{\omega}_k^N = \frac{b-a}{2}\omega_k^N$ are the nodes and weights of Gauss-Legendre quadrature rule in the interval $[a,b]$. Approximation of the unknown function $u(x)$ with the orthonormal polynomial wavelet basis $\mathcal{B}_{j_0 J}(x)$

$$u(x) \simeq u_{j_0 J}^{\text{Approx}}(x) = \mathcal{B}_{j_0 J}(x) \cdot \mathcal{CD}_{j_0 J}. \tag{36}$$

and its substitution in Eq. (35) yields

$$\mathcal{A}_{j_0 J;NjQ}(x) \cdot \mathcal{CD}_{j_0 J} \simeq f(x) \tag{37a}$$

where

$$\begin{aligned} \mathcal{A}_{j_0 J;NjQ}(x) = \Big[ a_0(x)\mathcal{B}_{j_0 J}(x) + \sum_{k=1}^{N} \bar{\omega}_k^N \Omega_k^N \left(\frac{2x-b-a}{b-a}\right) a_{VS}(x,t_k)\mathcal{B}_{j_0 J}(t_k) + \frac{1}{2^{jQ}} \\ \sum_{k=-2^{jQ+2}}^{2^{jQ+2}} G\left(x, \theta\left(\frac{k}{2^{jQ}}\right)\right) \mathcal{B}_{j_0 J}\left(\theta\left(\frac{k}{2^{jQ}}\right)\right) \theta'\left(\frac{k}{2^{jQ}}\right) \Big]. \end{aligned} \tag{37b}$$

For the reduction of Eq. (37a) into a system of linear simultaneous equations we adopt here wavelet-Galerkin scheme. So, we multiply both sides of Eq. (37a) by $\mathcal{B}_{j_0 J}^T(x)$ and integrate within the domain $[a,b]$ to get,

$$\mathfrak{A}_{j_0 J;NjQ} \cdot \mathcal{CD}_{j_0 J} \simeq \mathcal{F}_{j_0 J} \tag{38}$$

where

$$\mathfrak{A}_{j_0 J;NjQ} = \int_a^b \mathcal{B}_{j_0 J}^T(x) \mathcal{A}_{j_0 J;NjQ}(x) dx \tag{39a}$$

$$\mathcal{F}_{j_0 J} = \int_a^b \mathcal{B}_{j_0 J}^T(x) f(x) dx. \tag{39b}$$

To evaluate the integrals in (39a) and (39b), we use (DE) or $\sinh - \tanh$ quadrature formula and get

$$\mathfrak{A}_{j_0 J;NjQ}^{Approx} = \frac{1}{2^{jQ}} \sum_{k=-2^{jQ+2}}^{2^{jQ+2}} \mathcal{B}_{j_0 J}^T\left(\theta\left(\frac{k}{2^{jQ}}\right)\right) \mathcal{A}_{j_0 J;NjQ}\left(\theta\left(\frac{k}{2^{jQ}}\right)\right) \theta'\left(\frac{k}{2^{jQ}}\right) \tag{40a}$$

$$\mathcal{F}_{j_0 J;jQ}^{Approx} = \frac{1}{2^{jQ}} \sum_{k=-2^{jQ+2}}^{2^{jQ+2}} \mathcal{B}_{j_0 J}^T\left(\theta\left(\frac{k}{2^{jQ}}\right)\right) f\left(\theta\left(\frac{k}{2^{jQ}}\right)\right) \theta'\left(\frac{k}{2^{jQ}}\right). \tag{40b}$$

so that Eq. (38) can be approximately written as

$$\mathfrak{A}^{Approx}_{j_0J;NjQ} \cdot \mathcal{CD}_{j_0J} = \mathcal{F}^{Approx}_{j_0J;jQ}. \tag{41}$$

Here, $j_0$ and $J$ (appearing in the approximation in (36)) is coarsest and finest scale in the wavelet approximation respectively, while $N$ (present in (33a)) is the number of nodes in Gauss-Legendre quadrature formula and $jQ$ is resolution in (DE) or $\sinh - \tanh$ quadrature formula.

