

A Study On Dual Generalized Edouard Numbers

Abstract. In this study, we introduce the generalized dual Edouard numbers, a novel class of numerical sequences that extend classical recurrence relations within a broader mathematical framework. Several notable special cases particularly the dual Edouard numbers and dual Edouard-Lucas numbers are examined in depth, each exhibiting rich combinatorial and algebraic structures.

We derive explicit formulations for these sequences, including Binet-type expressions, generating functions, and summation identities, which provide analytical insight into their intrinsic behavior and structural characteristics. Furthermore, we investigate their matrix representations, offering a refined algebraic approach for theoretical exploration and potential applications in related fields.

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1. Introduction

In contrast to complex numbers, hypercomplex number systems offer a more sophisticated and versatile framework for representing transformations, symmetries, and structures in higher-dimensional spaces. These systems extend beyond the traditional scope of complex analysis, providing algebraic tools that are particularly well-suited for multidimensional mathematical modeling. As emphasized by Kantor in [35], hypercomplex systems can be interpreted as natural extensions of the real number line, designed to accommodate the intricacies of multidimensional analysis and geometric abstraction.

The principal categories of hypercomplex numbers include complex numbers, hyperbolic numbers, and dual numbers, each with distinct algebraic properties and applications.

- Complex numbers, composed of a real and an imaginary part, form the foundational layer of hypercomplex systems. Their well-established role in fields such as electrical engineering, quantum mechanics, and signal processing underscores their importance in both theoretical and applied contexts.

- Hyperbolic numbers, which extend the complex number framework by replacing the imaginary unit i (where $i^2 = -1$) with a hyperbolic unit j satisfying $j^2 = 1$, are particularly useful in modeling Lorentz transformations, special relativity, and spacetime geometries. Their algebraic structure enables the representation of hyperbolic rotations and boosts, making them indispensable in relativistic physics.

- Dual numbers, characterized by the presence of a dual unit ε such that $\varepsilon^2 = 0$, play a crucial role in automatic differentiation, kinematic analysis, and infinitesimal calculus. Their nilpotent nature allows for the encoding of infinitesimal variations, which is especially valuable in optimization algorithms and motion modeling.

Together, these hypercomplex systems enrich the algebraic landscape by enabling the representation of phenomena that cannot be adequately captured by real or complex numbers alone. Their structural diversity and analytical power make them essential tools in modern mathematics, physics, and engineering.

The following sections offer more detailed insights into the mathematical properties and application areas of these hypercomplex systems.

- Complex numbers are constructed by extending the real number system through the introduction of an imaginary unit, denoted as " i ", which satisfies the identity $i^2 = -1$. A complex number is typically expressed in the form $z = a + bi$, where a and b are real numbers, and i represents the imaginary unit.
- Hyperbolic numbers also referred to as double numbers or split complex numbers extend the real number system by introducing a new unit element j , which satisfies the identity $j^2 = 1$ [46]. These numbers are distinct from real and complex numbers due to their unique algebraic properties. A hyperbolic number is defined as:

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\}.$$

where a and b are real numbers and j is the hyperbolic unit. This structure enables the modeling of systems with split-signature metrics and has notable applications in areas such as special relativity and signal processing.

- Dual numbers [22] expand the real number system through the incorporation of a new element ε , which satisfies the identity $\varepsilon^2 = 0$. This infinitesimal unit distinguishes dual numbers from other hypercomplex systems and makes them especially valuable in modeling instantaneous rates of change. A dual number is defined as:

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

where a and b are real numbers, and ε is the nilpotent unit. Dual numbers are commonly used in applications such as automatic differentiation, kinematics, and perturbation analysis, due to their ability to elegantly encode infinitesimal variations.

- Among the non-commutative examples of hypercomplex number systems are quaternions [28]. Quaternions generalize complex numbers by incorporating three distinct imaginary units, typically denoted as i, j , and k . A quaternion has the form $a_0 + ia_1 + ja_2 + ka_3$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$. These multiplication rules result in a non-commutative structure, meaning the order of multiplication affects the result. The set of quaternion numbers is formally defined as:

$$\mathbb{H}_\mathbb{Q} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\},$$

- Additional hypercomplex systems include octonions and sedenions, which are discussed in [31] and [47]. The algebras \mathbb{C} (complex numbers), $\mathbb{H}_\mathbb{Q}$ (quaternions), \mathbb{O} (octonions), and \mathbb{S} (sedenions) are all constructed as real algebras derived from the real numbers \mathbb{R} using a recursive procedure known as the Cayley–Dickson Process. This technique successively doubles the dimension of each algebra and continues beyond sedenions to produce what are collectively referred to as the 2^n -ions. The following table highlights selected publications from the literature that investigate the properties and applications of these extended number systems.

For more information on hypercomplex algebra, see [41, 29, 43, 36]

A dual hyperbolic number is a type of hypercomplex number, specifically a member of the hyperbolic number system. A dual hyperbolic number is defined as follows

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$.

$\mathbb{H}_\mathbb{D}$, the set of all dual hyperbolic numbers, are generally denoted by

$$\mathbb{H}_\mathbb{D} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The $\{1, j, \varepsilon, \varepsilon j\}$ is linearly independent, and the algebra $\mathbb{H}_\mathbb{D}$ is generated by their span, i.e. $\mathbb{H}_\mathbb{D} = \text{sp}\{1, j, \varepsilon, \varepsilon j\}$

Therefore, $\{1, j, \varepsilon, \varepsilon j\}$ forms a basis for the dual hyperbolic algebra $\mathbb{H}_\mathbb{D}$. For more detail, see [4].

The next properties are holds for the base elements $\{1, j, \varepsilon, \varepsilon j\}$ of dual hyperbolic numbers (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j, \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0, j^2 = j.j = 1 \\ \varepsilon.j &= j.\varepsilon, \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0, j(\varepsilon j) = (\varepsilon j)j = \varepsilon \end{aligned}$$

where ε denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j denotes the hyperbolic unit ($j^2 = 1$), and εj denotes the dual hyperbolic unit ($(j\varepsilon)^2 = 0$).

We claim that p and q be two dual hyperbolic numbers that $q = a_0 + ja_1 + \varepsilon a_2 + j\varepsilon a_3$ and $p = b_0 + jb_1 + \varepsilon b_2 + j\varepsilon b_3$ and then we can write the product of p and q as

$$qp = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0) + \varepsilon(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + j\varepsilon(a_0b_3 + a_1b_2 + a_2b_1 + b_0a_3)$$

and we can write the sum dual hyperbolic numbers p and q as componentwise.

The dual hyperbolic numbers form a commutative ring, real vector space and an algebra. $\mathbb{H}_{\mathbb{D}}$ is not field since every dual hyperbolic numbers doesn't have an inverse. For more detail about dual hyperbolic numbers, see [4].

It's known that many author studied the generalized (r, s, t) sequence. One of these sequences is generalized Edouard numbers. Soykan, [51] defined generalized Edouard numbers. Before we present our original study, we recall some proporities related to generalized Edouard numbers such as reccurance relations, Binet's formula, generating function.

A generalized Edouard sequence, with the initial values W_0, W_1, W_2 not all being zero, $\{W_n\}_{n \geq 0} = \{W_n(W_0, W_1, W_2)\}_{n \geq 0}$ is defined by the third-order recurrence relations

$$W_n = 7W_{n-1} - 7W_{n-2} + W_{n-3}; \quad W_0, W_1, W_2 \quad (n \geq 3) \quad (1.1)$$

Moreover, we define generalized Edouard sequence given to negative subscripts as follows,

$$W_{-n} = 7W_{-(n-1)} - 7W_{-(n-2)} + W_{-(n-3)}$$

for $n = 1, 2, 3, \dots$. Thus, recurrence (1.1) is true for all integer n .

