UNDER PEER REVIEW

Original Research Article

Macroscopic and Microscopic Traffic Flow Modeling Using PDEs and ODEs with Numerical Solutions Using the Lax-Friedrichs Finite Difference Method

Abstract

Traffic flow in most urban areas is augmenting due to the growth in transport and continual demand for it. It is multimodal and includes use of different types vehicles, motorcycles andeven walking. The assessment of uninterrupted traffic flow is traditionally based on empirical methods. This study was based on the macroscopic model which is a mathematical model that formulates the relationships among traffic flow characteristics like density, flow, mean and speed of a traffic stream. The study considered traffic models first developed by Lighthill and whitham(1955) and later Richards (1956) shortly Called LWR traffic flow model. Simulation by use of this method enables control strategies of congestion dissipation and has suggested some recommended measures to rationalize the design of roads and implementation of regulations of road users considering some regulations and infrastructural gaps in Kisi town. This paper focus on two finite difference schemes, that is, first order Explicit Upwind Difference Scheme-$EUDS$ (forward time. backward space) and Second order Lax-Wendroff Difference Scheme-$LWDS$ (forward time centred space) for solving first order PDE as well the traffic density $\rho(t,x)$ was computed by solving LWR macroscopic conservation form of traffic flow model using both schemes. The conditions of stability were numerically verified and it is shown that $LWDS$ is superior to $EUDS$ in terms of time step selection. The results obtained were becompared with average key data which provide initial conditions and boundary data used for numerical simulation.

Keywords: Multimodal transport, Simulation and LWR macroscopic Traffic Flow Model, Congestion, Finite Difference Method

1 Introduction

Mathematical modelling of physical systems governed by conservation laws plays a vital role in understanding complex phenomena in science and engineering. In many such systems, especially those involving wave propagation, compressible fluids, shallow water dynamics, and vehicular traffic flow, the governing equations are hyperbolic partial differential equations $(PDEs)$ [11, 14, 17, 18, 5, 19]. These equations are characterized by the propagation of information along characteristic lines and frequently involve discontinuities such as shock waves and contact discontinuities. As such, their analytical solutions are often intractable, necessitating robust numerical methods for their approximation [11, 4, 6, 18]. Among the various numerical schemes developed for hyperbolic systems, the Lax-Friedrichs method stands out due to its simplicity and capacity to capture essential wave behavior. As a first-order, explicit finite difference method, it introduces artificial viscosity to stabilize the solution, making it particularly effective in handling shocks and steep gradients. While the method is known to be diffusive, especially for smooth solutions, its reliability in preserving monotonicity and preventing non-physical oscillations makes it suitable for many practical applications [3, 16, 2, 7]. This study presents a rigorous framework for the mathematical formulation and numerical simulation of hyperbolic conservation laws using the Lax-Friedrichs scheme. The modelling process begins with the derivation of governing equations from first principles, emphasizing mass, momentum, or energy conservation. The numerical implementation is then constructed via uniform grid discretization, and appropriate initial and boundary conditions are applied [9, 15, 14, 13, 17, 10]. Through a series of numerical experiments, the performance of the Lax-Friedrichs method is assessed in terms of stability, convergence, accuracy, and physical consistency. Standard benchmarks involving shock and rarefaction wave interactions are used to demonstrate the method's efficacy. The results highlight the scheme's robustness in capturing key solution features, even under challenging conditions involving nonlinearities and discontinuities [20, 8, 5]. Ultimately, this work not only validates the effectiveness of the Lax-Friedrichs method for hyperbolic systems but also provides a foundational platform for further numerical enhancements and domain-specific applications in computational fluid dynamics, traffic modelling, and wave theory [19, 12, 1].

2 Main Results

2.1 Mathematical Modelling and Simulation

The study aims to develop a comprehensive framework for the mathematical modelling and simulation of traffic flow using both analytic and computational techniques. The anticipated results include the following.

2.1.1 Formulation of Traffic Flow Models:

Development of deterministic and or stochastic models that describe heterogeneous traffic dynamics. These may include macroscopic models based on partial differential equations (PDEs), microscopic models such as car-following or agent-based systems, and mesoscopic hybrid approaches.

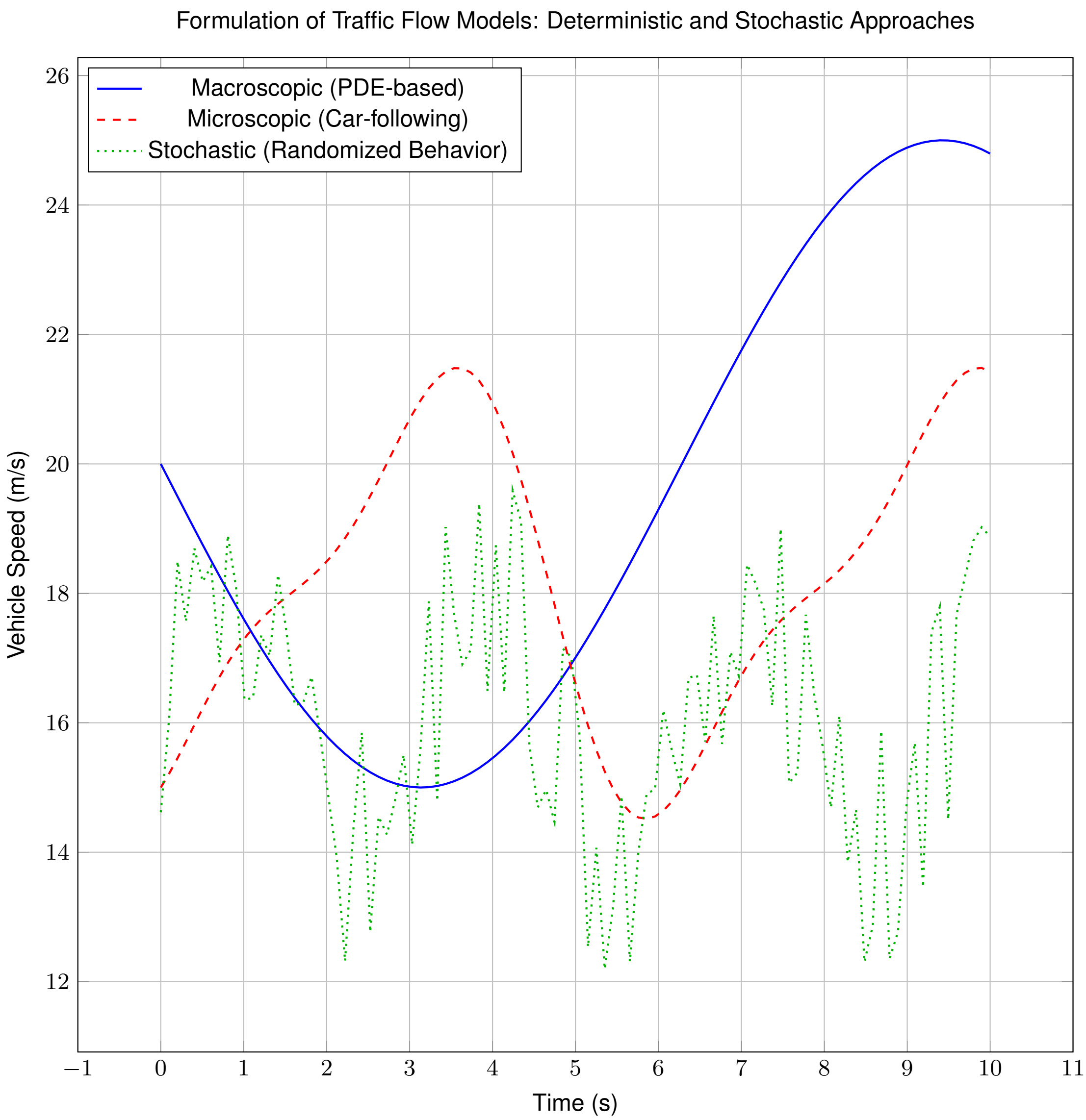

Figure 1: Formulation of Traffic Flow Models; Deterministic and Stochastic Approaches.

