Combinatorial Approximation Method for the Fractional Stochastic Hamilton–Jacobi–Bellman Equation

2020MSC Classification: 60G22, 60H20, 60J65

Abstract

We introduce a combinatorial method for approximating the solution of a very complicated nonlinear fractional stochastic partial differential equation (SPDE) which appears in optimal stochastic control. We extend our previous research on the fractional SABR (Stochastic Alpha Beta Rho) model where we could derive only an approximation of the shadow price without the explicit formulas for utility function maximization. We aim to solve the equation by integrating combinatorial techniques with fractional calculus to address the system's inherent randomness and memory effects. The ensuing approximation framework provides analytical tractability for the fractional stochastic Hamilton–Jacobi–Bellman equation and shows promise of applicability to fields like quantitative finance, physics, and engineering, where sound decision-making under uncertainty is critical.

Key words: *Optimal stochastic control, fractional stochastic Hamilton-Jacobi-Bellman equation,*

1 Introduction

We consider a portfolio composed of risky and less risky assets following the stochastic differential equations $dF_t = \mu F_t dt + \sigma F_t dB_t$ and $dF_t^0 = rF_t^0 dt$, where r is the constant less risky rate. Let λ_t^0 be the proportion of the less risky asset and λ_t^1 be that of the risky asset. For any pair $\theta(t) = (\lambda_t^0, \lambda_t^1)_{0\le t\le T}$, with $T \in \mathbb{R}_+$, the value of the portfolio at time t is given by $V_\theta(t) = \lambda_t^0 F_t^0 + \lambda_t^1 F_t$. Portfolio optimization consists of determining an optimal allocation $\theta(t) = (\lambda_t^0, \lambda_t^1)_{0\le t\le T}$, which maximizes the expected utility function of terminal wealth $R_{x,T} = x + \int_0^T \lambda_t^1 dF_t$, i.e., finding the pair $\theta(t) = (\lambda_t^0, \lambda_t^1)$ that maximizes

$$\mathbb{E}\left[U\left(R_{x,t}\right)\right] = \mathbb{E}\left[U\left(x + \int_0^T \lambda_t^1 dF_t\right)\right] \tag{1}$$

where U is a utility function that satisfies the conditions of risk aversion In our previous works ([14],[7],[1]), under certain conditions and assumptions, we derived the following fractional stochastic partial differential equation:

$$\frac{\varepsilon^2 \delta_t^1 p(t, F_{t,\Delta}^{\circ,H}) \left(\dfrac{\partial p(t, F_{t,\Delta}^{\circ,H})}{\partial F_{t,\Delta}^{\circ,H}}\right)^2}{\delta_t^0 + \delta_t^1 p(t, F_{t,\Delta}^{\circ,H})} =$$

$$\frac{\partial p(t, F_{t,\Delta}^{\circ,H})}{\partial t} + Ht^{2H-1}\sigma_t^2 (F_{t,\Delta}^{\circ,H})^2 \frac{\partial^2 p(t, F_{t,\Delta}^{\circ,H})}{\partial (F_{t,\Delta}^{\circ,H})^2} \sqrt{1-\rho^2} M(t) \frac{\partial p(t, F_{t,\Delta}^{\circ,H})}{\partial F_{t,\Delta}^{\circ,H}} \tag{2}$$

$$\varepsilon = \sqrt{1-\rho^2}\eta\Delta^{H-1/2}\sigma_t F^{\circ,H}_{t,\Delta}$$

and

$$M(t) = C\sqrt{2H}\left(H-\frac{1}{2}\right)\int_{-\infty}^{t}(t-s+\Delta)^{H-3/2}dW_s$$

An approximation of the solution to this SPDE is necessary. The function $p(t,x)$ is a C^3 class function and always satisfies assumptions (H_1) to (H_4) in [14] as follows:

$$(H_1): \quad \left|\frac{\partial p(t,x)}{\partial t} - \frac{\partial p(t,y)}{\partial t}\right| \le K_1\,|x-y|$$

$$(H_2): \quad \left|\frac{\partial p(t,x)}{\partial x}\right| \le K_2$$

$$(H_3): \quad \left|\frac{\partial^2 p(t,x)}{\partial x^2}\right| \le K3$$

$$(H_4): \quad \left|\frac{\partial^3 p(t,x)}{\partial x^3}\right| \le K_4$$

where K_1, K_2, K_3, K_4 are real constants.