## 3.2. Approximation of unknown sources (f(x)) when the exact solutions are known

Whenever the exact solution $u^{\text{Exact}}(x)$ of Eq. (28) is known but the inhomogeneous or the source term is not given, its estimation $f^{\text{Est}}(x)$ can be obtained by substituting $u^{\text{Exact}}(x)$ in place of $u(x)$ in Eq. (35). Thus,

$$f^{\text{Est}}(x) \simeq \mathcal{L}_{NjQ}(x), \tag{42a}$$

where

$$\begin{aligned}\mathcal{L}_{NjQ}(x) = a_0(x)u^{\text{Exact}}(x) + \sum_{k=1}^{N} \bar{\omega}_k^N \Omega_k^N \left(\frac{2x-b-a}{b-a}\right) a_{VS}(x,t_k)u^{\text{Exact}}(t_k) \\ + \frac{1}{2^{jQ}} \sum_{k=-2^{jQ+2}}^{2^{jQ+2}} G\left(x, \theta\left(\frac{k}{2^{jQ}}\right)\right) u^{\text{Exact}}\left(\theta\left(\frac{k}{2^{jQ}}\right)\right) \theta'\left(\frac{k}{2^{jQ}}\right).\end{aligned} \tag{42b}$$

To get the coefficient of the projection of $f^{\text{Est}}(x)$ to $V_{j_0} \bigoplus_{j=j_0}^{\infty} W_j$, we multiply both sides of Eq. (42a) by $\mathcal{B}^T_{j_0J}(x)$ and integrate over the domain $[a,b]$ to get,

$$\mathcal{F}^{\text{Approx}}_{j_0J} \simeq \mathcal{L}_{j_0J;NjQ}, \tag{43a}$$

where

$$\mathcal{L}_{j_0J;NjQ} = \int_a^b \mathcal{B}^T_{j_0J}(x)\mathcal{L}_{NjQ}(x)\,dx. \tag{43b}$$

For the evaluation of the integral in (43b), we use the (DE) or sinh-tanh quadrature formula and get

$$\mathcal{L}_{j_0J;NjQ} \simeq \mathcal{L}^{\text{Approx}}_{j_0J;NjQ} = \frac{1}{2^{jQ}} \sum_{k=-2^{jQ+2}}^{2^{jQ+2}} \mathcal{B}^T_{j_0J}\left(\theta\left(\frac{k}{2^{jQ}}\right)\right) \mathcal{L}_{NjQ}\left(\theta\left(\frac{k}{2^{jQ}}\right)\right) \theta'\left(\frac{k}{2^{jQ}}\right). \tag{44}$$

Substituting (44) into Eq. (43a) gives

$$\mathcal{F}_{j_0J} \simeq \mathcal{L}^{\text{Approx}}_{j_0J;NjQ}. \tag{45}$$

The approximate expression for the unknown source $f(x)$ in $V_{j_0} \bigoplus_{j=j_0}^{\infty} W_j$ is then given by

$$f^{\text{Approx}}_{j_0J}(x) = \mathcal{B}^T_{j_0J}(x) \cdot \mathcal{L}^{\text{Approx}}_{j_0J;NjQ}. \tag{46}$$

To assess the efficiency of the multiresolution approximation $f^{\text{Approx}}_{j_0J}(x)$ of the unknown source whenever the exact solution is given, successive steps of the previous section can be taken for Eq. (28) with $f(x)$ given by its approximation in (46). As a result, a system of linear equations

$$\mathfrak{A}^{\text{Approx}}_{j_0J;NjQ} \cdot \mathcal{CD}_{j_0J} = \mathcal{L}^{\text{Approx}}_{j_0J;NjQ}. \tag{47}$$

for the unknown coefficients $\mathcal{CD}_{j_0 J}$ can be obtained. Substitution of the solution of the derived equation to (36) provides the approximate solution of Eq. (28) for the estimated source $f^{\mathrm{Est}(x)}(x)$ in Eq. (42a).

**Table 1** Inputs $[a, b]$, $a_{VS}(x,t)$, $g(x)$, $a_{FS}(x,t)$, $a_R(x,t)$ and $f(x)$ in Eq. (28) with $a_0(x) = 1$ and corresponding exact solution ($u^{Exact}(x)$).