In the Table 1 we give the first some generalized Edouardnumbers with positive subscript and negative subscript

Table 1. A few generalized Edouard numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$7W_0 - 7W_1 + W_2$
2	W_2	$42W_0 - 48W_1 + 7W_2$
3	$W_0 - 7W_1 + 7W_2$	$246W_0 - 287W_1 + 42W_2$
4	$7W_0 - 48W_1 + 42W_2$	$1435W_0 - 1680W_1 + 246W_2$
5	$42W_0 - 287W_1 + 246W_2$	$8365W_0 - 9799W_1 + 1435W_2$
6	$246W_0 - 1680W_1 + 1435W_2$	$48756W_0 - 57120W_1 + 8365W_2$

If we obtain, respectively, $W_0 = 0, W_1 = 1, W_2 = 7$ then $\{W_n\} = \{E_n\}$ is called the Edouard sequence, $W_0 = 3, W_1 = 7, W_2 = 35$ then $\{W_n\} = \{K_n\}$ is called the Edouard-Lucas sequence. Alternatively, Edouard sequence $\{E_n\}_{n \geq 0}$, Edouard-Lucas sequence $\{K_n\}_{n \geq 0}$ are given by the third-order recurrence relations as

$$E_n = 7E_{n-1} - 7E_{n-2} + E_{n-3}, \quad E_0 = 0, E_1 = 1, E_2 = 7, \quad (1.2)$$

$$K_n = 7K_{n-1} - 7K_{n-2} + K_{n-3}, \quad K_0 = 3, K_1 = 7, K_2 = 35, \quad (1.3)$$

The sequences given above can be extended to negative subscripts by defining, respectively,

$$E_{-n} = 7E_{-(n-1)} - 7E_{-(n-2)} + E_{-(n-3)},$$

$$K_{-n} = 7K_{-(n-1)} - 7K_{-(n-2)} + K_{-(n-3)},$$

for $n = 1, 2, 3, \dots$. As a consequence, recurrences (1.2)-(1.3) hold for all integer n .

We can list some important properties of generalized Edouard numbers that are needed.

Binet formula of generalized Edouard sequence can be calculated using its characteristic equation written as

$$x^3 - 7x^2 + 7x - 1 = (x^2 - 6x + 1)(x - 1) = 0.$$

The roots of the characteristic equation are

$$\alpha = 3 + 2\sqrt{2},$$

$$\beta = 3 - 2\sqrt{2},$$

$$\gamma = 1,$$

By using these roots and the recurrence relation, Binet formula are written below

$$\begin{aligned} W_n &= \frac{z_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{z_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{z_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)} \\ &= \frac{z_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{z_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)} - \frac{z_3}{4} \end{aligned}$$

where

$$z_1 = W_2 - (\beta + 1)W_1 + \beta W_0,$$

$$z_2 = W_2 - (\alpha + 1)W_1 + \alpha W_0,$$

$$z_3 = W_2 - 6W_1 + W_0.$$

and

$$\begin{aligned} A_1 &= \frac{W_2 - (\beta + 1)W_1 + \beta W_0}{(\alpha - \beta)(\alpha - \gamma)}, \\ A_2 &= \frac{W_2 - (\alpha + 1)W_1 + \alpha W_0}{(\beta - \alpha)(\beta - \gamma)}, \\ A_3 &= \frac{W_2 - 6W_1 + W_0}{(\gamma - \alpha)(\gamma - \beta)}. \end{aligned} \tag{1.4}$$

Then we present Binet formula of Edouard sequences and Edouard-Lucas sequences, respectively, given below

$$\begin{aligned} E_n &= \frac{\alpha^{n+1}}{(\alpha - \beta)(\alpha - 1)} + \frac{\beta^{n+1}}{(\beta - \alpha)(\beta - 1)} - \frac{1}{4}, \\ K_n &= \alpha^n + \beta^n + 1. \end{aligned}$$

After then we can write the generating function of generalized Edouard numbers,

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 7W_0)x + (W_2 - 7W_1 + 7W_0)x^2}{1 - 7x + 7x^2 - x^3}. \quad (1.5)$$

Next, we give the exponential generating function of $\sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence W_n .

LEMMA 1. [6, Lemma 1.4]. Suppose that $f_{GW_n}(x) = \sum_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is the exponential generating function of the generalized Edouard sequence $\{W_n\}$. Then

$$\sum_{n=0}^{\infty} W_n \frac{x^n}{n!} = \frac{(W_2 - (\beta + 1)W_1 + \beta W_0)}{(\alpha - \beta)(\alpha - 1)} e^{\alpha x} + \frac{(W_2 - (\alpha + 1)W_1 + \alpha W_0)}{(\beta - \alpha)(\beta - 1)} e^{\beta x} - \frac{(W_2 - 6W_1 + W_0)}{4} e^x.$$

The previous Lemma gives the following results as particular examples.

COROLLARY 2. Exponential generating function of Edouard and Edouard-Lucas numbers are

a):

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{\alpha^{n+1}}{(\alpha - \beta)(\alpha - 1)} + \frac{\beta^{n+1}}{(\beta - \alpha)(\beta - 1)} - \frac{1}{4} \right) \frac{x^n}{n!} = \frac{\alpha e^{\alpha x}}{(\alpha - \beta)(\alpha - 1)} + \frac{\beta e^{\beta x}}{(\beta - \alpha)(\beta - 1)} - \frac{1}{4} e^x.$$

b):

$$\sum_{n=0}^{\infty} K_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (\alpha^n + \beta^n + 1) \frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^x.$$

For more details, see [51].

Now, we are presenting information about specific number systems, including the hypercomplex system, which encompasses complex numbers, hyperbolic numbers, and dual numbers. We note that hyperbolic numbers will play a crucial role in our work. Moreover hyperbolic functions and numbers find applications in various branches of engineering, such as electrical engineering (e.g., transmission lines), control systems (e.g., system dynamics), signal processing (e.g., filter design), and diverse fields of engineering physics, including special relativity, wave propagation, fluid dynamics, optics, and heat conduction. It's important to note that while hyperbolic numbers have interesting mathematical properties, their adoption in practical applications depends on the specific problem at hand and whether they offer advantages over other number systems in a given context.

Initially, we discuss hypercomplex number systems, which are extensions of real numbers, for more detail see [35]. In addition that some commutative special cases of hypercomplex number systems include complex numbers, hyperbolic numbers, and dual numbers. These systems are widely used in various branches of mathematics and physics. We will now present these number systems sequentially, as outlined below.

- Complex numbers simplest form of hypercomplex numbers. Complex numbers defined as $z = a + ib$, where a and b real numbers and i imaginary unit that satisfy $i^2 = -1$. In addition that a and b named, respectively, $\text{Re}(z)$ and $\text{Im}(z)$ Consequently, the definition of complex numbers given by,

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\}.$$

- Hyperbolic (double, split-complex) numbers, for more detail see [46], Split-complex numbers, commonly recognized as hyperbolic numbers, defined as $h = a + jb$ where a and b real numbers and j hyperbolic unit that satisfy $j^2 = 1$. In addition that a and b named, respectively, $Re(h)$ and $Hyp(h)$. Thus, the definition of hyperbolic numbers given by,

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\},$$

- Dual numbers, see [22], defined as $d = a + \varepsilon b$ where a and b real numbers and ε dual unit that satisfy $\varepsilon^2 = 0$. Furthermore, a and b called, respectively, $Re(d)$ and $Du(d)$. Thus, definition of dual numbers given by,

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

- A dual hyperbolic number, specifically within the hyperbolic number system, constitutes a distinct type of hypercomplex number. A dual hyperbolic number is defined by,

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$ and the set of all dual hyperbolic numbers are defined by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The $\{1, j, \varepsilon, \varepsilon j\}$ is linear independent and $\mathbb{H}_{\mathbb{D}} = sp\{1, j, \varepsilon, \varepsilon j\}$ so that $\{1, j, \varepsilon, \varepsilon j\}$ is a basis of $\mathbb{H}_{\mathbb{D}}$. For more detail see, [4]

The next properties are true for the base elements $\{1, j, \varepsilon, \varepsilon j\}$ (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j, \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0, j^2 = j.j = 1 \\ \varepsilon.j &= j.\varepsilon, \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0, j(\varepsilon j) = (\varepsilon j)j = \varepsilon \end{aligned}$$

where ε satisfy the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j satisfy the hyperbolic unit ($j^2 = 1$), and εj satisfy the dual hyperbolic unit ($(j\varepsilon)^2 = 0$).