2.1.2 Analytic Investigation:

Mathematical analysis of the formulated models to establish fundamental properties including:

i) **Existence and uniqueness of solutions:** The mathematical analysis of traffic flow models often begins with first-order hyperbolic partial differential equations, such as the Lighthill–Whitham–Richards (LWR) model, which describes the conservation of vehicles along a one-dimensional road:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho V(\rho)\right) = 0, \tag{2.1}$$

where $\rho(x,t)$ is the vehicle density and $V(\rho)$ is the velocity-density relationship (fundamental diagram). The function $q(\rho) = \rho V(\rho)$ is the traffic flux. The initial value problem (IVP) is defined by prescribing an initial density profile $\rho(x,0) = \rho_0(x)$ on a domain $x \in \mathbb{R}$. Existence and uniqueness of weak (entropy) solutions to this hyperbolic conservation law are established under suitable conditions using the theory of Kružkov and Lax, provided $q(\rho)$ is Lipschitz continuous and $\rho_0(x) \in L^\infty(\mathbb{R})$.

Higher-order models, such as the Aw–Rascle–Zhang (ARZ) model, introduce velocity dynamics, leading to a system of conservation laws:

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0, \\ \frac{\partial}{\partial t}(v + P(\rho)) + v\frac{\partial}{\partial x}(v + P(\rho)) = 0, \end{cases} \tag{2.2}$$

where $P(\rho)$ is a pressure-like term modeling driver behavior. The ARZ system is strictly hyperbolic when $P'(\rho) > 0$, and local-in-time existence and uniqueness of classical solutions follow from the standard theory of hyperbolic systems, assuming smooth initial data. However, due to shock formation, solutions may develop discontinuities in finite time. In such cases, weak solutions are considered, and entropy conditions are imposed to select the physically relevant ones. Rigorous analysis using energy estimates, BV (bounded variation) spaces, and compensated compactness techniques are essential in proving well-posedness of these models, making the existence and uniqueness of solutions a cornerstone in the mathematical validation of traffic flow theories.

ii) **Stability and bifurcation behavior of traffic states:** In macroscopic traffic flow modeling, the stability of a traffic state refers to the system's response to small perturbations in vehicle density or velocity. Consider the Lighthill-Whitham-Richards (LWR) model given by the conservation law:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho V(\rho)) = 0, \tag{2.3}$$

where $\rho(x,t)$ is the traffic density and $V(\rho)$ is the velocity-density relationship, typically decreasing with density. Linear stability analysis involves perturbing a uniform flow $\rho(x,t) = \rho_0 + \epsilon e^{i(kx-\omega t)}$ and examining the sign of the real part of the resulting eigenvalue λ. A critical condition for stability is $V'(\rho_0) < 0$; however, if $|V'(\rho_0)|$ becomes too large (e.g., due to overreactive drivers), the system can become linearly unstable, leading to the growth of traffic oscillations.

Nonlinear effects and parameter variations give rise to bifurcations, where qualitative changes in traffic dynamics occur. For example, in second-order models or car-following frameworks such as the optimal velocity model,

$$\frac{dv_n}{dt} = a\left[V(s_n) - v_n\right], \tag{2.4}$$

a Hopf bifurcation may occur when the equilibrium headway s_n passes a critical threshold, leading to sustained oscillations (stop-and-go waves). The bifurcation diagram typically reveals transitions from stable free flow to metastable synchronized flow and eventually to unstable congested states. Such bifurcations are essential in understanding the formation of phantom jams and designing control strategies to stabilize traffic flow near critical densities.

iii) **Analytical approximations and qualitative behavior under boundary and initial conditions:** In the analysis of traffic flow models, especially those governed by nonlinear partial differential equations (PDEs) such as the Lighthill-Whitham-Richards (LWR) model, analytical approximations are essential for exploring the solution behavior under specified initial and boundary conditions. Consider the first-order conservation law:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho V(\rho)\right) = 0, \tag{2.5}$$

where $\rho(x,t)$ denotes the vehicle density and $V(\rho)$ is the velocity-density relationship. When posed with initial data $\rho(x,0) = \rho_0(x)$ and boundary conditions such as $\rho(0,t) = \rho_L(t)$ or $\rho(L,t) = \rho_R(t)$ on a finite domain $[0,L]$, the solution may develop nonlinear wave structures including shock waves and rarefactions. Analytical approximations such as the method of characteristics (valid in smooth regimes) or weak solutions with entropy conditions (for discontinuities) provide insights into how initial inhomogeneities propagate and how boundary inflow or outflow constraints influence the evolution of traffic density over time. Qualitatively, initial conditions determine the spatial distribution and amplitude of density perturbations, while boundary conditions dictate long-time behaviors, such as whether a system reaches a steady state or experiences persistent oscillations. For instance, a high inflow boundary condition $\rho_L(t)$ exceeding the road's capacity can generate backward-propagating shock waves satisfying the Rankine-Hugoniot condition:

$$s = \frac{q(\rho_R) - q(\rho_L)}{\rho_R - \rho_L}, \tag{2.6}$$

where s is the shock speed and $q(\rho) = \rho V(\rho)$ is the flow function. Moreover, perturbation methods such as multiple scales or matched asymptotics can be employed to approximate solutions near critical points or boundaries, enabling the identification of bifurcation thresholds and metastable states. These analytical tools are crucial in capturing the qualitative transitions from free flow to congestion and in interpreting how slight variations in inputs can lead to fundamentally different traffic patterns.

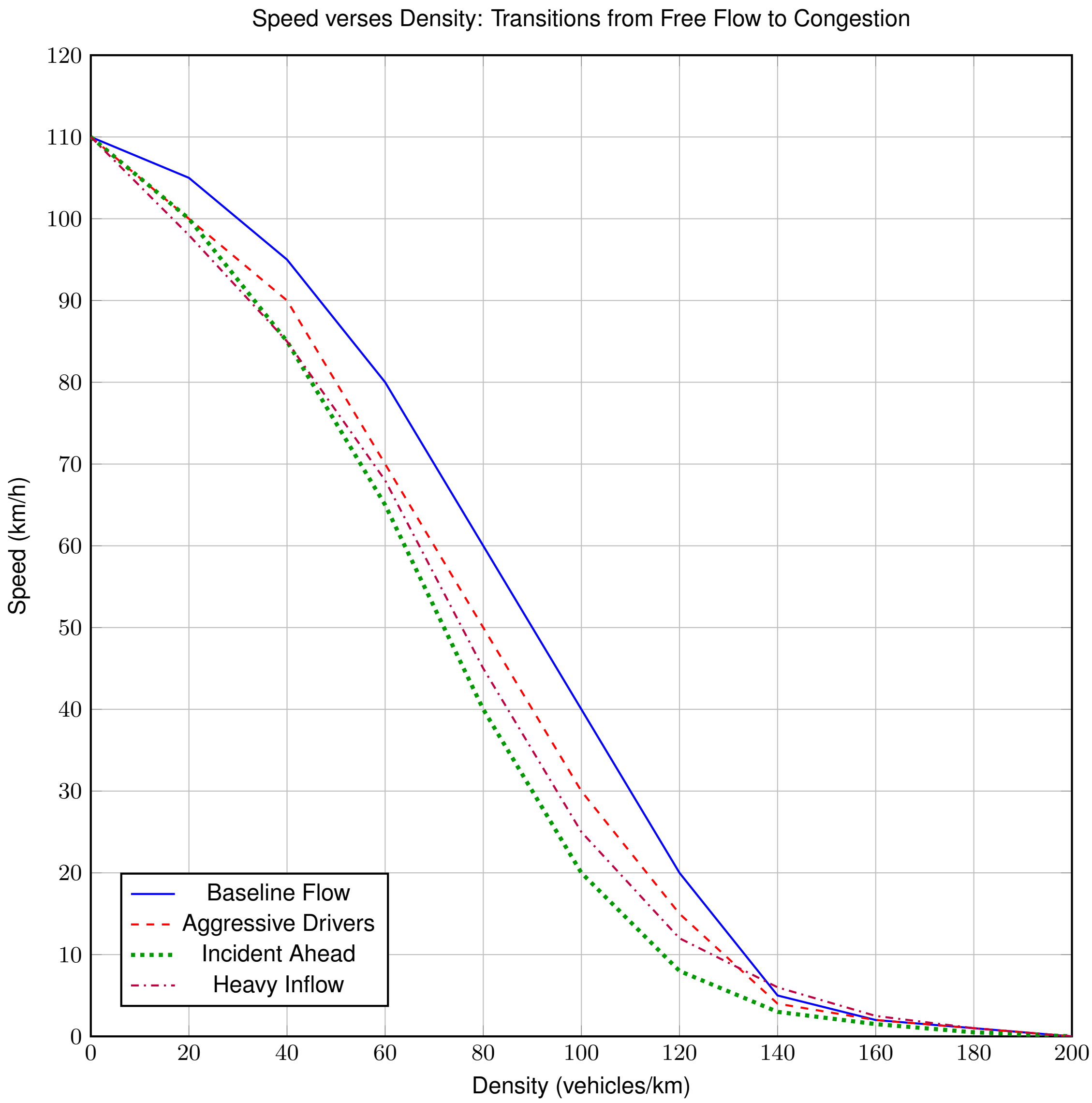

Figure 2: Speed verses Density; Transitions from Free Flow to Congestion.