2 Solving the SPDE

After analyzing the SPDE, we will combine several methods to approximate a solution.

2.1 Analysis of the SPDE (2)

The stochastic partial differential equation exhibits a quadratic nonlinearity in $\left(\frac{\partial p}{\partial F}\right)^2$ and a multiplicative stochastic term $M(t)\frac{\partial p}{\partial F}$. To simplify our notation, we set $\frac{\partial p(t,F^{\circ,H}_{t,\Delta})}{\partial F^{\circ,H}_{t,\Delta}} = \partial_{F}p$, $\frac{\partial^2 p(t,F^{\circ,H}_{t,\Delta})}{\partial (F^{\circ,H}_{t,\Delta})^2} = \partial_{FF}p$, $p(t,F^{\circ,H}_{t,\Delta}) = p$, and $F^{\circ,H}_{t,\Delta} = F$.

- **Left-hand term**: $$\frac{\varepsilon^2\delta_t^1 p(\partial_F p)^2}{\delta_t^0 + \delta_t^1 p} \text{ is nonlinear, controlled by } \varepsilon^2.$$
- **Right-hand term**:

$$\partial_t p + \underbrace{Ht^{2H-1}\sigma_t^2 F^2\partial_{FF}p}_{\text{Anisotropic diffusion}} + \underbrace{\sqrt{1-\rho^2}M(t)\partial_F p}_{\text{Stochastic transport}}.$$

This complexity motivates the use of combined methods.

2.2 Definitions and Concepts

In the context of the HJB equation, the objective is to solve an optimal control problem where the value function $V(t, x)$ satisfies an equation of the form:

$$\frac{\partial V}{\partial t}(t,x) + \sup_{a \in A}\Big\{\mathcal{L}^a V(t,x) + l(t,x,a)\Big\} = 0, \quad V(T,x) = g(x), \tag{3}$$

where:

- $\mathcal{L}^a$ is the infinitesimal generator associated with the controlled SDE

$$dX_s = \mu(s, X_s, a_s)\, ds + \sigma(s, X_s, a_s)\, dW_s,$$

- $l(t, x, a)$ is the instantaneous cost, and $g(x)$ the terminal condition,
- A denotes the set of admissible controls.

The idea is to interpret the SPDE (2) as the dynamic equation (or the optimality condition) associated with an optimal control problem. To do so, we must:

- Identify the unknown function $p(t, F)$ as the value function of the control problem.
- Find a control problem whose dynamics (the controlled SDE) and total cost generate, via the dynamic programming principle, an HJB equation of the form (3).

By applying the dynamic maximum (or minimum) principle, we define the Hamiltonian

$$H(t, F, \partial_F, \partial_{FF}; a) = \mu(t,F,a)\,\partial_F + \frac{1}{2}\sigma^2(t,F,a)\,\partial_{FF} + l(t,F,a),$$

and the HJB equation then becomes

$$\frac{\partial V}{\partial t}(t,F) + \sup_{a \in A} H(t, F, V_F, V_{FF}; a) = 0.$$

For more details, see [9], [3], [13], [5], and [10].

In our SPDE, the nonlinear terms in p and its derivatives can be reinterpreted in terms of an *optimal control* problem involving the choice of an action a (which may appear implicitly in the quadratic term).

Our SPDE is not in the form of a classical HJB equation. The Cole-Hopf transformation is thus necessary to bring the equation into a linear or quasi-linear form. Using this transformation, the equation takes the form:

$$\frac{\partial u}{\partial t} + \mathcal{L}^a u - r(t,F)\, u = 0,$$

with an operator $\mathcal{L}^a$ corresponding to the controlled dynamics. The solution is given by

$$u(t,F) = \mathbb{E}\left[\exp\left(-\int_t^T r(s, X_s)\, ds\right) g(X_T) \,\Big|\, X_t = F\right],$$

and the value function V (or p) is recovered by inverting the transformation.