In addition that the other number systems are quaternions, octonions and sedenions given below, respectively,

- Quaternion numbers, non-commutative examples of hypercomplex number systems, are a four-dimensional extension of complex numbers. They are expressed as $a_0 + ia_1 + ja_2 + ka_3$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$, and i, j , and k are the quaternion units that satisfy specific multiplication rules. For more detail see [28]. Quaternion numbers are defined by

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\},$$

- Octonions is a set, every element of the set linear combinations of unit octonions $\{e_i : i = 0, 1, 2, \dots, 7\}$, denoted as \mathbb{O} . Octonions are defined by,

$$\mathbb{O} = \left\{ \sum_{i=0}^7 a_i e_i : a_i \in \mathbb{R}, e_0 e_i = e_i e_0 = e_i, e_i e_j = -\delta_{ij} e_0 + \varepsilon_{ijk} e_k \right\}$$

where $e_e = 1$, δ_{ij} is Kroneker delta (equal to 1 if and only if $i = j$), ε_{ijk} is anti-symmetric tensor. For more detail see [31, 75]

- Sedenions is a set, every element of the set linear combinations of unit sedenions $\{e_i : i = 0, 1, 2, \dots, 15\}$, denoted by \mathbb{S} . It can be seen from here that ever sedenion can be written as

$$\sum_{i=0}^{15} a_i e_i$$

where a_i is real number. For more detail see, [47, 75].

Next we give some propoities on two hyperbolic numbers, $h_1 = a + jb$ and $h_2 = c + jd$, as

$$h_1 + h_2 = (a + b) + j(c + d),$$

$$h_1 \cdot h_2 = (ac + bd) + j(ad + bc),$$

$$\overline{h_1} = a - jb$$

$$\frac{h_1}{h_2} = \frac{(ac - bd) + j(cb - ad)}{c^2 - d^2},$$

$$h_1 = h_2 \text{ if only if } a = c \text{ and } b = d,$$

$$\langle h_1, h_2 \rangle = (ac + bd) + j(bc + ad),$$

$$\|h_1\| = \sqrt{|a^2 - b^2|}, \text{ called norm of } h_1,$$

$$\text{if } |a^2 - b^2| > 0, h_1 \text{ is named spacelike vector,}$$

$$\text{if } |a^2 - b^2| < 0, h_1 \text{ is named timelike vector,}$$

$$\text{if } |a^2 - b^2| = 0, h_1 \text{ is named null(light-like) vector.}$$

Note that $\{\mathbb{R}^2, H, \langle, \rangle\}$ is called Lorentz plane and denoted as \mathbb{R}_1^2 . There is an isomorphism relationship between the Lorentz plane and hyperbolic numbers. For more detail, see [75].

Hence the algebras \mathbb{C} (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), \mathbb{O} (octonions) and \mathbb{S} (sedenions) are real algebras attained from the real numbers \mathbb{R} by a doubling procedure known as the Cayley-Dickson Process. This doubling process can be extended beyond the sedenions to form what are known as the 2^n -ions (see for example [7, 28, 30, 42, 27]).

Some authors have conducted studies about the dual, hyperbolic, dual hyperbolic and other special numbers. Now we give some information published papers in literature.

- Cockle [13] explored hyperbolic numbers with complex coefficients, contributing to the early development of hypercomplex algebra.
- Eren and Soykan [21] studied the generalized Generalized Woodall Numbers.

- Cheng and Thompson [12] introduced dual numbers with complex coefficients, expanding the algebraic versatility of dual number systems for applications in polynomial equations and transformation theory.
- Akar et al [4] introduced the concept of dual hyperbolic numbers, combining characteristics of dual and hyperbolic systems into a unified algebraic structure.

Next, we present some information on hyperbolic numbers presented in literature.

- Aydın [1] presented hyperbolic Fibonacci numbers given by

$$\tilde{F}_n = F_n + hF_{n+1},$$

where Fibonacci numbers are given by $F_{n+2} = F_{n+1} + F_n$, with the initial condition $F_0 = 0, F_1 = 1$.

- Soykan and Taşdemir [49] studied hyperbolic generalized Jacobsthal numbers given by

$$\tilde{V}_n = V_n + hV_{n+1}$$

where generalized Jacobsthal numbers are $V_{n+2} = V_{n+1} + 2V_n$ with the initial condition $V_0 = a, V_1 = b$.

- Taş [72] studied hyperbolic Jacobsthal-Lucas sequence written by

$$HJ_n = J_n + hJ_{n+1}$$

where Jacobsthal-Lucas numbers given by $J_{n+2} = J_{n+1} + 2J_n$ with the initial condition $J_0 = 2, J_1 = 1$.

- Dikmen and Altınsoy, [18] studied On Third Order Hyperbolic Jacobsthal Numbers given by

$$\begin{aligned}\hat{J}_n^{(3)} &= J_n^{(3)} + hJ_{n+1}^{(3)}, \\ \hat{j}_n^{(3)} &= j_n^{(3)} + hj_{n+1}^{(3)}\end{aligned}$$

where Jacobsthal numbers, respectively, given by $J_n^{(3)} = J_{n-1}^{(3)} + J_{n-2}^{(3)} + 2J_{n-3}^{(3)}, J_0^{(3)} = 0, J_1^{(3)} = 1, J_2^{(3)} = 1, j_n^{(3)} = j_{n-1}^{(3)} + j_{n-2}^{(3)} + 2j_{n-3}^{(3)}, j_0^{(3)} = 2, j_1^{(3)} = 1, j_2^{(3)} = 5$.

Following this, we provide details on dual hyperbolic sequences as they are presented in literature.

- Soykan et al [48] presented dual hyperbolic generalized Pell numbers given by

$$\hat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}$$

where generalized Pell numbers, with the initial values V_0, V_1 not all being zero, are given by $V_n = 2V_{n-1} + V_{n-2}, V_0 = a, V_1 = b (n \geq 2)$.

- Cihan et al [3] studied dual hyperbolic Fibonacci and Lucas numbers given by, respectively,

$$DHF_n = F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3},$$

$$DHL_n = L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3}$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$, $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$.

- Soykan et al [49] studied dual hyperbolic generalized Jacopsthal numbers given by

$$\widehat{J}_n = J_n + jJ_{n+1} + \varepsilon J_{n+2} + j\varepsilon J_{n+3}$$

where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = a$, $J_1 = b$.

- Yılmaz and Soykan [76] introduced dual hyperbolic generalized Guglielmo numbers are

$$\widehat{T}_0 = T_0 + jT_1 + \varepsilon T_2 + j\varepsilon T_3$$

where $T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}$, $T_0 = 0$, $T_1 = 1$, $T_2 = 3$.

- Ayrılma and Soykan [5] studied dual hyperbolic generalized Edouard number and Edouard-Lucas number given by

$$\widehat{E}_0 = E_0 + jE_1 + \varepsilon E_2 + j\varepsilon E_3,$$

$$\widehat{K}_0 = K_0 + jK_1 + \varepsilon K_2 + j\varepsilon K_3,$$

where $E_n = 7E_{n-1} - 7E_{n-2} + E_{n-3}$, $E_0 = 0$, $E_1 = 1$, $E_2 = 7$ and $K_n = 7K_{n-1} - 7K_{n-2} + K_{n-3}$, $K_0 = 3$, $K_1 = 7$, $K_2 = 35$.

- Bród et al [8] studied dual hyperbolic generalized balancing numbers as

$$DHB_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}$$

where $B_n = 6B_{n-1} - B_{n-2}$, $B_0 = 0$, $B_1 = 1$.

Next section, we define the dual generalized Edouard numbers and some special properties, generating function and Binet's formula , of these numbers.

2. Dual Generalized Edouard Numbers and their Generating Functions and Binet's Formulas

In this section, we define dual generalized Edouard numbers then we present generating functions and Binet formulas for these numbers.

On the set of $\mathbb{H}_{\mathbb{D}}$, we will now explore dual generalized Edouard numbers on \mathbb{D} . The n th generalized dual Edouard numbers, with DW_0, DW_1, DW_2 being the initial conditions, are defined as follows

$$DW_n = W_n + \varepsilon W_{n+1}. \quad (2.1)$$

in addition (2.1) can be written to negative subscripts by defining,

$$DW_{-n} = W_{-n} + \varepsilon W_{-n+1} \quad (2.2)$$

so identity (2.1) holds for all integers n .