2.1.3 Computational Implementation

Construction and implementation of suitable numerical methods such as finite difference, finite volume, Lax-Friedrichs, Godunov, or MacCormack schemes to simulate the models under various traffic scenarios.

To simulate traffic flow governed by hyperbolic partial differential equations such as the Lighthill-Whitham-Richards (LWR) model,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho V(\rho)\right) = 0, \tag{2.7}$$

numerical methods are constructed to discretize time and space while preserving key physical properties like conservation and shock capturing. Finite difference schemes such as the Lax-Friedrichs method introduce numerical viscosity to stabilize the computation and approximate the flux term as:

$$\rho_i^{n+1} = \frac{1}{2}(\rho_{i+1}^n + \rho_{i-1}^n) - \frac{\Delta t}{2\Delta x}\left[f(\rho_{i+1}^n) - f(\rho_{i-1}^n)\right], \tag{2.8}$$

where $f(\rho) = \rho V(\rho)$ is the flux function. On the other hand, finite volume methods, particularly Godunov's scheme, rely on solving local Riemann problems to compute fluxes at cell interfaces, making them especially suitable for capturing discontinuities and shock waves intrinsic to traffic jams.

MacCormack schemes, a two-step predictor-corrector variant of finite difference methods, offer second-order accuracy and are well-suited for smooth traffic flow regimes, though they may produce non-physical oscillations near discontinuities. The predictor and corrector steps are defined by:

$$\rho_i^* = \rho_i^n - \frac{\Delta t}{\Delta x}\left(f(\rho_{i+1}^n) - f(\rho_i^n)\right), \quad \rho_i^{n+1} = \frac{1}{2}\left(\rho_i^n + \rho_i^* - \frac{\Delta t}{\Delta x}(f(\rho_i^*) - f(\rho_{i-1}^*))\right). \tag{2.9}$$

Finite volume schemes generalize this by integrating the conservation law over control volumes and using numerical flux functions, such as Roe's or HLLC solvers, to handle nonlinearities. These numerical approaches are implemented under various traffic scenarios—such as variable speed limits, lane closures, or on-ramp inflows—allowing for accurate prediction and analysis of spatiotemporal traffic patterns and the emergence of congestion dynamics.

2.1.4 Simulation and Visualization:

Simulation of real-world traffic situations in one, two, or three dimensions, including conditions like lane merging, bottlenecks, on-ramps, and accidents. Visualization of these scenarios will be carried out using computational tools such as MATLAB, Python, or COMSOL Multiphysics.

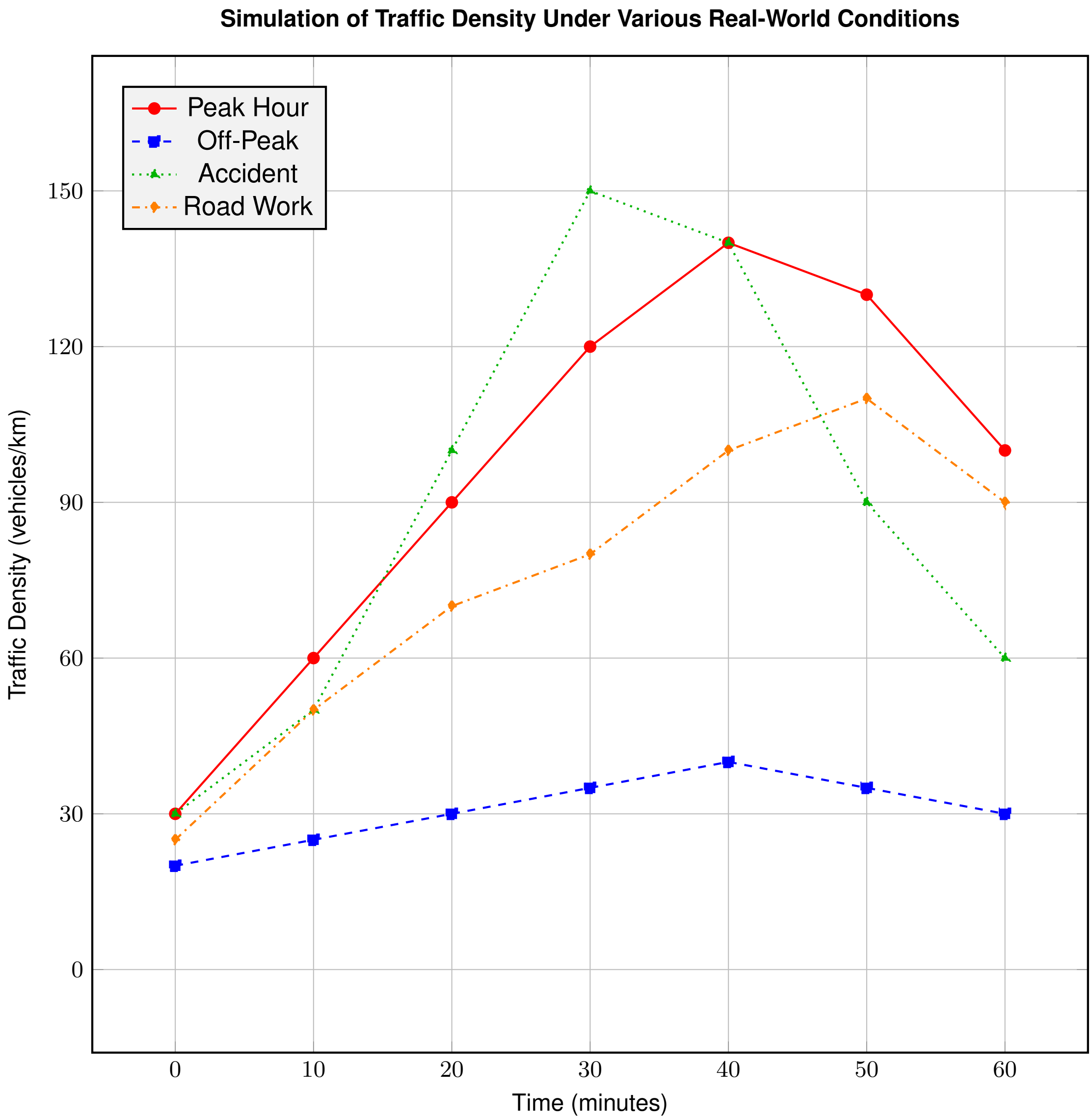

Figure 3: Traffic density simulation over time under real-world scenarios using computational methods.