| Ex. | $[a,b]$ | $a_{VS}(x,t)$ | $g(x)$ | $a_{FS}(x,t)$ | $a_R(x,t)$ | $f(x)$ | $u^{Exact}(x)$ |
|---|---|---|---|---|---|---|---|
| 1.1 | $[0,1]$ | $\sqrt{\frac{1+t}{1+x}}$ | $x$ | $0$ | $\frac{1}{x+t+2}$ | $\frac{1}{\sqrt{1+x}}\Big\{x\log x$ $+(1-x)\log(1-x)$ $+2\tan^{-1}\left(\sqrt{\frac{2}{1+x}}\right)$ $-2\tan^{-1}\left(\sqrt{\frac{1}{1+x}}\right)\Big\}$ | $\frac{1}{\sqrt{1+x}}$ |
| 1.2 | $[0,\pi]$ | $-e^{x-t}$ | | $-e^{x-t}\times$ $\log\left\lvert\sin\frac{x+t}{2}\right\rvert$ | $-\sin(x-t)$ | $e^x\left(1+\pi\log 2\right)$ $+\frac{1+e^{\pi}}{2}\left(\cos x+\sin x\right)$ | $e^x$ |
| 1.3 | | $\frac{1}{\pi}$ | | | $\frac{(x-t)^2}{\pi}$ | $-4x$ | $\sin x$ |
| 1.4 | | $-\frac{1}{\pi}\left(1+e^{xt}\sin^2\left(\frac{x-t}{2}\right)\right)$ | | | $-\frac{1}{\pi(x^2+t^2+1)}$ | | $\cosh\left(\frac{1}{x^2+8x+16}\right)$ |
| 1.5 | $[-\pi,\pi]$ | $-1-\frac{x+t}{\pi(x+t+\pi)}\sin^2\left(\frac{x-t}{2}\right)$ | $2\sin\frac{x}{2}$ | $0$ | $-\frac{1}{\pi}\left(1+x+\cos t\right)$ | Unknown | $\sqrt{x^2+x+\pi}$ |
| 1.6 | | $-\frac{1}{\pi}\left(\frac{2}{3}+\sqrt{\frac{1}{x+t+3\pi}}\sin^2\left(\frac{x-t}{2}\right)\right)$ | | | $-\frac{1}{\pi}\left(1+\frac{7}{2}x\right)$ | | $\log\left(2\pi+\frac{1+x}{x+\pi}\right)$ |
| 1.7 | | $-\frac{1}{\pi}\sinh\left(\frac{x+t}{2}\right)\sin^2\left(\frac{x-t}{2}\right)$ | | | $-\frac{1}{\pi}\log(10+x)$ | | $\frac{xe^x}{x^2+e}$ |
| 1.8 | $[0,2\pi]$ | $-2$ | | | $-1$ $-\log\left(\frac{5-3\cos(x+t)}{8}\right)$ | $4\pi-2\pi e^{\cos x}\cos(\sin x)$ $-2\pi e^{\frac{\cos x}{3}}\cos\left(\frac{\sin x}{3}\right)$ | Unknown |

## 4. Computational Results

The computational scheme developed in the previous section has been applied to various examples to examine its accuracy and efficiency in solving Fredholm integral equations of the second kind with logarithmic or pseudo-logarithmic kernels whenever either the solution or the source term is unknown. We have considered the examples, Ex. 1.1 – Ex. 1.8, which are of the form of Eq. (28) with $(a_0(x),\, a_S(x,t),\, b_R(x,t),\, g(x))$ provided in Table 1. Details of the specific inputs for several examples are provided in Cols. 2–7, with the corresponding exact solutions listed in Col. 8 of the same table. It has been verified that all the examples listed in the table satisfy the conditions of Prop. 1 stated at the beginning of the previous section. In our exercise, the following parameter values have been used: $N = 20$, $jQ = 5$, $j_0 = 1$ with different $J = 1, \cdots, 5$. The final system of algebraic equations Eq. (41) has been solved by using the library function Solve[·, ·] available in the MATHEMATICA, and all calculations are performed on a PC with an Intel Core™ i5 processor (2.50 GHz) and 4 GB of RAM.

All the steps discussed in Sect. 2.1 have been exercised to obtain the approximate solution $u^{\mathrm{Approx}}_{j_0 J}(x)$ for Exs. 1.1–1.3. While for the Exs. 1.4–1.7 (whose sources are not known), we exercise the steps discussed in Sect. 2.2; in the first step, we use the exact solution as input in the proposed scheme to obtain the approximate expression of source appearing in (46). The obtained approximate expression of sources $\left(f_{j_0J}^{\text{Approx}}(x)\right)$ for the examples (Exs. 1.4–1.7) are provided in the Appendix.

To exhibit the qualitative behavior of the sources presented by analytic expressions (A1)–(A5), their plots are presented in Fig. 1. Closely observing these figures reveals that the projection of unknown sources gradually converges with the increase in the resolution of the detail spaces. These approximate sources have been used further to obtain $u_{j_0J}^{\text{Approx}}(x)$ of the known solution (regarded as unknown) to the problem. To exhibit the accuracies and the rate of convergence of the approximate solution with the variation of the resolution $J$ of the detail spaces, plots of absolute error (in $\log_{10}$ scale) in $u_{j_0J}^{\text{Approx}}(x)$ have been provided in Fig. 2. It is observed that the approximate solution converges rapidly for Exs. 1.1–1.3, but the convergence rates of Exs. 1.4– 1.7 are relatively slow. Such a slow rate of convergence is suspected due to the approximation of the source terms.