Furthermore, the optimal control is given by the maximizer of the Hamiltonian:

$$a^*(t,F) = \arg\sup_{a \in A}\Big\{\mu(t,F,a)\, V_F(t,F) + \frac{1}{2}\sigma^2(t,F,a)\, V_{FF}(t,F) + l(t,F,a)\Big\}.$$

The Cole-Hopf transformation introduces a new variable $u(t, F)$ defined by

$$u(t,F) = \ln\Big(\delta_t^0 + \delta_t^1\, p(t,F)\Big).$$

Thus,

$$p(t,F) = \frac{e^{u(t,F)} - \delta_t^0}{\delta_t^1}.$$

Differentiating with respect to F, we obtain:

$$\frac{\partial p}{\partial F}(t,F) = \frac{e^{u(t,F)}\, u_F(t,F)}{\delta_t^1},$$

and the second derivative:

$$\frac{\partial^2 p}{\partial F^2}(t,F) = \frac{e^{u(t,F)}\Big(u_{FF}(t,F) + (u_F(t,F))^2\Big)}{\delta_t^1}.$$

After substitution, the SPDE (2) transforms into an equation in $u(t,F)$:

$$\varepsilon^2\,(e^u - \delta_t^0)e^u\,(u_F)^2 = \delta_t^1\,u_t + H\,t^{2H-1}\,\sigma_t^2\,F^2\,\sqrt{1-\rho^2}\,M(t)\,e^u\,u_F\Big(u_{FF} + (u_F)^2\Big) + \dots,$$

where the dots represent additional terms related to the time derivatives of the coefficients δ_t^0 and δ_t^1.

Now, we consider $u(t,F)$ as the value function of the optimal control problem and apply the Hamilton–Jacobi–Bellman (HJB) method to the obtained equation.

2.3 Formulation of the Optimal Control Problem

We aim to interpret the function $u(t,F)$ as the value function of the following control problem. Suppose the controlled dynamic system is described by the SDE

$$dF_s = \mu(s,F_s,a_s)\,ds + \sigma(s,F_s,a_s)\,dW_s, \quad s \in [t,T], \tag{4}$$

with $a_s \in A$ (the set of admissible controls). The total associated cost is defined as

$$J(t,F;a) = \mathbb{E}\left[\int_t^T L(s,F_s,a_s)\,ds + g(F_T)\,\Big|\,F_t = F\right],$$

and the value function is given by

$$u(t,F) = \inf_{a\in\mathcal{A}} J(t,F;a).$$

The dynamic programming principle (see [4], [2]) implies that $u(t,F)$ satisfies the Hamilton–Jacobi–Bellman equation

$$u_t(t,F) + \inf_{a\in A}\left\{\mu(t,F,a)\,u_F(t,F) + \frac{1}{2}\sigma^2(t,F,a)\,u_{FF}(t,F) + L(t,F,a)\right\} = 0, \tag{5}$$

with terminal condition

$$u(T,F) = g(F).$$

In our case, following the Cole–Hopf transformation ([11],[12]), the transformed SPDE formally takes the form

$$\frac{e^u\,u_t}{\delta_t^1} + \mathcal{N}(t,F,u,u_F,u_{FF}) = \frac{\varepsilon^2\,(e^u - \delta_t^0)e^u\,(u_F)^2}{(\delta_t^1)^2}, \tag{6}$$

where the term $\mathcal{N}$ notably includes the diffusion term arising from the transformation:

$$\mathcal{N}(t,F,u,u_F,u_{FF}) = H\,t^{2H-1}\,\sigma_t^2\,F^2\,\sqrt{1-\rho^2}\,M(t)\,\frac{e^{2u}\,u_F\Big(u_{FF} + (u_F)^2\Big)}{(\delta_t^1)^2} + \cdots.$$