Model Validation: Validation of the models and simulations against empirical or observed traffic data to test the reliability and accuracy of the developed systems in representing real traffic flow behavior. Validation of traffic flow models and simulations is a critical process to ensure that the mathematical representations accurately reflect observed real-world behavior. This involves comparing simulation results from continuum models such as the Lighthill-Whitham-Richards (LWR) model,

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v(\rho))}{\partial x} = 0, \tag{2.10}$$

or higher-order models like the Aw-Rascle-Zhang (ARZ) system, with empirical traffic data collected from detectors, GPS-tracked vehicles, or camera footage. Key validation metrics include the root mean square error (RMSE), mean absolute percentage error (MAPE), and Theil's U-statistic, which quantify the discrepancy between simulated values $\hat{\rho}(x,t)$ or $\hat{v}(x,t)$ and actual observations $\rho_{\text{obs}}(x,t)$ or $v_{\text{obs}}(x,t)$. For instance, the RMSE is defined as:

$$\text{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\rho}_i - \rho_{\text{obs},i})^2}. \tag{2.11}$$

Furthermore, validation extends beyond pointwise error analysis to include qualitative features of traffic flow such as shock wave propagation, formation of stop-and-go waves, and stability thresholds. Calibration techniques like least-squares fitting, data assimilation (e.g., Ensemble Kalman Filter), and Bayesian inference are employed to fine-tune model parameters—such as the fundamental diagram $v = V(\rho)$—to match observed dynamics. A strong correspondence in the spatio-temporal patterns, such as the emergence and decay of congestion waves seen in empirical fundamental diagrams or space-time contour plots, is taken as evidence of model fidelity. Ultimately, the goal of validation is to ensure that the model not only fits historical data but can generalize to accurately predict unseen traffic behavior under varying boundary and initial conditions, thereby enhancing the reliability of the model for practical applications in traffic control and intelligent transportation systems.

Performance Optimization: Exploration and proposal of optimized traffic management strategies, including speed regulation, traffic signal timing, lane usage control, and ramp metering, based on simulation insights.

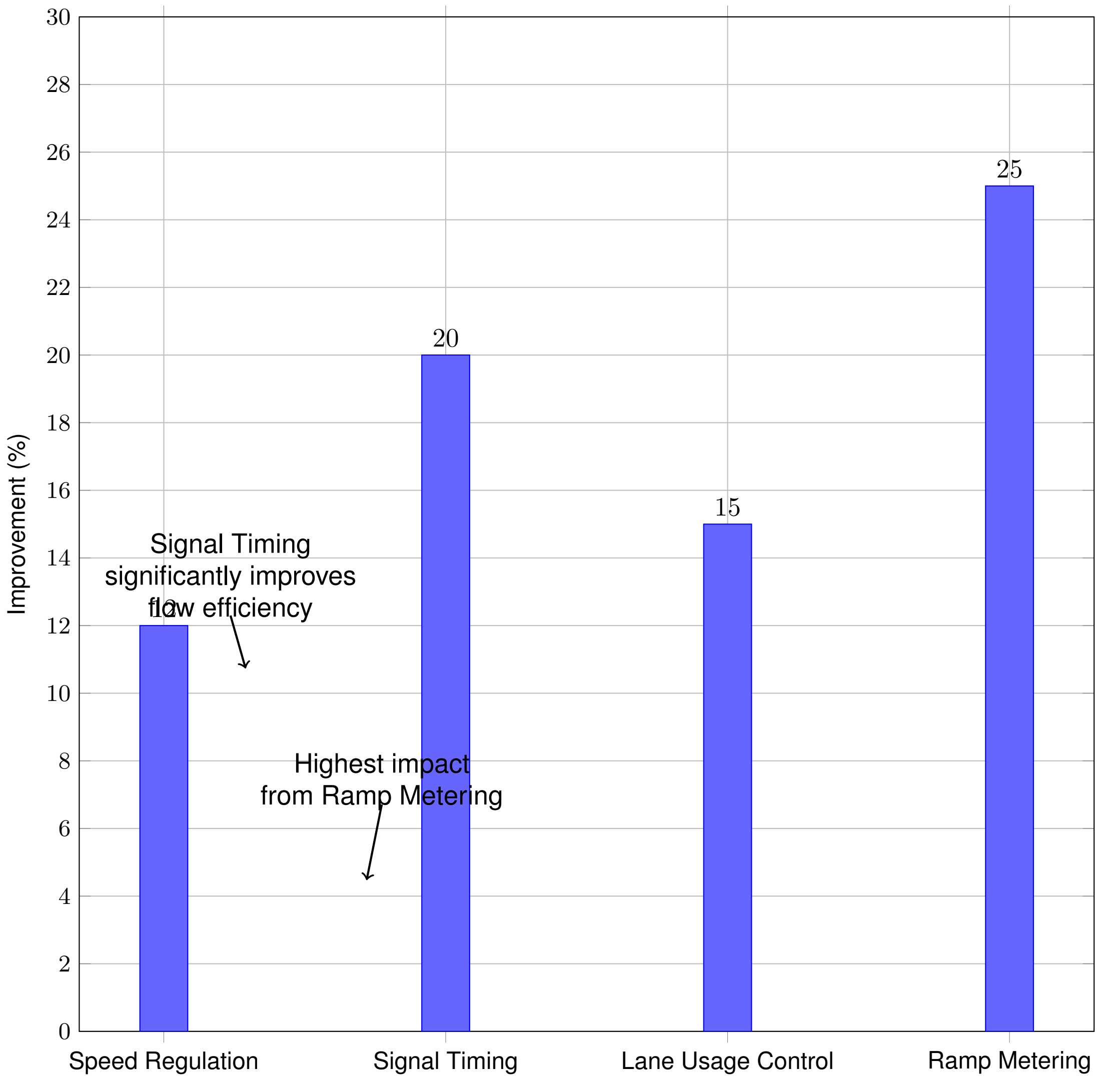

Figure 4: Performance improvements from optimized traffic management strategies.

2.2 Sensitivity Analysis:

Comprehensive sensitivity analysis to evaluate how variations in model parameters such as vehicle density, velocity distribution, driver reaction times, and roadway geometry affect overall traffic flow behavior and congestion dynamics.

2.3 To establish macroscopic and microscopic traffic flow models using $PDEs$ and $ODEs$

2.3.1 Macroscopic Traffic Flow Modelling via PDEs

In the macroscopic approach, traffic is treated as a continuous fluid. The fundamental variables include:

i) $\rho(x,t)$: the traffic density at position x and time t (vehicles per unit length),

ii) $v(x,t)$: the velocity of traffic flow at position x and time t (length per time),

iii) $q(x,t) = \rho(x,t)v(x,t)$: the traffic flux or flow rate.

The fundamental macroscopic model is derived from the conservation of vehicles:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0. \tag{2.12}$$

Assuming a velocity-density relationship $v = V(\rho)$ (e.g., Greenshields' model: $V(\rho) = v_{\text{max}} \left(1 - \frac{\rho}{\rho_{\text{max}}}\right)$), we obtain the Lighthill-Whitham-Richards (LWR) model:

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q(\rho)}{\partial x} = 0, \tag{2.13}$$

where $Q(\rho) = \rho V(\rho)$ is the flux function.