To examine the reliability of estimation of global *a posteriori* error provided in the formula (23) of Sect. 2.2, the exponent $n$ in the $O(10^{-n})$ of the *a posteriori* error and $L^{\infty}-$ error

$$\text{Err}_{j0J}^{L^{\infty}} = \sup_{x\in[a,b]} \left| u^{\text{Exact}}(x) - u_{j_0J}^{\text{Approx}}(x) \right|, \tag{48}$$

in the $u_{j_0J}^{\text{Approx}}(x)$ (of Exs. 1.1–1.7 for $J = 1, \cdots, 4$) have been provided in Col. 3 and Col. 4 of Table 2. Their comparison reveals that the estimate of *a posteriori* error suggested in formula (23) is quite efficient. The exponent $n$ in the order of accuracies in approximate solutions obtained by other available methods have been cited in Cols. $5-6$ of the same table. A close examination of the exponents provided in Col. 3 (achieved in the proposed scheme) and Cols. $5-6$ (for other methods now available) establishes the proposed scheme's better efficiency over the available approximation schemes.

Seeing the efficiency of the proposed scheme for examples with known exact solutions, we applied the scheme to Ex. 1.8 (listed in the last row of Table 1), whose analytic solution is not yet available. The plot for the approximate solution at $J = 5$ is presented in Fig. 3. A comparison of the exponent $n$ of the *a posteriori* error with that of another method (given in Col. 4 and Col. 7 of Table 2) suggests that our approach succeeds in providing a highly accurate approximate solution in comparison to approximation methods now available.

## 5. Conclusion

A computational scheme using an orthogonal polynomial wavelet basis has been developed here to obtain the accurate approximate source/solution for Fredholm integral equations of the second kind with logarithmic/pseudo-logarithmic kernels. The method demonstrated high accuracy by testing multiple examples (Exs. 1.1–1.8). For problems with known sources, it provided solutions that converged quickly. For problems with unknown sources, the method still delivered reliable results with a slightly slower convergence rate because of the source approximation. The global *a posteriori* error estimates provided by the scheme are validated, showing consistency with the observed $L^{\infty}-$errors, making it worthwhile to get approximate solutions for those problems whose exact solutions are unknown. Furthermore, a comparison of the accuracy with other available methods confirms that the proposed scheme achieves higher accuracy. These findings show its ability as an efficient tool for solving integral equations with logarithmic/pseudo-logarithmic kernels and can be used to handle complex problems in science and engineering. Furthermore, the extension of this wavelet-based approximation scheme for solving linear and nonlinear singular integral/integro-differential equations with weakly, Cauchy and hyper-singular kernels [25–29] are in progress, will be reported elsewhere.

**Table 2** Exponent in the order ($O(10^{-n})$) of the $L^\infty-$(48) and estimated *a posteriori* errors (23) in the Approximate solutions $\left(u^{\text{Approx}}_{j_0 J}(x)\right)$ of examples cited in Table 1 for $N = 20$, $jQ = 5$, $j_0 = 1$ with different $J$.

| Example | $J$ | Order $n$ in errors | | | | |
|---|---|---|---|---|---|---|
| | | Present scheme | | Other schemes | | |
| | | $L^\infty$ | *a posteriori* | MDGM [22] | LTPSM [23] | THWGM [24] |
| 1.1 | 1 | 4 | 5 | 3 | NA | NA |
| | 2 | 6 | 8 | 4 | | |
| | 3 | 14 | 15 | 5 | | |
| | 4 | 25 | | 6 | | |
| 1.2 | 1 | 1 | 2 | 2 | NA | NA |
| | 2 | 5 | 6 | 3 | | |
| | 3 | 15 | 16 | 4 | | |
| | 4 | 25 | | 5 | | |
| 1.3 | 1 | 1 | 2 | 2 | NA | NA |
| | 2 | 3 | 4 | 3 | | |
| | 3 | 10 | 12 | 4 | | |
| | 4 | 29 | | 5 | | |
| 1.4 | 1 | $\frac{1}{2}$ | 2 | NA | NA | NA |
| | 2 | 1 | 2 | | | |
| | 3 | 3 | 4 | | | |
| | 4 | 6 | | | | |
| 1.5 | 1 | 1 | 2 | NA | NA | NA |
| | 2 | 2 | 3 | | | |
| | 3 | 5 | 6 | | | |
| | 4 | 8 | | | | |
| 1.6 | 1 | 2 | 3 | NA | NA | NA |
| | 2 | 3 | 4 | | | |
| | 3 | 5 | 6 | | | |
| | 4 | 9 | | | | |
| 1.7 | 1 | 1 | 2 | 4 | 6 | NA |
| | 2 | 2 | 3 | | | |
| | 3 | 4 | 5 | | | |
| | 4 | 7 | | | | |
| 1.8 | 1 | – | 1 | NA | NA | 1 |
| | 2 | – | 2 | | | 2 |
| | 3 | – | 5 | | | 2 |
| | 4 | – | 8 | | | 3 |
| | 5 | – | | | | 3 |

MDGM–Meshless discrete Galerkin method, THWGM–Trigonometric Hermite wavelet Galerkin method and LTPSM–Local thin plate spline method.
Results for Ex. 1.7 from MDGM is provided in [23]

## Supplementary information :

We do not have any research data outside of the submitted manuscript file.