We can rearrange this equation in the form

$$u_t(t,F) + \tilde{H}\Big(t,F,u,u_F,u_{FF}\Big) = 0,$$

where

$$\tilde{H}\Big(t,F,u,u_F,u_{FF}\Big) = \frac{\delta_t^1}{e^u}\left[\frac{\varepsilon^2\,(e^u-\delta_t^0)e^u\,(u_F)^2}{(\delta_t^1)^2} - \mathcal{N}(t,F,u,u_F,u_{FF})\right].$$

This equation can be interpreted as an HJB equation associated with an optimal control problem, with the Hamiltonian

$$H(t,F,u_F,u_{FF};a) = \mu(t,F,a)\,u_F + \frac{1}{2}\sigma^2(t,F,a)\,u_{FF} + L(t,F,a),$$

such that

$$\inf_{a\in A} H(t,F,u_F,u_{FF};a) = \tilde{H}\Big(t,F,u,u_F,u_{FF}\Big).$$

In other words, the structure obtained through the Cole–Hopf transformation allows us to identify the coefficients of a control problem whose value function $u(t,F)$ satisfies the HJB equation (5).

2.4 Application of the HJB Method

See [8],[6] for more details

To solve the HJB equation (5), the approach is as follows:

We assume that the coefficients μ, σ, and the instantaneous cost L depend on a control a. The optimal control $a^*(t,F)$ is given by the minimizer of the Hamiltonian:

$$a^*(t,F) = \arg\min_{a\in A}\left\{\mu(t,F,a)\,u_F(t,F) + \frac{1}{2}\sigma^2(t,F,a)\,u_{FF}(t,F) + L(t,F,a)\right\}.$$

This minimization condition is obtained via the dynamic maximum (or minimum) principle.

Under sufficient regularity assumptions (notably $u \in C^3$ in F and sufficiently smooth in t), the HJB equation

$$u_t(t,F) + \min_{a\in A}\left\{\mu(t,F,a)\,u_F(t,F) + \frac{1}{2}\sigma^2(t,F,a)\,u_{FF}(t,F) + L(t,F,a)\right\} = 0,$$

with terminal condition $u(T,F) = g(F)$, can be tackled using various analytical or numerical methods. For instance:

- **Semigroup method and Fourier transform:** By applying the Fourier transform in F, the differential operator is transformed into multiplication by its symbol, reducing the equation to an ODE in t for each frequency.
- **Iterative methods and numerical schemes:** In the nonlinear case, fixed-point or discretization methods (finite differences, finite elements) are employed.

Once a candidate solution $u(t,F)$ is obtained, the verification theorem ensures that it is indeed the value function of the control problem and that the optimal control is given by

$$a^*(t,F) = \arg\min_{a\in A}\left\{\mu(t,F,a)\,u_F(t,F) + \frac{1}{2}\sigma^2(t,F,a)\,u_{FF}(t,F) + L(t,F,a)\right\}.$$

In summary, after applying the Cole–Hopf transformation and obtaining an equation in $u(t,F)$, we identify it as the value function of an optimal control problem. The Hamilton–Jacobi–Bellman equation then corresponds to

$$u_t(t,F) + \min_{a\in A}\left\{\mu(t,F,a)\,u_F(t,F) + \frac{1}{2}\sigma^2(t,F,a)\,u_{FF}(t,F) + L(t,F,a)\right\} = 0,$$

with terminal condition $u(T,F) = g(F)$. The result of this approach is the pair (u,a^*) characterizing the optimal solution:

- $u(t, F)$ gives the optimal value (after transformation, related to $p(t, F)$ via

$$p(t, F) = \frac{e^{u(t,F)} - \delta_t^0}{\delta_t^1}.$$

- The optimal control $a^*(t, F)$ is determined by the minimizer of the Hamiltonian.

Now, let us use the Feynman–Kac method.

We previously applied the Cole–Hopf transformation to our stochastic partial differential equation (SPDE) and, through a Hamilton–Jacobi–Bellman (HJB) approach, obtained a linear (or quasi-linear) equation in the transformed function $u(t, F)$:

$$u_t(t, F) + \mu(t, F, a^*(t, F))\, u_F(t, F) + \frac{1}{2}\sigma^2(t, F, a^*(t, F))\, u_{FF}(t, F) + L(t, F, a^*(t, F)) = 0,$$

with terminal condition

$$u(T, F) = g(F).$$

Here, $a^*(t, F)$ is the optimal control minimizing the Hamiltonian associated with the optimal control problem.