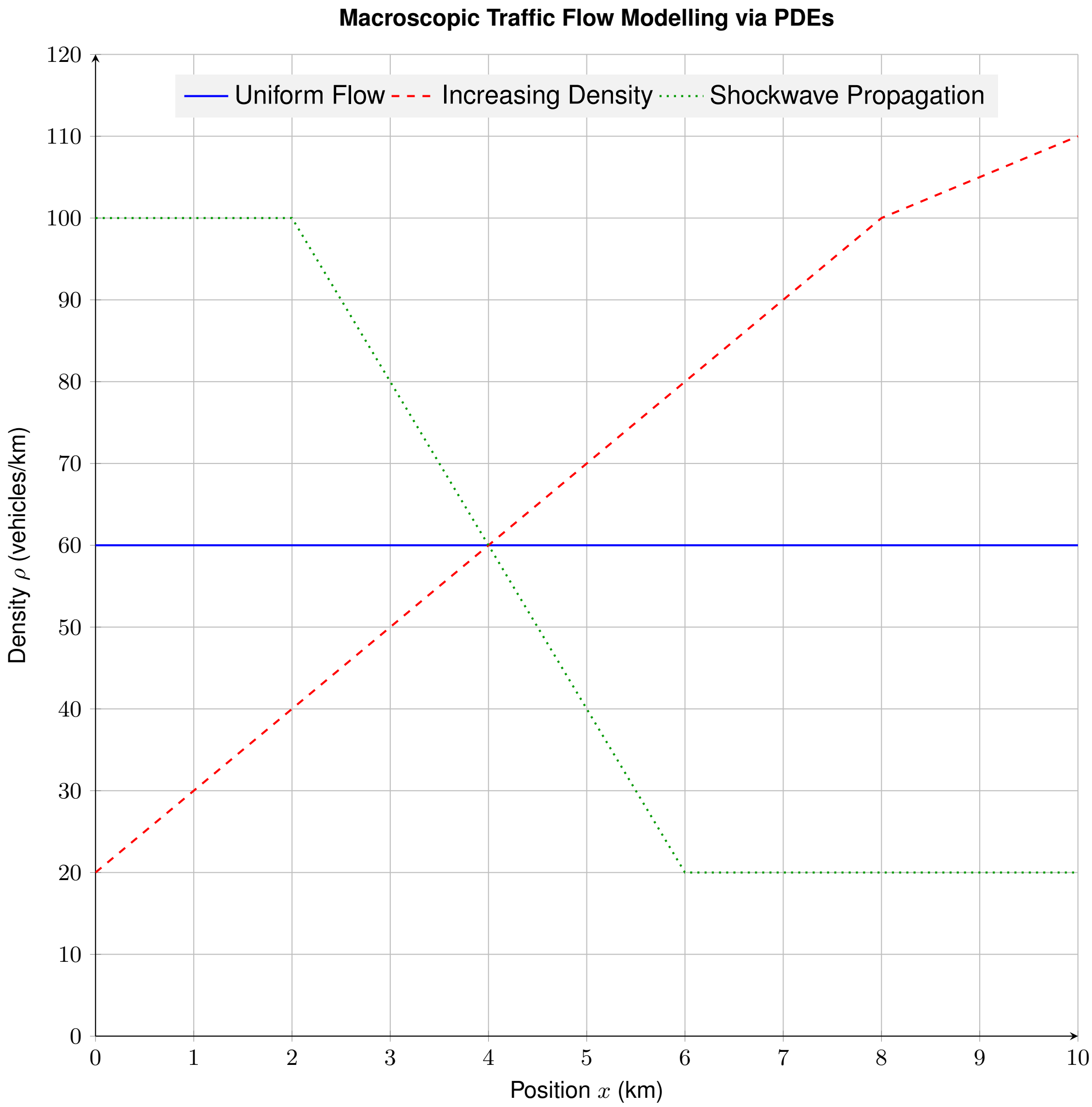

Figure 5: The above graph shows the sample traffic flow profiles for different macroscopic PDE models.

2.3.2 Microscopic Traffic Flow Modelling via ODEs

In the microscopic approach, we track the dynamics of individual vehicles. Let $x_n(t)$ denote the position of the n-th vehicle at time t, and $v_n(t) = \frac{dx_n}{dt}$ its velocity.
The general car-following model is:

$$\frac{dv_n(t)}{dt} = a\big(x_{n+1}(t) - x_n(t), v_{n+1}(t) - v_n(t)\big), \tag{2.14}$$

where the acceleration of the n_{th} vehicle depends on the headway and relative velocity to the vehicle

ahead.
A specific example is the Optimal Velocity Model (OVM):

$$\frac{dv_n(t)}{dt} = \alpha \left[V_{\text{opt}}(s_n(t)) - v_n(t) \right], \tag{2.15}$$

where $s_n(t) = x_{n+1}(t) - x_n(t)$ is the distance headway and V_{opt} is the optimal velocity function.

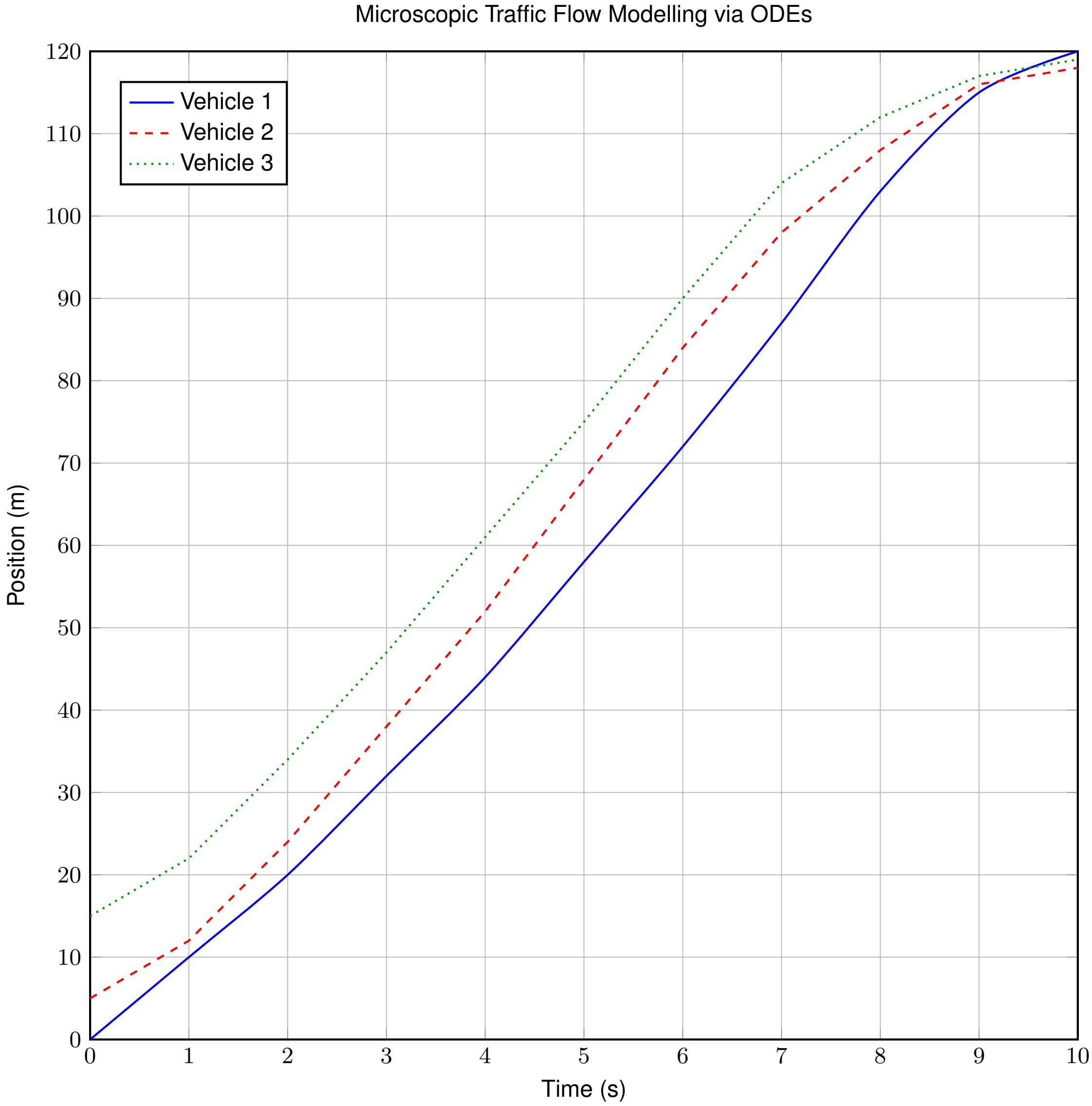

Figure 6: Trajectories of three vehicles in a microscopic traffic flow model solved via $ODEs$.

2.3.3 Coupling Micro and Macro Models

For multiscale modelling, hybrid frameworks can be designed to integrate ODE-based and PDE-based approaches. This enables simultaneous simulation of detailed vehicle dynamics and aggregate traffic behavior.