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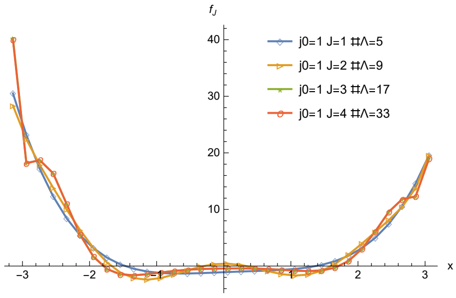
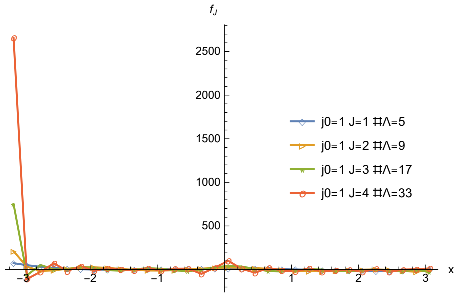
**Fig. 1** Figures of approximate source ($f^{\text{Approx}}_{j_0 J}(x)$) of Exs. 1.4–1.7 (cited in Table 1) for $N = 20$, $jQ = 5$, $j_0 = 1$ with $J = 1, \cdots, 4$.

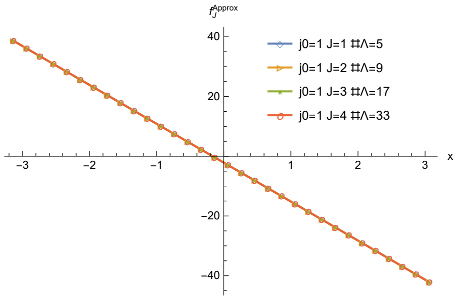
Ex. 1.4



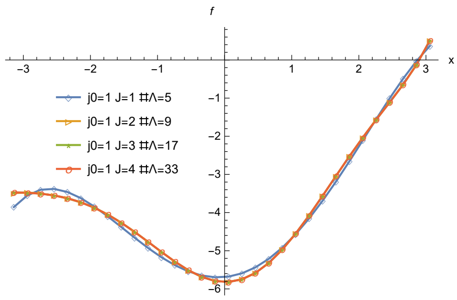

Ex. 1.5



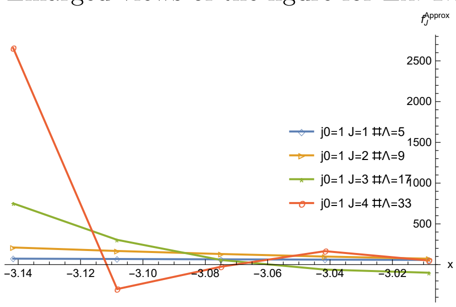

Ex. 1.6



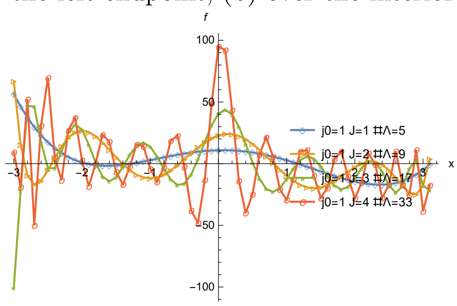

Ex. 1.7




**Enlarged views of the figure for Ex. 1.5: (a) near the left endpoint, (b) over the interior range.




(a)




(b)

# Declarations

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## Author's contributions:

All authors have equal contributions

## Conflict of interest.:

The authors have no conflict of interest that could be perceived to influence the results and/or discussion reported in this document.

## Ethical approval:

We do not have research data outside the submitted manuscript file.

## Consent for publication:

Not applicable.

## Data availability:

Not applicable.

## Material availability:

Not applicable.

## Code availability:

The code used in this study is available upon request from the corresponding author.

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**Fig. 2** Plots of absolute errors (in $\log_{10}$ scale) in approximation of solutions $u_{j_0 J}^{\text{Approx}}(x)$ of examples (Exs.1.1–1.7) listed in Table 1 for $N = 20$, $jQ = 5$, $j_0 = 1$ with $J = 1, \cdots, 4$.