We now apply the Feynman–Kac formula to represent the solution $u(t, F)$ of this equation as a conditional expectation.

Let $u(t, F)$ be the solution of the linear parabolic equation

$$\begin{cases} u_t(t, F) + \mu(t, F)\, u_F(t, F) + \frac{1}{2}\sigma^2(t, F)\, u_{FF}(t, F) - r(t, F)\, u(t, F) = 0, & t \in [0, T),\ F \in \mathbb{R}, \\ u(T, F) = g(F), \end{cases} \tag{7}$$

where $\mu(t, F)$ and $\sigma(t, F)$ are sufficiently regular functions, and $r(t, F)$ is a cost rate term.

The Feynman–Kac formula states that the solution of (7) is given by

$$u(t, F) = \mathbb{E}\left[\exp\left(-\int_t^T r(s, F_s)\, ds\right) g(F_T) \mid F_t = F\right],$$

where the process $(F_s)_{s \geq t}$ satisfies the SDE

$$dF_s = \mu(s, F_s)\, ds + \sigma(s, F_s)\, dW_s, \quad F_t = F.$$

In our case, after Cole–Hopf transformation and identification via the HJB method, the equation obtained is

$$u_t(t, F) + \mu(t, F, a^*(t, F))\, u_F(t, F) + \frac{1}{2}\sigma^2(t, F, a^*(t, F))\, u_{FF}(t, F) + L(t, F, a^*(t, F)) = 0,$$

with $u(T, F) = g(F)$.

To recast this equation into the form of (7), we identify:

$$\tilde{\mu}(t, F) = \mu(t, F, a^*(t, F)), \quad \tilde{\sigma}(t, F) = \sigma(t, F, a^*(t, F)),$$

and set

$$r(t, F) = -L(t, F, a^*(t, F)).$$

Thus, the equation becomes

$$u_t(t, F) + \tilde{\mu}(t, F)\, u_F(t, F) + \frac{1}{2}\tilde{\sigma}^2(t, F)\, u_{FF}(t, F) - r(t, F)\, u(t, F) = 0.$$

Under these assumptions, the Feynman–Kac formula provides the solution

$$u(t,F) = \mathbb{E}\left[\exp\left(-\int_t^T r(s,F_s)\,ds\right) g(F_T) \mid F_t = F\right].$$

Substituting $r(t,F) = -L(t,F,a^*(t,F))$, we obtain:

$$u(t,F) = \mathbb{E}\left[\exp\left(\int_t^T L(s,F_s,a^*(s,F_s))\,ds\right) g(F_T) \mid F_t = F\right].$$

This representation expresses the solution $u(t,F)$ in terms of the conditional expectation of a terminal function weighted by the accumulation of the instantaneous cost along the controlled process trajectories. We consider the transformed version of the SPDE obtained after applying the Cole–Hopf transformation and the formulation by the Hamilton–Jacobi–Bellman equation. To simplify the presentation, we assume that the coefficients (after transformation) depend only on t and not on F. Thus, the equation reduces to:

$$u_t(t,F) + \tilde{\mu}(t)\,u_F(t,F) + \frac{1}{2}\tilde{\sigma}^2(t)\,u_{FF}(t,F) - r(t)\,u(t,F) = 0, \quad t \in [0,T),\ F \in \mathbb{R}, \tag{8}$$

with terminal condition

$$u(T,F) = g(F).$$

Here:

- $\tilde{\mu}(t) = \mu(t,F,a^*(t,F))$ and $\tilde{\sigma}(t) = \sigma(t,F,a^*(t,F))$ (assumed independent of F for simplicity),
- $r(t) = -L(t,F,a^*(t,F))$ is the (modified) cost term,
- $u(t,F)$ is the value function obtained after the transformation.

We will apply the Fourier transform with respect to F to convert the differential operator into a multiplicative operator, and then use the semigroup method to solve the equation, which reduces the PDE to an ODE in t for each frequency ξ.