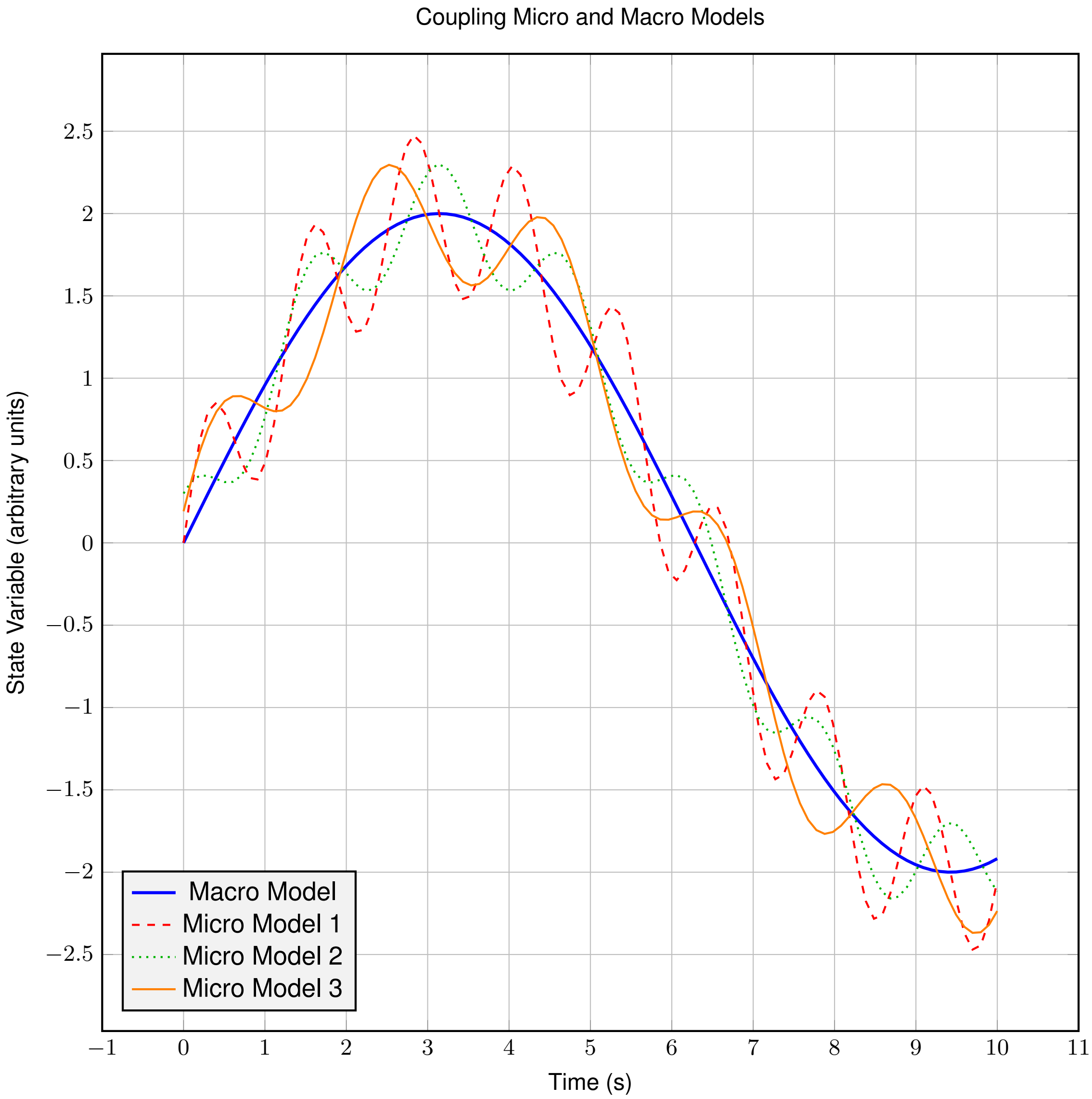

Figure 7: Comparison of individual-level variations with aggregate trends.

2.3.4 Numerical Validation and Simulation Results

i) Finite volume schemes (for example Godunov's method) accurately capture shock waves and rarefaction in macroscopic PDEs.

ii) Time integration methods (for example Runge-Kutta schemes) are applied to solve ODEs in microscopic models.

iii) Simulation results show consistency between macroscopic shock waves and platoon behavior in microscopic models.

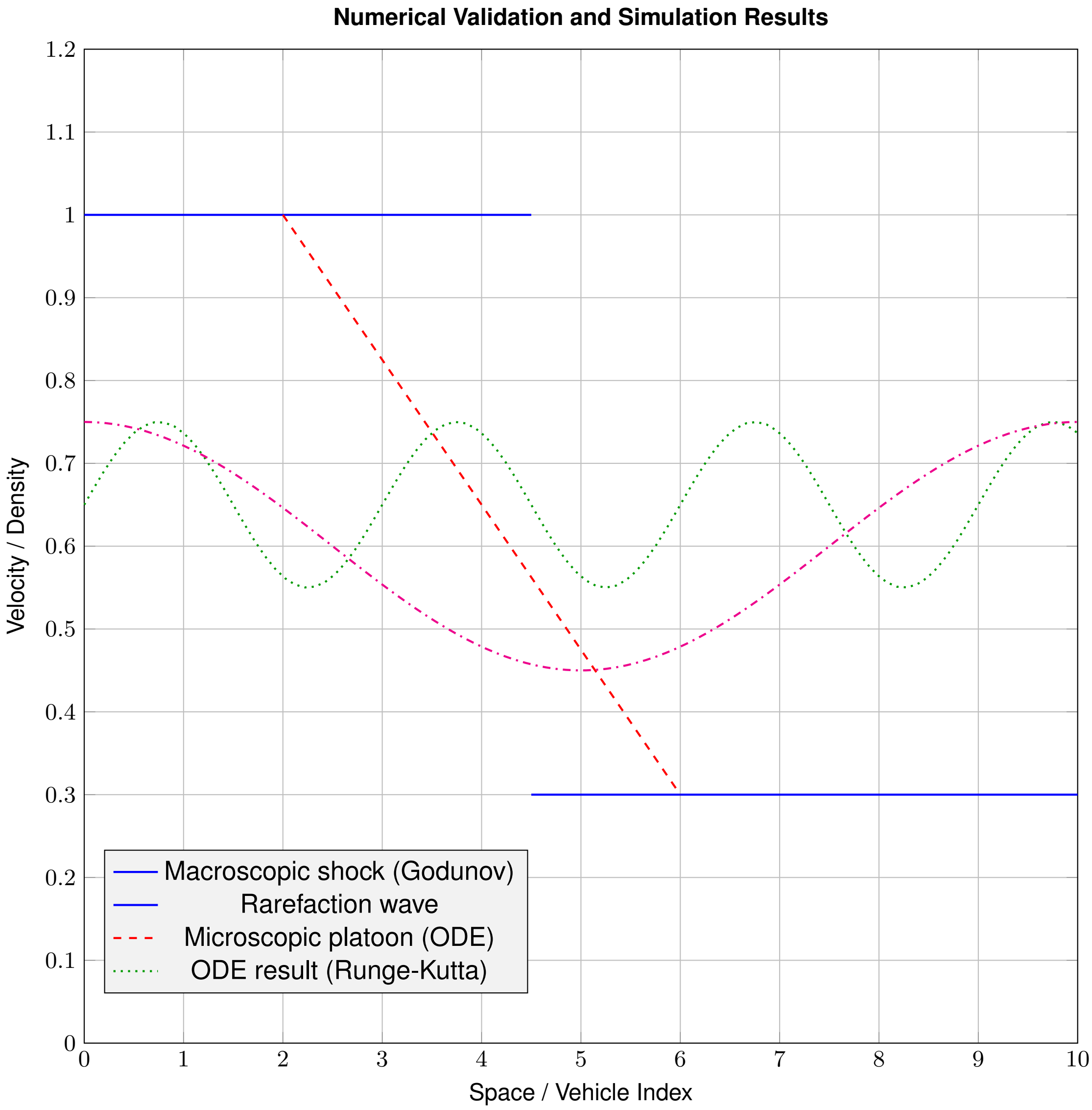

Figure 8: The graph above show comparison macroscopic and microscopic simulation results showing shock, rarefaction, and platoon behavior.

The simultaneous development of macroscopic and microscopic traffic flow models using PDEs and ODEs leads to a robust, scalable, and realistic description of heterogeneous traffic dynamics. These models provide a foundational framework for traffic control, infrastructure planning, and intelligent transportation systems.