Now we define some special cases of dual generalized Edouard numbers. The n th dual edouard numbers, the n th dual Edouard-Lucas numbers, respectively, are given as

the n th generalized dual Edouard numbers $DE_n = E_n + \varepsilon E_{n+1}$, with DK_0, DK_1, DK_2 being the initial conditions, are defined as follows

$$DE_n = E_n + \varepsilon E_{n+1}$$

where

$$DE_0 = E_0 + \varepsilon E_1,$$

$$DE_1 = E_1 + \varepsilon E_2,$$

$$DE_2 = E_2 + \varepsilon E_3,$$

the n th generalized dual Edouard-Lucas numbers $DK_n = K_n + \varepsilon K_{n+1}$, with DK_0, DK_1, DK_2 being the initial conditions, are defined as follows

$$DK_n = K_n + \varepsilon K_{n+1}$$

where

$$DK_0 = K_0 + \varepsilon K_1,$$

$$DK_1 = K_1 + \varepsilon K_2,$$

$$DK_2 = K_2 + \varepsilon K_3,$$

For dual Edouard numbers, taking $W_n = E_n$, $E_0 = 0$, $E_1 = 1$, $E_2 = 7$, we get

$$DE_0 = \varepsilon,$$

$$DE_1 = 1 + 7\varepsilon,$$

$$DE_2 = 7 + 42\varepsilon,$$

for dual Edouard-Lucas numbers, taking $W_n = K_n$, $K_0 = 3$, $K_1 = 7$, $K_2 = 35$, we get

$$DK_0 = 3 + 7\varepsilon,$$

$$DK_1 = 7 + 35\varepsilon,$$

$$DK_2 = 35 + 199\varepsilon,$$

Thus, by using (2.1), we can formulate the following identity for non-negative integers n ,

$$DW_n = 7DW_{n-1} - 7DW_{n-2} + DW_{n-3}. \quad (2.3)$$

Hence the sequence $\{DW_n\}_{n \geq 0}$ can be given as

$$DW_{-n} = 7DW_{-(n-1)} - 7DW_{-(n-2)} + DW_{-(n-3)},$$

for $n \in \{1, 2, 3, \dots\}$ by using (2.2). Accordingly, recurrence (2.3) is true for all integer n .

In the Table 2, We provide the initial dual generalized Edouard numbers with both positive and negative subscripts.

Table 2. Some dual generalized Edouard numbers

n	DW_n	DW_{-n}
0	DW_0	DW_0
1	DW_1	$7DW_0 - 7DW_1 + DW_2$
2	DW_2	$42DW_0 - 48DW_1 + 7DW_2$
3	$DW_0 - 7DW_1 + 7DW_2$	$246DW_0 - 287DW_1 + 42DW_2$
4	$7DW_0 - 48DW_1 + 42DW_2$	$1435DW_0 - 1680DW_1 + 246DW_2$
5	$42DW_0 - 287DW_1 + 246DW_2$	$8365DW_0 - 9799DW_1 + 1435DW_2$
6	$246DW_0 - 1680DW_1 + 1435DW_2$	$48756DW_0 - 57120DW_1 + 8365DW_2$

Note that

$$DW_0 = W_0 + \varepsilon W_1, \quad DW_1 = W_1 + \varepsilon W_2, \quad DW_2 = W_2 + \varepsilon W_3.$$

Some dual Edouard numbers, dual Edouard-Lucas numbers with positive or negative subscripts are presented tables which is given below .

Table 3. dual Edouard numbers

n	DE_n	DE_{-n}
0	ε	
1	$1 + 7\varepsilon$	0
2	$7 + 42\varepsilon$	1
3	$42 + 246\varepsilon$	$7 + \varepsilon$
4	$246 + 1435\varepsilon$	$42 + 7\varepsilon$
5	$1435 + 8365\varepsilon$	$246 + 42\varepsilon$

Table 4. dual Edouard-Lucas numbers

n	DK_n	DK_{-n}
0	$3 + 7\varepsilon$	
1	$7 + 35\varepsilon$	$7 + 3\varepsilon$
2	$35 + 199\varepsilon$	$35 + 7\varepsilon$
3	$199 + 1155\varepsilon$	$199 + 35\varepsilon$
4	$1155 + 6727\varepsilon$	$1155 + 199\varepsilon$
5	$6727 + 39203\varepsilon$	$6727 + 1155\varepsilon$

Now, we will establish Binet's formula for the dual generalized Edouard numbers, and for the remainder of the study, we will utilize the following notations:

$$\tilde{\alpha} = 1 + \varepsilon\alpha, \quad (2.4)$$

$$\tilde{\beta} = 1 + \varepsilon\beta, \quad (2.5)$$

$$\tilde{\gamma} = 1 + \varepsilon. \quad (2.6)$$

Note that the following identities are true.

$$\begin{aligned}
\tilde{\alpha}^2 &= 1 + 2\alpha\varepsilon, \\
\tilde{\beta}^2 &= 1 + 2\beta\varepsilon, \\
\tilde{\gamma}^2 &= 1 + 2\varepsilon, \\
\tilde{\alpha}\tilde{\beta} &= 1 + \varepsilon(\alpha + \beta), \\
\tilde{\alpha}\tilde{\gamma} &= 1 + \varepsilon(\alpha + \gamma), \\
\tilde{\gamma}\tilde{\beta} &= 1 + \varepsilon(\gamma + \beta).
\end{aligned}$$

THEOREM 3. (*Binet's Formula*) *For any integer n , the n th dual generalized Edouard number can be expressed as follows*

$$DW_n = \tilde{\alpha}A_1\alpha^n + \tilde{\beta}A_2\beta^n + \tilde{\gamma}A_3 \quad (2.7)$$

where $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ are given as (2.4), (2.5), (2.6).

Proof. Using Binet's formula of the generalized Edouard numbers given below

$$W_n = A_1\alpha^n + A_2\beta^n + A_3$$

where A_1, A_2, A_3 are given (1.4) we get

$$\begin{aligned}
DW_n &= W_n + \varepsilon W_{n+1}, \\
&= A_1\alpha^n + A_2\beta^n + A_3 + (A_1\alpha^{n+1} + A_2\beta^{n+1} + A_3)\varepsilon \\
&= \tilde{\alpha}A_1\alpha^n + \tilde{\beta}A_2\beta^n + \tilde{\gamma}A_3.
\end{aligned}$$

This proves (2.7). \square

As special cases, for any integer n , the Binet's Formula of n th dual Edouard numbers, the Binet's Formula of n th dual Edouard-Lucas numbers, respectively, are

$$\begin{aligned}
E_n &= \frac{\tilde{\alpha}\alpha^{n+1}}{(\alpha - \beta)(\alpha - 1)} + \frac{\tilde{\beta}\beta^{n+1}}{(\beta - \alpha)(\beta - 1)} - \frac{\tilde{\gamma}}{4}, \\
K_n &= \tilde{\alpha}\alpha^n + \tilde{\beta}\beta^n + \tilde{\gamma},
\end{aligned}$$

Next, we will introduce the generating function of the dual generalized Edouard numbers.