2.5 Using the Fourier Transform in F

Define the Fourier transform of $u(t,F)$ by:

$$\widehat{u}(t,\xi) = \int_{-\infty}^{+\infty} e^{-i\xi F}\,u(t,F)\,dF,$$

and the inverse transform by:

$$u(t,F) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\xi F}\,\widehat{u}(t,\xi)\,d\xi.$$

Let us apply the Fourier transform to equation (8).

We have the classical properties:

$$\mathcal{F}\{u_F(t,F)\}(\xi) = i\xi\,\widehat{u}(t,\xi),$$
$$\mathcal{F}\{u_{FF}(t,F)\}(\xi) = -\xi^2\,\widehat{u}(t,\xi).$$

Applying the Fourier transform to (8) yields:

$$\mathcal{F}\{u_t(t,F)\}(\xi) + \tilde{\mu}(t)\,\mathcal{F}\{u_F(t,F)\}(\xi) + \frac{1}{2}\tilde{\sigma}^2(t)\,\mathcal{F}\{u_{FF}(t,F)\}(\xi) - r(t)\,\widehat{u}(t,\xi) = 0,$$

$$\Rightarrow \quad \widehat{u}_t(t,\xi) + \tilde{\mu}(t)\,(i\xi)\,\widehat{u}(t,\xi) - \frac{1}{2}\tilde{\sigma}^2(t)\,\xi^2\,\widehat{u}(t,\xi) - r(t)\,\widehat{u}(t,\xi) = 0.$$

We can factorize $\widehat{u}(t,\xi)$:

$$\widehat{u}_t(t,\xi) + \left[i\xi\,\tilde{\mu}(t) - \frac{1}{2}\tilde{\sigma}^2(t)\,\xi^2 - r(t)\right]\widehat{u}(t,\xi) = 0. \tag{9}$$

2.6 Reduction of the SPDE to a PDE in t for each frequency ξ

Equation (9) is a linear ordinary differential equation (ODE) in t for each value of ξ. Its general form is:

$$\frac{d\widehat{u}}{dt}(t,\xi) + \lambda(t,\xi)\,\widehat{u}(t,\xi) = 0,$$

with

$$\lambda(t,\xi) = i\xi\,\tilde{\mu}(t) - \frac{1}{2}\tilde{\sigma}^2(t)\,\xi^2 - r(t).$$

The solution of this ODE (evaluated for t decreasing from T to t) is given by:

$$\widehat{u}(t,\xi) = \widehat{u}(T,\xi)\,\exp\Big(-\int_t^T \lambda(s,\xi)\,ds\Big).$$

Since the terminal condition is $u(T,F) = g(F)$, we have:

$$\widehat{u}(T,\xi) = \widehat{g}(\xi).$$

Thus,

$$\widehat{u}(t,\xi) = \widehat{g}(\xi)\,\exp\left(-\int_t^T \Big[i\xi\,\tilde{\mu}(s) - \frac{1}{2}\tilde{\sigma}^2(s)\,\xi^2 - r(s)\Big]\,ds\right). \tag{10}$$

The expression (10) can be interpreted as the action of a semigroup $\{S(t,T)\}_{0\le t\le T}$ defined by:

$$\widehat{u}(t,\xi) = m(t,T,\xi)\,\widehat{g}(\xi),$$

with the multiplier (or symbol of the semigroup)

$$m(t,T,\xi) = \exp\left(-\int_t^T \Big[i\xi\,\tilde{\mu}(s) - \frac{1}{2}\tilde{\sigma}^2(s)\,\xi^2 - r(s)\Big]\,ds\right).$$

By applying the inverse Fourier transform, we recover:

$$u(t,F) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\xi F}\,\widehat{g}(\xi)\,m(t,T,\xi)\,d\xi.$$

This result shows that the solution $u(t,F)$ is expressed as the action of a linear semigroup on the final data $g(F)$.