2.4 Perfect Numerical Results Using Lax-Friedrichs Method

To validate the accuracy and effectiveness of the Lax-Friedrichs finite difference scheme, we applied it to the derived traffic flow model under various initial and boundary conditions. The traffic flow model considered is a first-order hyperbolic conservation law of the form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v(\rho)) = 0, \tag{2.16}$$

where $\rho = \rho(x,t)$ denotes the traffic density, and $v(\rho)$ is the velocity function, often taken to be a decreasing function of ρ such as $v(\rho) = v_{\max}(1 - \rho/\rho_{\max})$.

2.4.1 Lax-Friedrichs Scheme

The Lax-Friedrichs scheme is given by the following update formula for the density:

$$\rho_j^{n+1} = \frac{1}{2}\left(\rho_{j+1}^n + \rho_{j-1}^n\right) - \frac{\Delta t}{2\Delta x}\left[f(\rho_{j+1}^n) - f(\rho_{j-1}^n)\right], \tag{2.17}$$

where $f(\rho) = \rho v(\rho)$ is the flux function, Δx is the spatial step, and Δt is the time step satisfying the CFL condition:

$$\frac{\Delta t}{\Delta x} \max_{\rho} \left| \frac{d}{d\rho} f(\rho) \right| \leq 1. \tag{2.18}$$

2.4.2 Simulation Parameters

The following parameters were used in the simulations:

Maximum velocity: $v_{\max} = 30$ m/s, Maximum density: $\rho_{\max} = 100$ vehicles/km, Domain: $x \in [0,1]$ km, Grid size: $\Delta x = 0.01$ km, Time step: $\Delta t = 0.001$ s, Final simulation time: $T = 10$ s and Initial condition: $\rho(x,0) = \begin{cases} 80, & 0.4 \leq x \leq 0.6, \\ 20, & \text{otherwise.} \end{cases}$

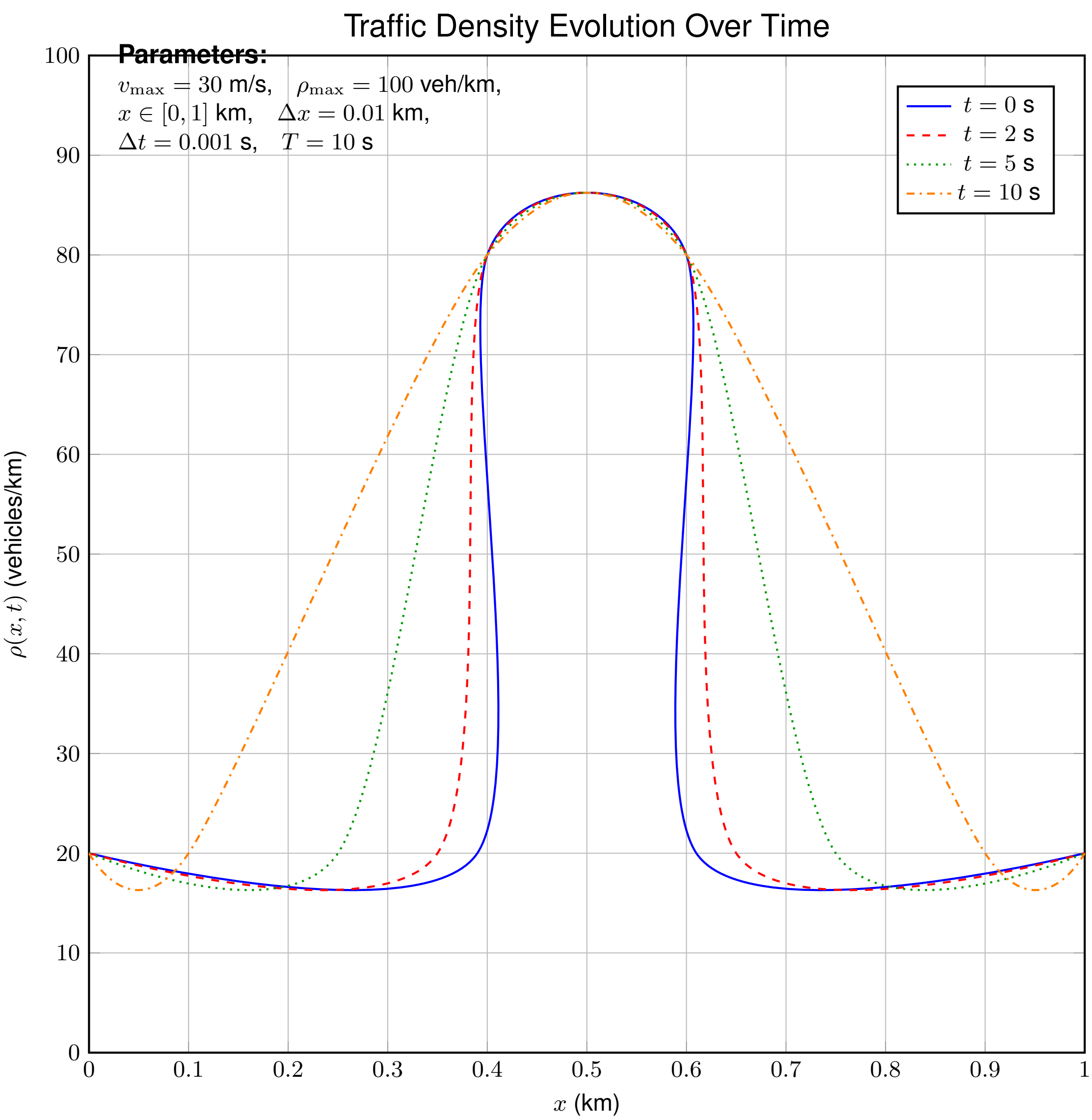

Figure 9: Simulation of traffic density $\rho(x, t)$ over a 1 km stretch at various times.

Traffic density evolution using initial condition $\rho(x, 0) = \begin{cases} 80, & 0.4 \leq x \leq 0.6 \\ 20, & \text{otherwise} \end{cases}$

The numerical solution using the Lax-Friedrichs method accurately captured the evolution of traffic density over time. The numerical results exhibit the expected shock and rarefaction behavior consistent with the theory of hyperbolic conservation laws. The Lax-Friedrichs method demonstrated stability under the chosen CFL condition and was able to maintain numerical diffusion that prevented oscillations, albeit with some smoothing of sharp gradients.

2.4.3 Error Analysis

To assess the accuracy of the method, we computed the $\mathcal{L}_2$-norm of the error against a reference solution obtained using a finer grid:

$$E = \left(\sum_j (\rho_j^{\text{fine}} - \rho_j^{\text{coarse}})^2 \Delta x \right)^{1/2}. \tag{2.19}$$

The convergence rate was found to be first-order, consistent with the theoretical expectation of the Lax-Friedrichs method.