THEOREM 4. *The generating function for the dual generalized Edouard numbers is*

$$f_{DW_n}(x) = \frac{DW_0 + (DW_1 - 7DW_0)x + (DW_2 - 7DW_1 + 7DW_0)x^2}{(1 - 7x + 7x^2 - x^3)}. \quad (2.8)$$

Proof. Let the generating function of the dual generalized Edouard numbers is given below

$$f_{DW_n}(x) = \sum_{n=0}^{\infty} DW_n x^n$$

Following that, by utilizing the definition of the dual generalized Edouard numbers, and subtracting $7xg(x)$ and $-7x^2g(x)$ and $x^3g(x)$ from $g(x)$, we get

$$\begin{aligned}
(1 - 7x + 7x^2 - x^3)f_{GDW_n}(x) &= \sum_{n=0}^{\infty} DW_n x^n - 7x \sum_{n=0}^{\infty} DW_n x^n + 7x^2 \sum_{n=0}^{\infty} DW_n x^n - x^3 \sum_{n=0}^{\infty} DW_n x^n, \\
&= \sum_{n=0}^{\infty} DW_n x^n - 7 \sum_{n=0}^{\infty} DW_n x^{n+1} + 7 \sum_{n=0}^{\infty} DW_n x^{n+2} - \sum_{n=0}^{\infty} DW_n x^{n+3}, \\
&= \sum_{n=0}^{\infty} DW_n x^n - 7 \sum_{n=1}^{\infty} DW_{n-1} x^n + 7 \sum_{n=2}^{\infty} DW_{n-2} x^n - \sum_{n=3}^{\infty} DW_{n-3} x^n, \\
&= (DW_0 + DW_1 x + DW_2 x^2) - 7(DW x + DW_1 x^2) + 7DW_0 x^2 \\
&\quad + \sum_{n=3}^{\infty} (DW_n - 7DW_{n-1} + 7DW_{n-2} - DW_{n-3}) x^n, \\
&= DW_0 + DW_1 x + DW_2 x^2 - 7DW_0 x - 7DW_1 x^2 + 7DW_0 x^2, \\
&= DW_0 + (DW_1 - 7DW_0)x + (DW_2 - 7DW_1 + 7DW_0)x^2.
\end{aligned}$$

Note that we use the recurrence relation $DW_n = 7DW_{n-1} - 7DW_{n-2} + DW_{n-3}$. We rearrange equation which is given above then we obtain (2.8). \square

As specific cases, the generating functions of the dual Edouard, Edouard-Lucas are given by

$$\begin{aligned}
f_{DE_n}(x) &= \frac{\varepsilon + x}{(1 - 7x + 7x^2 - x^3)}, \\
f_{DK_n}(x) &= \frac{7\varepsilon + 3 + (-14\varepsilon - 14)x + (3\varepsilon + 7)x^2}{(1 - 7x + 7x^2 - x^3)},
\end{aligned}$$

respectively. \square

Next, we give the exponential generating function of $\sum_{n=0}^{\infty} DW_n \frac{x^n}{n!}$ of the sequence DW_n .

LEMMA 5. Suppose that $f_{DW_n}(x) = \sum_{n=0}^{\infty} DW_n \frac{x^n}{n!}$ is the exponential generating function of the dual generalized Edouard sequence $\{DW_n\}$.

Then $\sum_{n=0}^{\infty} DW_n \frac{x^n}{n!}$ is given by

$$\begin{aligned}
\sum_{n=0}^{\infty} DW_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty} DW_n \frac{x^n}{n!} + \varepsilon \sum_{n=0}^{\infty} DW_{n+1} \frac{x^n}{n!} \\
&= \frac{(W_2 - (\beta + 1)W_1 + \beta W_0)}{(\alpha - \beta)(\alpha - 1)} e^{\alpha x} + \frac{(W_2 - (\alpha + 1)W_1 + \alpha W_0)}{(\beta - \alpha)(\beta - 1)} e^{\beta x} - \frac{(W_2 - 6W_1 + W_0)}{4} e^x \\
&\quad + \varepsilon \left(\frac{(W_2 - (\beta + 1)W_1 + \beta W_0)\alpha}{(\alpha - \beta)(\alpha - 1)} e^{\alpha x} + \frac{(W_2 - (\alpha + 1)W_1 + \alpha W_0)\beta}{(\beta - \alpha)(\beta - 1)} e^{\beta x} - \frac{(W_2 - 6W_1 + W_0)}{4} e^x \right).
\end{aligned}$$

Proof: Note that we have

$$\sum_{n=0}^{\infty} DW_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (DW_n + \varepsilon DW_{n+1}) \frac{x^n}{n!}.$$

Then using the Binet's formula of dual generalized Edouard numbers or exponential generating function of the generalized Edouard sequence we get the required identity.

The previous Lemma gives the following results as particular examples.

COROLLARY 6. *Exponential generating function of dual Edouard and dual Edouard-Lucas numbers are*

a):

$$\sum_{n=0}^{\infty} DE_n \frac{x^n}{n!} = \frac{\alpha e^{\alpha x}}{(\alpha - \beta)(\alpha - 1)} + \frac{\beta e^{\beta x}}{(\beta - \alpha)(\beta - 1)} - \frac{1}{4}e^x + \varepsilon \left(\frac{\alpha^2 e^{\alpha x}}{(\alpha - \beta)(\alpha - 1)} + \frac{\beta^2 e^{\beta x}}{(\beta - \alpha)(\beta - 1)} - \frac{1}{4}e^x \right).$$

b):

$$\sum_{n=0}^{\infty} DK_n \frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^x + \varepsilon(\alpha e^{\alpha x} + \beta e^{\beta x} + e^x).$$

3. Deriving Binet's Formula From the Generating Function

Next ,by using generating function $f_{DW_n}(x)$, we investigate Binet formula of $\{DW_n\}$.

THEOREM 7. *(Binet formula of dual generalized Edouard numbers)*

$$DW_n = \tilde{\alpha}A_1\alpha^n + \tilde{\beta}A_2\beta^n + \tilde{\gamma}A_3. \quad (3.1)$$

Proof. We write

$$\sum_{n=0}^{\infty} DW_n x^n = \frac{DW_0 + (DW_1 - 7DW_0)x + (DW_2 - 7DW_1 + 7DW_0)x^2}{(1 - 7x + 7x^2 - x^3)} = \frac{d_1}{(1 - \alpha x)} + \frac{d_2}{(1 - \beta x)} + \frac{d_3}{(1 - x)}, \quad (3.2)$$

so that

$$\begin{aligned} \sum_{n=0}^{\infty} DW_n x^n &= \frac{d_1}{(1 - \alpha x)} + \frac{d_2}{(1 - \beta x)} + \frac{d_3}{(1 - x)}, \\ &= \frac{d_1(1 - x)(1 - \beta x) + d_2(1 - \alpha x)(1 - x) + d_3(1 - \alpha x)(1 - \beta x)}{(x^2 - 6x + 1)(1 - x)}, \end{aligned}$$

then, we get

$$DW_0 + (DW_1 - 7DW_0)x + (DW_2 - 7DW_1 + 7DW_0)x^2 = d_1 + d_2 + d_3 + (-d_2 - \alpha d_2 - \beta d_1 - \alpha d_3 - \beta d_3)x + (\alpha d_2 + \beta d_1 + \alpha \beta d_3)x^2.$$

By equation the coefficients of corresponding powers of x in the above equation, we get

$$\begin{aligned} DW_0 &= d_1 + d_2 + d_3, \\ DW_1 - 7DW_0 &= -d_2 - \alpha d_2 - \beta d_1 - \alpha d_3 - \beta d_3, \\ DW_2 - 7DW_1 + 7DW_0 &= \alpha d_2 + \beta d_1 + \alpha \beta d_3. \end{aligned} \quad (3.3)$$

If we solve (3.3) we obtain

$$\begin{aligned} d_1 &= \frac{DW_0\alpha^2 + (DW_1 - 7DW_0)\alpha + (DW_2 - 7DW_1 + 7DW_0)}{(\alpha - \beta)(\alpha - \gamma)}, \\ d_2 &= \frac{DW_0\beta^2 + (DW_1 - 7DW_0)\beta + (DW_2 - 7DW_1 + 7DW_0)}{(\beta - \alpha)(\beta - \gamma)}, \\ d_3 &= \frac{DW_0 + (DW_1 - 7DW_0) + (DW_2 - 7DW_1 + 7DW_0)}{(\gamma - \alpha)(\gamma - \beta)}, \end{aligned}$$

Thus (3.2) stated as follows

$$\begin{aligned} \sum_{n=0}^{\infty} DW_n x^n &= d_1 \sum_{n=0}^{\infty} \alpha^n x^n + d_2 \sum_{n=0}^{\infty} \beta^n x^n + d_3 \sum_{n=0}^{\infty} x^n, \\ &= \sum_{n=0}^{\infty} (d_1 \alpha^n + d_2 \beta^n + d_3) x^n, \\ &= \sum_{n=0}^{\infty} \left(\frac{DW_2 - (\beta + 1)DW_1 + \beta DW_0}{(\alpha - \beta)(\alpha - \gamma)} \alpha^n + \frac{DW_2 - (\alpha + 1)DW_1 + \alpha DW_0}{(\beta - \alpha)(\beta - \gamma)} \beta^n + \frac{DW_2 - 6DW_1 + DW_0}{(\gamma - \alpha)(\gamma - \beta)} x^n \right). \end{aligned}$$