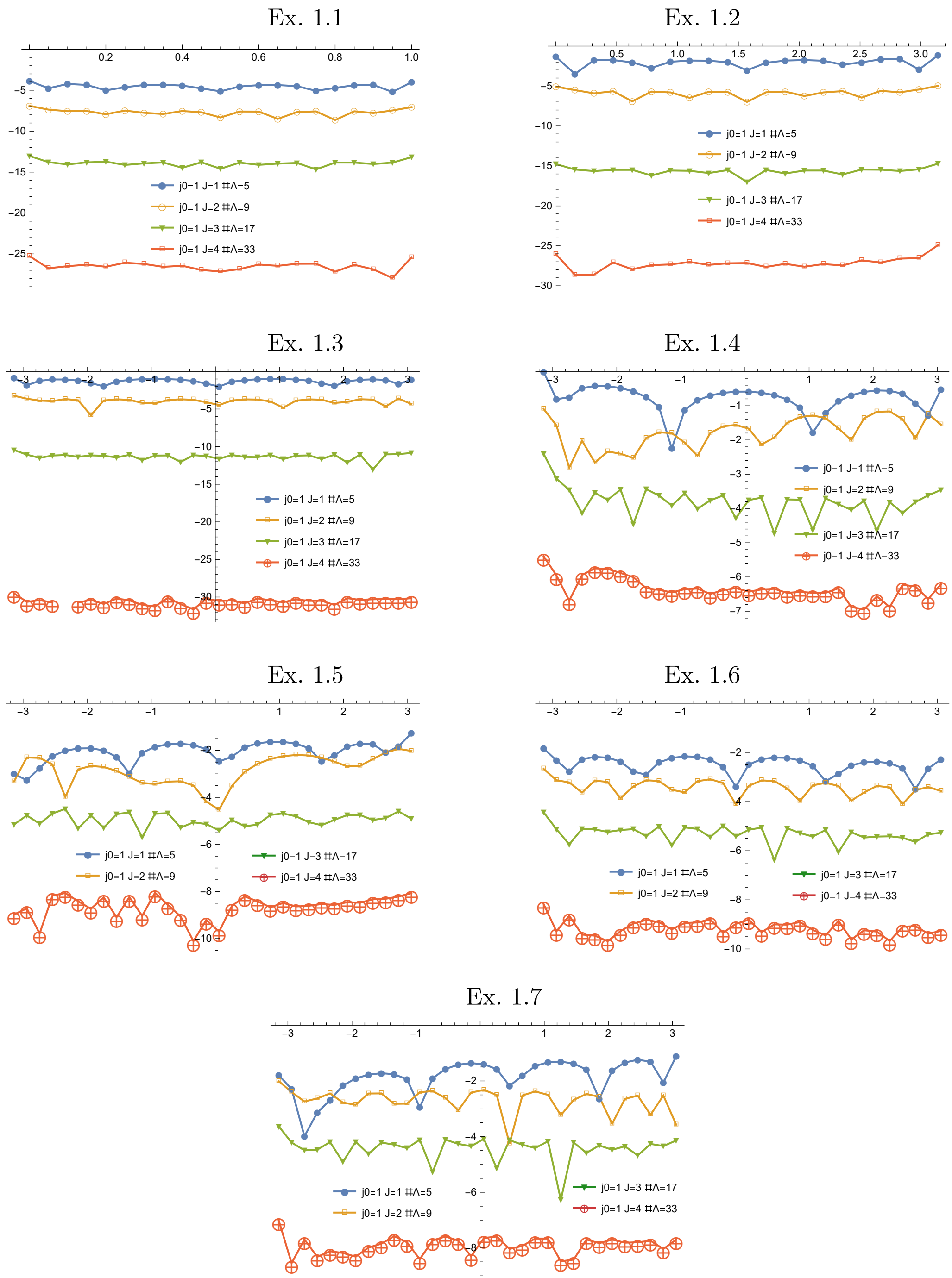
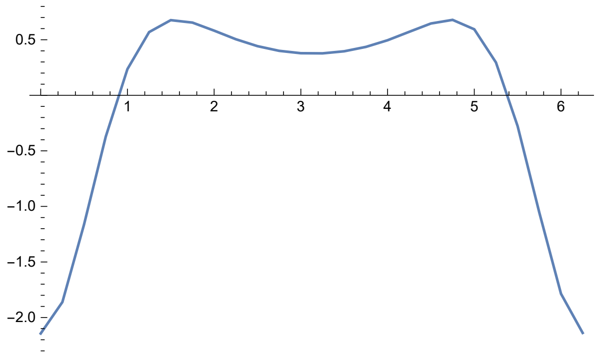


**Fig. 3** Plot of approximate solution ($u_{j_0 J}^{\text{Approx}}(x)$) of Ex. 1.8 (Table 1) for $N = 20$, $jQ = 5$, $j_0 = 1$ and $J = 5$.