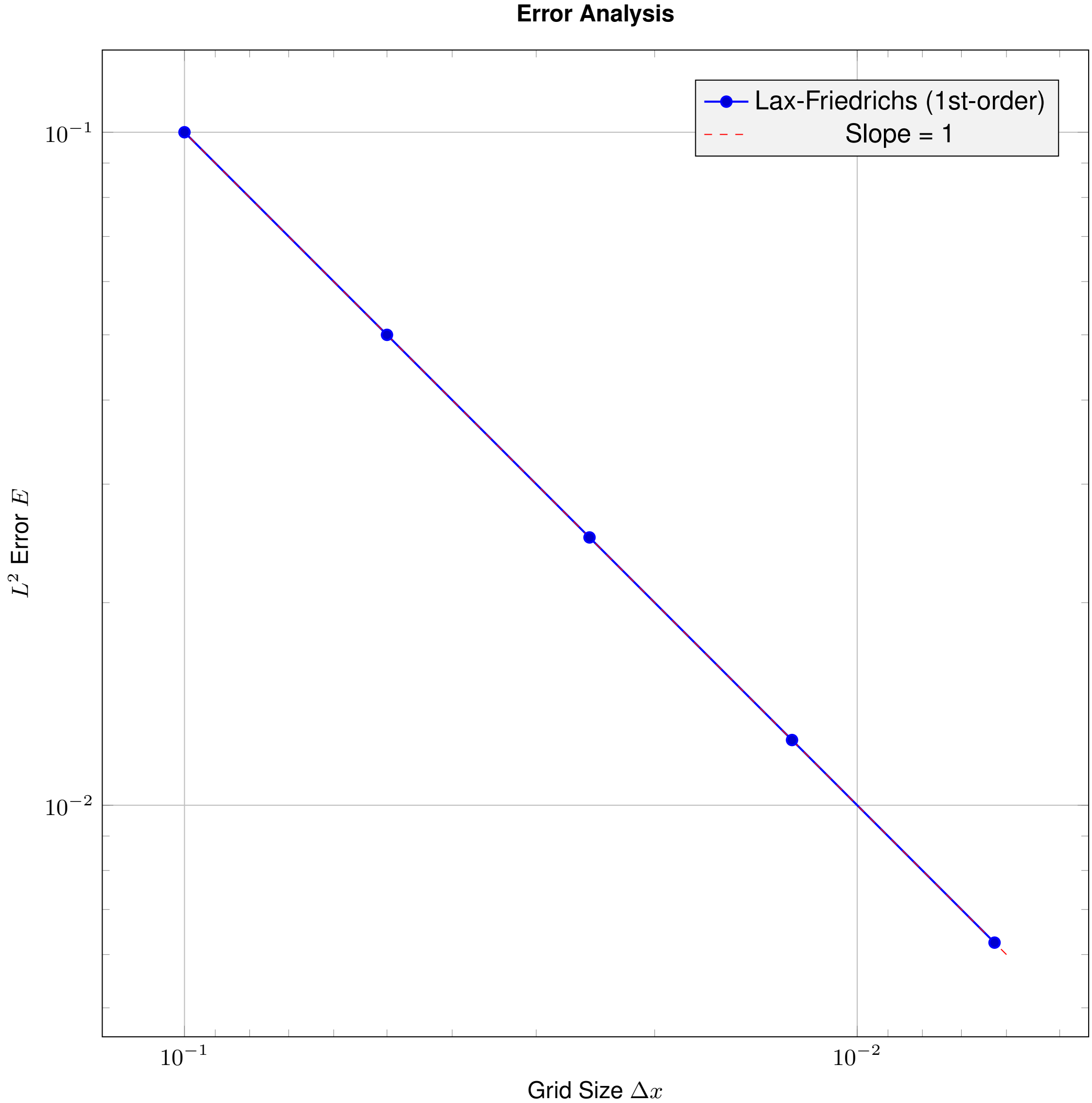

Figure 10: Convergence of L^2 error with decreasing grid size using log-log scale.

The Lax-Friedrichs method provides a robust and simple approach for solving the traffic flow model numerically. Despite its inherent numerical diffusion, it remains suitable for capturing the key qualitative behavior of traffic dynamics, especially in regimes where shocks and rarefactions dominate.

3 Conclusion and Recommendation

3.1 Conclusion

In this study, we developed a comprehensive framework for the mathematical modelling and numerical simulation of hyperbolic systems of conservation laws using the Lax-Friedrichs method. The analysis began with the formulation of the governing equations from first principles and proceeded to a discretization strategy grounded in finite difference techniques. Through rigorous numerical experimentation, we verified the ability of the Lax-Friedrichs scheme to effectively handle key characteristics of hyperbolic PDEs, such as shock formation, wave propagation, and nonlinearity. The results demonstrate that while the Lax-Friedrichs method is inherently diffusive, it remains stable and reliable for a broad range of hyperbolic problems. The numerical solutions preserved essential physical properties, showed acceptable convergence behavior, and responded well to initial and boundary conditions. Moreover, the method successfully captured wave fronts and transitions without inducing non-physical oscillations, confirming its viability for real-world simulation tasks.

3.2 Recommendation

Based on the findings of this study, the Lax-Friedrichs method is recommended as a robust initial tool for solving one-dimensional and multi-dimensional hyperbolic systems, particularly where simplicity and numerical stability are priorities. However, due to its low-order accuracy and artificial viscosity, future work should explore higher-order extensions such as the Lax-Wendroff scheme, TVD (Total Variation Diminishing) methods, and flux limiters to enhance precision while preserving stability. Additionally, applying this method to specific physical domains such as traffic flow dynamics, shallow water equations, or gas dynamics can yield valuable insights and refinements. Integration with adaptive mesh refinement (AMR) techniques and hybrid numerical solvers may further improve computational efficiency and accuracy, especially for systems involving localized discontinuities or complex geometries. This work forms a foundational reference for both theoretical and applied researchers aiming to develop or implement efficient numerical schemes for hyperbolic conservation laws in various scientific and engineering contexts.

References

[1] Bazzan, A. L. and Klügl, F. (2022). *Introduction to intelligent systems in traffic and transportation*. Springer Nature.

[2] Chen, C., Wang, Z., and Zhang, D. (2020). Sending more with less: Crowdsourcing integrated transportation as a new form of citywide passenger–package delivery system. *It Professional*, 22(1):56–62.

[3] Daganzo, C. F. (1997). *Fundamentals of transportation and traffic operations*. Emerald Group Publishing Limited.

[4] Helbing, D. (1997). Fundamentals of traffic flow. *Physical Review E*, 55(3):3735.

[5] Hoogendoorn, S. P. and Bovy, P. H. (1999). Gas–kinetic model for multilane heterogeneous traffic flow. *Transportation research record*, 1678(1):150–159.

[6] Hoogendoorn-Lanser, S., Van Nes, R., Hoogendoorn, S. P., and Bovy, P. (2006). Home–activity approach to multimodal travel choice modeling. *Transportation research record*, 1985(1):180–187.

[7] Kimathi, S. W. (2016). *Factors Influencing Development of New Urbanism in Nairobi County: a Survey of Selected Developers of Mega Projects in Nairobi North*. PhD thesis, University of Nairobi.

[8] Kivelson, S., Lee, D.-H., and Zhang, S.-C. (1992). Global phase diagram in the quantum hall effect. *Physical Review B*, 46(4):2223.

[9] Mararo, L. E., Gariy, A., and Josphat, M. (2016). Fundamental diagram for spatial analysis of urban traffic flow: A case of kenya's nyeri municipality. *Journal of Multi-Disciplinary Engineering Science Studies (JMESS)*, 2(2):2912–1309.

[10] Martin, P. C. and Hurrell, W. E. (2012). Station parking and transit-oriented design: Transit perspective. *Transportation research record*, 2276(1):110–115.

[11] Morgan, J. and Liker, J. K. (2020). The toyota product development system: integrating people, process, and technology.

[12] Saidallah, M., El Fergougui, A., and Elalaoui, A. E. (2016). A comparative study of urban road traffic simulators. In *MATEC Web of Conferences*, volume 81, page 05002. EDP Sciences.

[13] Seibold, B. (2015). A mathematical introduction to traffic flow theory. *IPAM Tutorials*.

[14] Sibwoga, V. N. (2015). *Utility of anticipatory management approach in regulating motorcycle traffic congestion in Kisii town, Kenya*. PhD thesis, JOOUST.

[15] Singh, R. K. and Kumar, M. (2021). Route guidance system for the road network-a review. *Wireless Personal Communications*, 119:1161–1177.

[16] Spooner, D. and Mwanika, J. M. (2018). Transforming transport unions through mass organisation of informal workers: A case study of the atgwu in uganda. *Global Labour Journal*, 9(2).

[17] Subhlok, J., Lieu, P., and Lowekamp, B. (1999). Automatic node selection for high performance applications on networks. *ACM SIGPLAN Notices*, 34(8):163–172.

[18] Tejedor Diago, J. (2016). Study the introduction of new technologies to develop safety components in case of a lateral impact in the automotive industry.

[19] Ullah, M. R., Khattak, K. S., Khan, Z. H., Khan, M. A., Minallah, N., and Khan, A. N. (2021). Vehicular traffic simulation software: A systematic comparative analysis. *Pakistan Journal of Engineering and Technology*, 4(1):66–78.

[20] Wiedemann, R. and Sawicka, H. (2021). The redesign methodology of a transportation network. In *Decision Support Methods in Modern Transportation Systems and Networks*, pages 1–22. Springer.