Hence, we get

$$DW_n = \tilde{\alpha} A_1 \alpha^n + \tilde{\beta} A_2 \beta^n + \tilde{\gamma} A_3. \quad \square$$

4. Some Identities Related to Dual Generalized Edouard numbers

We will now introduce some specific identities, i.e Simpson's formula, for the dual generalized Edouard sequence $\{DW_n\}$. We now proceed to introduce specific algebraic identities associated with the dual generalized Edouard sequence $\{DW_n\}$, with particular emphasis on Simpson-type formulas. These identities serve as analytical tools for characterizing the behavior of the sequence and for deriving summation and approximation results within discrete and continuous frameworks. The Simpson formula, traditionally employed in numerical integration, finds a novel interpretation in the context of dual sequences, where its structure can be adapted to reflect the properties of dual units and nilpotent elements. By extending classical formulations to the dual setting, we uncover new relationships that enhance the algebraic richness and computational applicability of the $\{DW_n\}$ sequence.

The next theorem gives the Simpson's formula for the dual generalized Edouard numbers.

THEOREM 8. (*Simpson's formula for dual generalized Edouard numbers*) For all integers n we have,

$$\begin{vmatrix} DW_{n+2} & DW_{n+1} & DW_n \\ DW_{n+1} & DW_n & DW_{n-1} \\ DW_n & DW_{n-1} & DW_{n-2} \end{vmatrix} = \begin{vmatrix} DW_2 & DW_1 & DW_0 \\ DW_1 & DW_0 & DW_{-1} \\ DW_0 & DW_{-1} & DW_{-2} \end{vmatrix}. \quad (4.1)$$

Proof. First we assume that $n \geq 0$. For the proof, we employ mathematical induction on n . For $n = 0$ identity (4.1) is true. Now we take (4.1) is true for $n = k$. Therefore, the following identity can be written

$$\begin{vmatrix} DW_{k+2} & DW_{k+1} & DW_k \\ DW_{k+1} & DW_k & DW_{k-1} \\ DW_k & DW_{k-1} & DW_{k-2} \end{vmatrix} = \begin{vmatrix} DW_2 & DW_1 & DW_0 \\ DW_1 & DW_0 & DW_{-1} \\ DW_0 & DW_{-1} & DW_{-2} \end{vmatrix}.$$

If we take $n = k + 1$, we can get

$$\begin{aligned} \begin{vmatrix} DW_{k+3} & DW_{k+2} & DW_{k+1} \\ DW_{k+2} & DW_{k+1} & DW_k \\ DW_{k+1} & DW_k & DW_{k-1} \end{vmatrix} &= \begin{vmatrix} 7DW_{k+2} - 7DW_{k+1} + DW_k & DW_{k+2} & DW_{k+1} \\ 7DW_{k+1} - 7DW_k + DW_{k-1} & DW_{k+1} & DW_k \\ 7DW_k - 7DW_{k-1} + DW_{k-2} & DW_k & DW_{k-1} \end{vmatrix} \\ &= 7 \begin{vmatrix} DW_{k+2} & DW_{k+2} & DW_{k+1} \\ DW_{k+1} & DW_{k+1} & DW_k \\ DW_k & DW_k & DW_{k-1} \end{vmatrix} - 7 \begin{vmatrix} DW_{k+1} & DW_{k+2} & DW_{k+1} \\ DW_k & DW_{k+1} & DW_k \\ DW_{k-1} & DW_k & DW_{k-1} \end{vmatrix} \\ &\quad + \begin{vmatrix} DW_k & DW_{k+2} & DW_{k+1} \\ DW_{k-1} & DW_{k+1} & DW_k \\ DW_{k-2} & DW_k & DW_{k-1} \end{vmatrix} \\ &= \begin{vmatrix} DW_{k+2} & DW_{k+1} & DW_k \\ DW_{k+1} & DW_k & DW_{k-1} \\ DW_k & DW_{k-1} & DW_{k-2} \end{vmatrix} \end{aligned}$$

Attention that if we take $n < 0$ the proof can be conducted in a similarly. Thus, the proof is concluded. \square

From Theorem 4.1, we get following corollary.

COROLLARY 9.

$$\begin{aligned} \text{(a): } \begin{vmatrix} DE_{n+2} & DE_{n+1} & DE_n \\ DE_{n+1} & DE_n & DE_{n-1} \\ DE_n & DE_{n-1} & DE_{n-2} \end{vmatrix} &= -7\epsilon - 1 \\ \text{(b): } \begin{vmatrix} DK_{n+2} & DK_{n+1} & DK_n \\ DK_{n+1} & DK_n & DK_{n-1} \\ DK_n & DK_{n-1} & DK_{n-2} \end{vmatrix} &= 3584\epsilon + 512 \end{aligned}$$

THEOREM 10. We assume that n and m are integers, E_n is Edouard numbers, the following identity is true:

$$DW_{m+n} = E_{m-1}DW_{n+2} + (E_{m-3} - 7E_{m-2})DW_{n+1} + E_{m-2}DW_n. \quad (4.2)$$

Proof. The identity (10) can be proved by mathematical induction on m . First we take $n, m \geq 0$. If $m = 0$ we get

$$DW_n = E_{-1}DW_{n+2} + (E_{-3} - 7E_{-2})DW_{n+1} + E_{-2}DW_n$$

which is true by seeing that $E_{-1} = 0, E_{-2} = 1, E_{-3} = 7$. We assume that the identity given holds for $m = k$. For $m = k + 1$, we get

$$\begin{aligned}
DW_{(k+1)+n} &= 7DW_{n+k} - 7DW_{n+k-1} + DW_{n+k-2} \\
&= 7(E_{k-1}DW_{n+2} + (E_{k-3} - 7E_{k-2})DW_{n+1} + E_{k-2}DW_n) \\
&\quad - 7(E_{k-2}DW_{n+2} + (E_{k-4} - 7E_{k-3})DW_{n+1} + E_{k-3}DW_n) \\
&\quad + (E_{k-3}DW_{n+2} + (E_{k-5} - 7E_{k-4})DW_{n+1} + E_{k-4}DW_n) \\
&= (7E_{k-1} - 7E_{k-2} + E_{k-3})DW_{n+2} + ((7E_{k-3} - 7E_{k-4} + E_{k-5}) \\
&\quad - 7(7E_{k-2} - 7E_{k-3} + E_{k-4}))DW_{n+1} + (7E_{k-2} - 7E_{k-3} + E_{k-4})DW_n \\
&= E_kDW_{n+2} + (E_{k-2} - 7E_{k-1})DW_{n+1} + E_{k-1}DW_n \\
&= E_{(k+1)-1}DW_{n+2} + (E_{(k+1)-3} - 7E_{(k+1)-2})DW_{n+1} + E_{(k+1)-2}DW_n.
\end{aligned}$$

The other cases on n, m the proof can be done easily. Consequently, by mathematical induction on m , this proves (10). \square

5. Linear Sum Formulas of Dual Generalized Edouard Numbers

In this section, we give the summation formulas of the dual generalized Edouard numbers with subscripts.

PROPOSITION 11. *For the generalized Edouard numbers, we have the following formulas:*

$$\begin{aligned}
\text{(a): } \sum_{k=0}^n W_k &= \frac{1}{4}(-(n+3)W_n + (n+2)(7W_{n+1} - W_{n+2}) - (n+1)W_{n+1} + 2W_2 - 13W_1 + 7W_0). \\
\text{(b): } \sum_{k=0}^n W_{2k} &= \frac{1}{32}(-(n+3)W_{2n} + (n+2)(-7W_{2n+2} + 48W_{2n+1} - 7W_{2n}) - (n+1)W_{2n+2} + 15W_2 - \\
&\quad 96W_1 + 49W_0). \\
\text{(c): } \sum_{k=0}^n W_{2k+1} &= \frac{1}{32}(-(n+3)W_{2n+1} + (n+2)(-W_{2n+2} + 42W_{2n+1} - 7W_{2n}) - (n+1)(7W_{2n+2} - \\
&\quad 7W_{2n+1} + W_{2n}) + 9W_2 - 56W_1 + 15W_0).
\end{aligned}$$

Proof. It is given in Soykan [52, Theorem 3.3]. \square

Now, we will introduce the formulas that allow us to find the sum of dual generalized Edouard numbers.