# Appendix A

The expressions for the approximation $f_{14}^{\text{Approx}}(x)$ to the (unknown) sources obtained by using the formula (46) for Ex. 1.3-Ex. 1.7.

$$\begin{aligned}
f_{Ex1.3}^{\text{Approx}}(x) &= -3.9999999999999999982875602161744x - 4.858810570540133 \times 10^{-16}x^3 \\
&+ 2.09158799914740860 \times 10^{-15}x^5 - 3.2537302167705157 \times 10^{-15}x^7 \\
&+ 2.6554690673120803 \times 10^{-15}x^9 - 1.3784784555915986 \times 10^{-15}x^{11} \\
&+ 5.237749649233799 \times 10^{-16}x^{13} - 1.601414950330776 \times 10^{-16}x^{15} \\
&+ 4.03289420142528 \times 10^{-17}x^{17} - 8.0875671699093 \times 10^{-18}x^{19} \\
&+ 1.2329079875872 \times 10^{-18}x^{21} - 1.373784179810 \times 10^{-19}x^{23} \\
&+ 1.07622724992 \times 10^{-20}x^{25} - 5.602689923 \times 10^{-22}x^{27} \\
&+ 1.73979025 \times 10^{-23}x^{29} - 2.439554 \times 10^{-25}x^{31},
\end{aligned} \tag{A1}$$

$$\begin{aligned}
f_{Ex1.4}^{\text{Approx}}(x) &= -0.4253390616299625969770794804360 2 + 0.1329742544366944472124445851462 8x \\
&- 0.38588015352242961253736071326443x^2 + 0.11957609598803011459926868234429x^3 \\
&- 0.45990975352311215122437488337075x^4 - 0.018303047655183152633191740311798x^5 \\
&+ 0.44466745994230334666544621905283x^6 + 0.033571311023070247866057615292453x^7 \\
&- 0.34372789436825171639120298460665x^8 - 0.060651949100270289422238417737264x^9 \\
&+ 0.30334873542187870909937957214362x^{10} + 0.052315295137658297472040650112024x^{11} \\
&- 0.19973785767335854706251257421515x^{12} - 0.030609348282218398386502933456887x^{13} \\
&+ 0.097187674035918304282034552037615x^{14} + 0.012589004544530699924645495973274x^{15} \\
&- 0.034989630985363320704390719551656x^{16} - 0.0036989821850187474104418143123830x^{17} \\
&+ 0.0093369608328499138145142346286141x^{18} + 0.00077929464727475644160584482934880x^{19} \\
&- 0.0018466561206280930425777250414344x^{20} - 0.00011680791800533221919906604578339x^{21} \\
&+ 0.00026885126737717450460781142868745x^{22} + 0.000012231966062757211985323335116315x^{23} \\
&- 0.000028349234146608257851394122088017x^{24} - 8.6576284244624322623832581585300 \times 10^{-7}x^{25} \\
&+ 2.1008607759101228977502132541349 \times 10^{-6}x^{26} + 3.9007349475273390341489 4 \times 10^{-8}x^{27} \\
&- 1.0358731048470561147166 93 \times 10^{-7}x^{28} - 9.9408322809006282609 82 \times 10^{-10}x^{29} \\
&+ 3.0484927702162776234 53 \times 10^{-9}x^{30} + 1.0643142231781590490 2 \times 10^{-11}x^{31} \\
&- 4.0508938624998977029 8 \times 10^{-11}x^{32},
\end{aligned} \tag{A2}$$

$$
\begin{aligned}
f_{Ex1.5}^{\text{Approx}}(x) = {} & 94.995295558172288575843562700072 4 + 247.30799798670279401830448491989x \\
& - 2634.7932752084380565740091010687x^2 - 3163.0023752583513818114558187892x^3 \\
& + 17953.961380123496386092841086331x^4 + 14137.779617234448376149298081174x^5 \\
& - 54459.746468487633963452571905625x^6 - 31725.586823615064968093615356719x^7 \\
& + 91776.631735721885396776907275250x^8 + 41744.882454694374609339894391657x^9 \\
& - 96535.216849252200167737451886831x^{10} - 35210.673453176199122440143515731x^{11} \\
& + 67956.755277316441285105440038493x^{12} + 20135.198934958441323982226135593x^{13} \\
& - 33467.851447262190100555122502335x^{14} - 8092.1697852088571125757835256706x^{15} \\
& + 11861.766636166742885556825723344x^{16} + 2336.4579263963797920622817727614x^{17} \\
& - 3076.4088389621614874188061965879x^{18} - 489.96580474507120891132972642895x^{19} \\
& + 588.01198571838430923678530410523x^{20} + 74.666673892335789436512966630660x^{21} \\
& - 82.640235907329215222126816364852x^{22} - 8.1784106401079202750026312675171x^{23} \\
& + 8.4287682112428365864901538052357x^{24} + 0.62694395673464725980327956988794x^{25} \\
& - 0.60649642563397015786211567822436x^{26} - 0.031911545018419223242512711116367x^{27} \\
& + 0.029172052944526810690102832068280x^{28} + 0.00096842289112048201253538790181069x^{29} \\
& - 0.00084134592459307975967380942641415x^{30} - 0.000013255999472044831595820093036456x^{31} \\
& + 0.000010998279543500795592974484282866x^{32}, \qquad \text{(A3)}
\end{aligned}
$$