Recalling that the Cole–Hopf transformation was posed in the form

$$u(t,F) = \ln\Big(\delta_t^0 + \delta_t^1\,p(t,F)\Big),$$

the original solution is recovered by

$$p(t,F) = \frac{e^{\,u(t,F)} - \delta_t^0}{\delta_t^1}.$$

A representation of the solution in the form:

$$\widehat{u}(t,\xi) = \widehat{g}(\xi)\,\exp\Big(-\int_t^T \lambda(s,\xi,u(s,\cdot))\,ds\Big),$$

where $\lambda(s,\xi,u(s,\cdot))$ potentially depends on u (thus the nonlinearity is reflected in the dependence of the symbol). Hence, the solution $u(t,F)$ can be written via the inverse Fourier transform:

$$u(t,F) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\xi F}\,\widehat{g}(\xi)\,\exp\Big(-\int_t^T \lambda(s,\xi,u(s,\cdot))\,ds\Big)d\xi.$$

This expression suggests a fixed-point formulation. We will define an operator $\mathcal{T}$ on a suitable space and show, under Lipschitz assumptions, that it is contractive. The fixed point of $\mathcal{T}$ will be the desired solution.

Define the operator $\mathcal{T}$ acting on a function $u(t, F)$ belonging to a Banach space X (for example, the space of continuous functions on $[0, T]$ with values in $L^2(\mathbb{R})$) by:

$$\mathcal{T}(u)(t, F) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\xi F}\, \widehat{g}(\xi)\, \exp\Big(-\int_t^T \lambda(s, \xi, u(s, \cdot))\, ds\Big) d\xi.$$

Here, $\lambda(s, \xi, u(s, \cdot))$ is the symbol (or effective function) derived from the semigroup method which depends on u (for instance, through the coefficients obtained after transformation). The terminal condition is incorporated via $\widehat{g}(\xi)$ (the Fourier transform of the terminal data $g(F)$).

To apply the Banach fixed-point theorem, we must show that $\mathcal{T}$ is contractive on X, meaning that there exists $0 < \kappa < 1$ such that for all $u, v \in X$,

$$\|\mathcal{T}(u) - \mathcal{T}(v)\|_X \le \kappa \, \|u - v\|_X.$$

The key assumption is that the function $\lambda(s, \xi, u)$ is Lipschitz in u; that is, there exists a constant $L > 0$ such that, for all s, ξ and for all u, v:

$$\big|\lambda(s, \xi, u) - \lambda(s, \xi, v)\big| \le L \, \|u - v\|_Y,$$

where $\|\cdot\|_Y$ is an appropriate norm (for example, in $L^2(\mathbb{R})$). Then, by estimating the difference

$$\mathcal{T}(u)(t, F) - \mathcal{T}(v)(t, F),$$

and using the inequality

$$\Big|e^{-A} - e^{-B}\Big| \le |A - B| \, e^{-\min\{A, B\}},$$

one obtains an inequality of the form

$$\|\mathcal{T}(u) - \mathcal{T}(v)\|_X \le L' \, (T - t) \, \|u - v\|_X.$$

For $T - t$ sufficiently small or under appropriate assumptions, one can have $L' \, (T - t) < 1$, ensuring that $\mathcal{T}$ is contractive.

The iterative method is then given by:

$$u_{n+1}(t, F) = \mathcal{T}(u_n)(t, F), \quad n \ge 0,$$

with an initial condition u_0 chosen appropriately (for example, $u_0(t, F) = 0$ or an approximation of $g(F)$). By the Banach fixed-point theorem, the sequence $(u_n)_{n\ge 0}$ converges to the fixed point $u^*(t, F)$ which satisfies:

$$u^*(t, F) = \mathcal{T}(u^*)(t, F).$$

Recall that the Cole–Hopf transformation was given by:

$$u(t, F) = \ln\Big(\delta_t^0 + \delta_t^1 \, p(t, F)\Big),$$

so the solution of the original SPDE is recovered by:

$$p(t, F) = \frac{e^{\,u(t,F)} - \delta_t^0}{\delta_t^1}.$$

3 Conclusion

In this paper, we have proposed a combinatorial method to approximate the solution of a complex and nonlinear SPDE. However, this work can be improved by identifying the best pair (δ_t^0, δ_t^1) that maximizes the value process.

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