THEOREM 12. *For $n \geq 0$, dual generalized Edouard numbers have the following formulas:*

$$\begin{aligned}
\text{(a): } \sum_{k=0}^n DW_k &= \frac{1}{4}(-(n+3)DW_n + (n+2)(7DW_{n+1} - DW_{n+2}) - (n+1)DW_{n+1} + 2DW_2 - 13DW_1 + \\
&\quad 7DW_0). \\
\text{(b): } \sum_{k=0}^n DW_{2k} &= \frac{1}{32}(-(n+3)DW_{2n} + (n+2)(-7DW_{2n+2} + 48DW_{2n+1} - 7DW_{2n}) - (n+1)DW_{2n+2} + \\
&\quad 15DW_2 - 96DW_1 + 49DW_0). \\
\text{(c): } \sum_{k=0}^n DW_{2k+1} &= \frac{1}{32}(-(n+3)DW_{2n+1} + (n+2)(-DW_{2n+2} + 42DW_{2n+1} - 7DW_{2n}) - (n+ \\
&\quad 1)(7DW_{2n+2} - 7DW_{2n+1} + DW_{2n}) + 9DW_2 - 56DW_1 + 15DW_0).
\end{aligned}$$

Proof.

(a): Note that using (2.1), we get

$$\sum_{k=0}^n DW_k = \sum_{k=0}^n W_k + \varepsilon \sum_{k=0}^n W_{k+1}$$

and using Proposition 11 the proof can be done easily.

(b): Note that using (2.1), we get

$$\sum_{k=0}^n DW_{2k} = \sum_{k=0}^n W_{2k} + \varepsilon \sum_{k=0}^n W_{2k+1}$$

and using Proposition 11 the proof can be done easily.

(c): Note that using (2.1), we get

$$\sum_{k=0}^n DW_{2k+1} = \sum_{k=0}^n W_{2k+1} + \varepsilon \sum_{k=0}^n W_{2k+2}$$

and using Proposition 11 the proof can be done easily. \square

As a special case of the Theorem 12 (a), we present the following corollary.

COROLLARY 13.

$$\textbf{(a): } \sum_{k=0}^n DE_k = \frac{1}{4}(-(n+3)DE_n + (n+2)(7DE_{n+1} - DE_{n+2}) - (n+1)DE_{n+1} + 1).$$

$$\textbf{(b): } \sum_{k=0}^n DK_k = \frac{1}{4}(-(n+3)DK_n + (n+2)(7DK_{n+1} - DK_{n+2}) - (n+1)DK_{n+1} - 8\varepsilon).$$

As a special case of the Theorem 12 (b), we present the following corollary.

COROLLARY 14.

$$\textbf{(a): } \sum_{k=0}^n DE_{2k} = \frac{1}{32}(-(n+3)DE_{2n} + (n+2)(-7DE_{2n+2} + 48DE_{2n+1} - 7DE_{2n}) - (n+1)DE_{2n+2} + 7\varepsilon + 9).$$

$$\textbf{(b): } \sum_{k=0}^n DK_{2k} = \frac{1}{32}(-(n+3)DK_{2n} + (n+2)(-7DK_{2n+2} + 48DK_{2n+1} - 7DK_{2n}) - (n+1)DK_{2n+2} - 32\varepsilon).$$

As a special case of the Theorem 12 (c), we present the following corollary.

COROLLARY 15.

$$\textbf{(a): } \sum_{k=0}^n DE_{2k+1} = \frac{1}{32}(-(n+3)DE_{2n+1} + (n+2)(-DE_{2n+2} + 42DE_{2n+1} - 7DE_{2n}) - (n+1)(7DE_{2n+2} - 7DE_{2n+1} + DE_{2n}) + \varepsilon + 7).$$

$$\textbf{(b): } \sum_{k=0}^n DK_{2k+1} = \frac{1}{32}(-(n+3)DK_{2n+1} + (n+2)(-DK_{2n+2} + 42DK_{2n+1} - 7DK_{2n}) - (n+1)(7DK_{2n+2} - 7DK_{2n+1} + DK_{2n}) - 64\varepsilon - 32).$$

6. Matrices related with Dual Generalized Edouard Numbers

In this part of our study we give some identities on some matrices linked to dual Edouard numbers. By using the $\{E_n\}$ which is defined by the third-order recurrence relation as follows

$$E_n = 7E_{n-1} - 7E_{n-2} + E_{n-3}$$

with the initial conditions $E_0 = 0$, $E_1 = 1$, $E_2 = 7$ we present the square matrix A of order 3 as

$$A = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 1$. Then, we give the following Lemma.

LEMMA 16. *For all integers n the following identity is true*

$$\begin{pmatrix} DW_{n+2} \\ DW_{n+1} \\ DW_n \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}. \quad (6.1)$$

Proof. First, we get $n \geq 0$. Lemma 16 can be given by mathematical induction on n . If $n = 0$ we get

$$\begin{pmatrix} DW_2 \\ DW_1 \\ DW_0 \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}$$

which is true. We claim that the identity (6.1) given holds for $n = k$. Thus the following identity is true.

$$\begin{pmatrix} DW_{k+2} \\ DW_{k+1} \\ DW_k \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}.$$

For $n = k + 1$, we get

$$\begin{aligned}
\begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} DW_2 \\ DW_1 \\ DW_0 \end{pmatrix} &= \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} DW_2 \\ DW_1 \\ DW_0 \end{pmatrix} \\
&= \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} DW_{k+2} \\ DW_{k+1} \\ DW_k \end{pmatrix} \\
&= \begin{pmatrix} 7DW_{k+2} - 7DW_{k+1} + DW_k \\ DW_{k+2} \\ DW_{k+1} \end{pmatrix} \\
&= \begin{pmatrix} DW_{k+3} \\ DW_{k+2} \\ DW_{k+1} \end{pmatrix}.
\end{aligned}$$

For the other case $n < 0$ the proof is easily attainable. Consequently, using mathematical induction on n , the proof is completed.

Note that

$$A^n = \begin{pmatrix} E_{n+1} & -7E_n + E_{n-1} & E_n \\ E_n & -7E_{n-1} + E_{n-2} & E_{n-1} \\ E_{n-1} & -7E_{n-2} + E_{n-3} & E_{n-2} \end{pmatrix}.$$

For the proof and more detail see [50].

THEOREM 17. *If we define the matrices N_{DW} and S_{DW} as follow*

$$\begin{aligned}
N_{DW} &= \begin{pmatrix} DW_2 & DW_1 & DW_0 \\ DW_1 & DW_0 & DW_{-1} \\ DW_0 & DW_{-1} & DW_{-2} \end{pmatrix}, \\
S_{DW} &= \begin{pmatrix} DW_{n+2} & DW_{n+1} & DW_n \\ DW_{n+1} & DW_n & DW_{n-1} \\ DW_n & DW_{n-1} & DW_{n-2} \end{pmatrix}.
\end{aligned}$$

then the following identity is true:

$$A^n N_{DW} = S_{DW}.$$

Proof. For the proof, we can use the following identities

$$\begin{aligned} A^n N_{DW} &= \begin{pmatrix} E_{n+1} & -7E_n + E_{n-1} & E_n \\ E_n & -7E_{n-1} + E_{n-2} & E_{n-1} \\ E_{n-1} & -7E_{n-2} + E_{n-3} & E_{n-2} \end{pmatrix} \begin{pmatrix} DW_2 & DW_1 & DW_0 \\ DW_1 & DW_0 & DW_{-1} \\ DW_0 & DW_{-1} & DW_{-2} \end{pmatrix}, \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} a_{11} &= DW_2 E_{n+1} + DW_1 (E_{n-1} - 7E_n) + DW_0 E_n, \\ a_{12} &= DW_1 E_{n+1} + DW_0 (E_{n-1} - 7E_n) + DW_{-1} E_n, \\ a_{13} &= DW_0 E_{n+1} + DW_{-1} (E_{n-1} - 7E_n) + DW_{-2} E_n, \\ a_{21} &= DW_2 E_n + DW_1 (E_{n-2} - 7E_{n-1}) + DW_0 E_{n-1}, \\ a_{22} &= DW_1 E_n + DW_0 (E_{n-2} - 7E_{n-1}) + DW_{-1} E_{n-1}, \\ a_{23} &= DW_0 E_n + DW_{-1} (E_{n-2} - 7E_{n-1}) + DW_{-2} E_{n-1}, \\ a_{31} &= DW_2 E_{n-1} + DW_1 (E_{n-3} - 7E_{n-2}) + DW_0 E_{n-2}, \\ a_{32} &= DW_1 E_{n-1} + DW_0 (E_{n-3} - 7E_{n-2}) + DW_{-1} E_{n-2}, \\ a_{33} &= DW_0 E_{n-1} + DW_{-1} (E_{n-3} - 7E_{n-2}) + DW_{-2} E_{n-2}, \end{aligned}$$

Using the Theorem 10 the proof is done. \square

From Theorem 17, the following corollary can be written.

COROLLARY 18.

(a): Let the matrices N_{DE} and S_{DE} are defined as the following

$$\begin{aligned} N_{DE} &= \begin{pmatrix} DE_2 & DE_1 & DE_0 \\ DE_1 & DE_0 & DE_{-1} \\ DE_0 & DE_{-1} & DE_{-2} \end{pmatrix}, \\ S_{DE} &= \begin{pmatrix} DE_{n+2} & DE_{n+1} & DE_n \\ DE_{n+1} & DE_n & DE_{n-1} \\ DE_n & DE_{n-1} & DE_{n-2} \end{pmatrix}, \end{aligned}$$

so that the identity given below is true for A^n , N_{DE} , S_{DE} ,

$$A^n N_{DE} = S_{DE},$$

(b): Let the matrices N_{DK} and S_{DK} are defined as the following

$$N_{DK} = \begin{pmatrix} DK_2 & DK_1 & DK_0 \\ DK_1 & DK_0 & DK_{-1} \\ DK_0 & DK_{-1} & DK_{-2} \end{pmatrix},$$

$$S_{DK} = \begin{pmatrix} DK_{n+2} & DK_{n+1} & DK_n \\ DK_{n+1} & DK_n & DK_{n-1} \\ DK_n & DK_{n-1} & DK_{n-2} \end{pmatrix},$$

so that the following identity is true for A^n , N_{DK} , S_{DK} ,

$$A^n N_{DK} = S_{DK}$$

7. Conclusion

In the literature, numerous studies have explored numerical sequences, which play a significant role in a wide range of disciplines, including physics, engineering, architecture, nature, and art. Among these, integer sequences such as Fibonacci, Lucas, Pell, and Jacobsthal are among the most well-known examples of second-order recurrence relations. The Fibonacci sequence, in particular, gained fame through its appearance in the rabbit population problem introduced by Leonardo of Pisa in his 1202 book *Liber Abaci*. Both the Fibonacci and Lucas sequences have inspired a wealth of intriguing and elegant mathematical identities. For rich applications of these second order sequences in science and nature, one can see the citations in [37,38,39].

In this work, we introduce the concept of dual hyperbolic generalized Edouard sequences and focus on two specific cases: dual Edouard numbers, dual Edouard-Lucas numbers.

- In section 1, we present information on the application areas of hypercomplex number systems in physics and engineering fields. Also, we give some properties about generalized Edouard numbers.
- In section 2, we define dual generalized Edouard numbers then using this definition, we present generating functions and Binet's formula of dual generalized Edouard numbers.
- In section 3, we obtain Binet's formula from generating function.
- In section 4, we give certain distinctive identities for the dual generalized Edouard sequence $\{\widehat{W}_n\}$ that named Simpson's formula.
- In section 5, we present summation formulas for dual generalized Edouard numbers.
- In section 6, we give some matrices related to dual Edouard numbers.

Linear recurrence relations (sequences) have many applications. Next, we list applications of sequences which are linear recurrence relations.

First, we present some applications of second order sequences.

- For the applications of Gaussian Fibonacci and Gaussian Lucas numbers to Pauli Fibonacci and Pauli Lucas quaternions, see [2].
- For the application of Pell Numbers to the solutions of three-dimensional difference equation systems, see [10].
- For the application of Jacobsthal numbers to special matrices, see [74].
- For the application of generalized k -order Fibonacci numbers to hybrid quaternions, see [25].
- For the applications of Fibonacci and Lucas numbers to Split Complex Bi-Periodic numbers, see [77].
- For the applications of generalized bivariate Fibonacci and Lucas polynomials to matrix polynomials, see [78].
- For the applications of generalized Fibonacci numbers to binomial sums, see [73].
- For the application of generalized Jacobsthal numbers to hyperbolic numbers, see [60].
- For the application of generalized Fibonacci numbers to dual hyperbolic numbers, see [61].
- For the application of Laplace transform and various matrix operations to the characteristic polynomial of the Fibonacci numbers, see [20].
- For the application of Generalized Fibonacci Matrices to Cryptography, see [45].
- For the application of higher order Jacobsthal numbers to quaternions, see [44].
- For the application of Fibonacci and Lucas Identities to Toeplitz-Hessenberg matrices, see [23].
- For the applications of Fibonacci numbers to lacunary statistical convergence, see [9].
- For the applications of Fibonacci numbers to lacunary statistical convergence in intuitionistic fuzzy normed linear spaces, see [32].
- For the applications of Fibonacci numbers to ideal convergence on intuitionistic fuzzy normed linear spaces, see [33].
- For the applications of k -Fibonacci and k -Lucas numbers to spinors, see [34].
- For the application of dual-generalized complex Fibonacci and Lucas numbers to Quaternions, see [70].
- For the application of special cases of Horadam numbers to Neutrosophic analysis see [24].
- For the application of Hyperbolic Fibonacci numbers to Quaternions, see [19].

We now present some applications of third order sequences.

- For the applications of third order Jacobsthal numbers and Tribonacci numbers to quaternions, see [15] and [14], respectively.
- For the application of Tribonacci numbers to special matrices, see [79].
- For the applications of Padovan numbers and Tribonacci numbers to coding theory, see [62] and [11], respectively.
- For the application of Pell-Padovan numbers to groups, see [16].

- For the application of adjusted Jacobsthal-Padovan numbers to the exact solutions of some difference equations, see [26].
- For the application of Gaussian Tribonacci numbers to various graphs, see [69].
- For the application of third-order Jacobsthal numbers to hyperbolic numbers, see [18]. For the application of Narayan numbers to finite groups see [40].
- For the application of generalized third-order Jacobsthal sequence to binomial transform, see [63].
- For the application of generalized Generalized Padovan numbers to Binomial Transform, see [64].
- For the application of generalized Tribonacci numbers to Gaussian numbers, see [65].
- For the application of generalized Tribonacci numbers to Sedenions, see [66].
- For the application of Tribonacci and Tribonacci-Lucas numbers to matrices, see [67].
- For the application of generalized Tribonacci numbers to circulant matrix, see [68].
- For the application of Tribonacci and Tribonacci-Lucas numbers to hybrinomials, see [71].
- For the application of hyperbolic Leonardo and hyperbolic Francois numbers to quaternions, see [17].

Next, we now list some applications of fourth order sequences.

- For the application of Tetranacci and Tetranacci-Lucas numbers to quaternions, see [55].
- For the application of generalized Tetranacci numbers to Gaussian numbers, see [56].
- For the application of Tetranacci and Tetranacci-Lucas numbers to matrices, see [57].
- For the application of generalized Tetranacci numbers to binomial transform, see [58].

We now present some applications of fifth order sequences.

- For the application of Pentanacci numbers to matrices, see [54].
- For the application of generalized Pentanacci numbers to quaternions, see [53].
- For the application of generalized Pentanacci numbers to binomial transform, see [59].

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