$$
\begin{aligned}
f_{Ex1.6}^{pprox}(x) = {} & -2.1435969250030013079212072726761 - 12.993473449061048493747172552187x \\
& - 0.020397782420490154903345399066774x^2 - 0.016587395365450849422477398735882x^3 \\
& + 0.0059863595317084625114314500974322x^4 + 0.0059344287469257593529500617793276x^5 \\
& - 0.0020565778565312581753891697307491x^6 - 0.0034945533879001622186740400753584x^7 \\
& + 0.00065902281474811109860614115957068x^8 + 0.0031903623036145027489085330722852x^9 \\
& - 0.00020396710478272294379811751865836x^{10} - 0.0025455574421552089069300900070871x^{11} \\
& + 0.000060679844472628157671483933722639x^{12} + 0.0014621195900220608632056791592542x^{13} \\
& - 0.000016623559513594330415892306697989x^{14} - 0.00059652789338299040482572155627516x^{15} \\
& + 3.9776013816159836590049878238720 \times 10^{-6}x^{16} + 0.00017503915298854976065223695711275x^{17} \\
& + 7.9269125048174425897568800610153 \times 10^{-7}x^{18} - 0.000037276537058220980720445026654265x^{19} \\
& + 1.2668634382135457570830 54 \times 10^{-7}x^{20} + 5.7635980349887861297551044 \times 10^{-6}x^{21} \\
& - 1.5731626349268425 1515354 \times 10^{-8}x^{22} - 6.4004369539704484313965009 \times 10^{-7}x^{23} \\
& + 1.4704851988814032548515 \times 10^{-9}x^{24} + 4.9717500690021717716577 1 \times 10^{-8}x^{25} \\
& - 9.9407566276234820853 1 \times 10^{-11}x^{26} - 2.5633724746876746968856 \times 10^{-9}x^{27} \\
& + 4.5708067704990083808 2 \times 10^{-12}x^{28} + 7.8780364702947856953 6 \times 10^{-11}x^{29} \\
& - 1.276090018196807157 \times 10^{-13}x^{30} - 1.0919687262091150624 \times 10^{-12}x^{31} \\
& + 1.6299522860073145 \times 10^{-15}x^{32}, \qquad \text{(A4)}
\end{aligned}
$$

$$
\begin{aligned}
f_{Ex1.7}^{\text{Approx}}(x) = {} & -5.8203153035496555320487850658416 - 0.034380326233149364417917759854266x \\
& + 1.1525807794130743814618028395161x^2 + 0.19880942365214364842354512538919x^3
\end{aligned}
$$

$$
\begin{aligned}
&- 0.1736319867398750096866597558325 8x^{4} - 0.030177526767387504269525174722971x^{5} \\
&+ 0.032500104992046098412435106447187x^{6} + 0.0026107666339909906907131034967145x^{7} \\
&- 0.010829014096455174747250213089941x^{8} - 0.00011148142668802387676223252360151x^{9} \\
&+ 0.0040178382920104669747061859226430x^{10} - 0.00010948276820087188631128914136845x^{11} \\
&- 0.0013955832208013342578232577842592x^{12} + 0.00010100410387921373333008829089285x^{13} \\
&+ 0.00043587145454259025042157435603655x^{14} - 0.000049164196420801595858861734759377x^{15} \\
&- 0.00011574291813397346171130728255320x^{16} + 0.000015947590201919924285621747194200x^{17} \\
&+ 0.000024983294798015299644764619526670x^{18} - 3.6280701714730055310907956 \times 10^{-6}x^{19} \\
&- 4.2446404228101957539163872 \times 10^{-6}x^{20} + 5.8726450863728138644056 53 \times 10^{-7}x^{21} \\
&+ 5.5294120106475455580836 37 \times 10^{-7}x^{22} - 6.7406807699388988003662 8 \times 10^{-8}x^{23} \\
&- 5.3724434200170772303465 7 \times 10^{-8}x^{24} + 5.3695248802699594305202 \times 10^{-9}x^{25} \\
&+ 3.7513276832503824710073 \times 10^{-9}x^{26} - 2.8262966896609419939 71 \times 10^{-10}x^{27} \\
&- 1.7737410358170015608 46 \times 10^{-10}x^{28} + 8.8480453956853749021 \times 10^{-12}x^{29} \\
&+ 5.0762720144297917774 \times 10^{-12}x^{30} - 1.248425173198603131 \times 10^{-13}x^{31} \\
&- 6.63181113177527444 \times 10^{-14}x^{32}. \qquad \text{(A5)}
\end{aligned}
$$