Coefficient Estimates for Initial Taylor-Maclaurin Coefficients for a New Family of Bi-Univalent and Analytic Functions Defined Using the Integral Operator

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ABSTRACT

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| In this research article, a new subclass of analytic and bi-univalent function is defined on the open unit disc using the integral operator and then the initial coefficient bounds for and are derived. Later on, the many interesting results are discussed which are obtained as the special cases of our main result. |

*Keywords: Bi-univalent functions, Coefficient bounds, Subordination, Integral Operator.*

2010 Mathematics Subject Classification: 30C45, 30C80

1. INTRODUCTION

Geometric function theory is very interesting and fascinating area of research. This field attracts researchers because of variety of ideas, methods and open challenges. The study of univalent, bi-univalent and multivalent functions are the branches of Geometric Function Theory and these are active fields of research even after more than century. In the study of bi-univalent functions, geometric behavior of functions can be analyzed by estimating coefficient bounds. In the early stages of development of the theory, Bieberbach conjecture (1916) was the open problem and it was successfully resolved by de Branges in 1985. Later on, researchers like Duren P.L. (1983), Nehari Z. (1953) and others gave remarkable contribution to this theory and resolved the issues related to the Bieberbach conjecture. Lewin M. (1967)extended the theory of univalent functions and introduced an interesting concept of bi-univalent functions. Later on, many researchers contributed to this area of research and attracted new researchers to contribute in this field.

In this research paper, a new subclass of bi-univalent and analytic functions is discussed and initial coefficient bounds are derived for this class of functions.

Let be an open unit disc in the complex plane . Let be a class of analytic functions defined on and which are normalized by the conditions and . Hence it can be expressed as

 Let be class of all normalized analytic functions defined on , which are univalent in .

 For two analytic functions and defined on , we say is subordinate to in and it is defined as

if there exists a Schwarz function , which is analytic in with

such that,

 (3)

If the function is univalent in , then we have the following equivalence.

.

For , we get is invertible. The Koebe one-quarter theorem [ Duren P.L. (1983)]ensures that, the image of under every contains a disc of radius .

 Thus for each , exists and define as

and

Here, the inverse function is given by

 (4)

 A function is said to be bi-univalent if both and its inverse, are univalent on . Let be a class of such bi-univalent functions defined on and which are in the form (1). The subclasses of such bi-univalent functions are discovered and studied by many researchers and initial coefficient bounds are also obtained. For the brief history and the recent work on interesting examples subclasses of , see Srivastava H.M., Mishra A.K. and Gochhayat (2010), Brannan D.A. and Taha T.S. (1988), Bulut S. (2013), Caglar M., Orhan H. and Yagmur N. (2013), Goyal S.P. and Goswami P. (2012), Hayani T. and Owa S. (2012), Muthaiyan E. and Wanas A.K. (2025), Al-Rawashdeh W. (2024) and Bakheet S.R., Atiyah M.A. and Muhammed M.S. (2025).

 Motivated by the work of Pathak R.P., Jadhav S.D., Khatu R.S. and Patil A.B. (2024) and Serap Bulut (2013), we introduce one new subclass of bi-univalent functions and further find initial coefficient bounds.

2. priliminaries, definitions and examples

 An integral operator defined on the class of analytic functions in introduced and studied by Pathak R.P., Jadhav S.D., Khatu R.S. and Patil A.B. (2024) and it is defined as follows.

**2.1 Definition:** Let and . The integral operator defined as

where

It can be easily observed that,

. (5)

where,

In general,

where. (6)

Now we define a new subclass of bi-univalent functions using above integral operator.

**2.2 Definition:** Let be a convex univalent function such that

, . (7)

A bi-univalent function given by equation (1) belongs to the class if the following condition holds.

 (8)

where is the operator defined by equation (6) and the function is given by the equation (4).

If we set , and a convex univalent function which satisfies the conditions mentioned in equation (7), then we can find a bi-univalent function with conditions and and in the form (1) such that, it satisfies the conditions mentioned in (7). It ensures that, the set defined in definition 2.2 is not empty.

**Remark 2.1:** If we set in Definition 2.2 then the class reduces to the class which is a class of bi-univalent functions given by equation (1) satisfying

 (9)

where is the operator defined by equation (6) and the function is given by the equation (4).

**Remark 2.2** If we set in Definition 2.2 then the class reduces to the class which is a class of bi-univalent functions given by equation (1) satisfying

 (10)

where is the operator defined by equation (6) and the function is given by the equation (4).

**Remark 2.3:** If we set in Definition 2.2 then the class reduces to the class which was introduced by Bulut S. (2013). It is a class of bi-univalent functions given by equation (1) satisfying

 (11)

where and the function is given by the equation (4).

**Remark 2.4:** If we set in Definition 2.2 then the class reduces to the class which was introduced by Bulut S. (2013). It is a class of bi-univalent functions given by equation (1) satisfying

 (12)

where and the function is given by the equation (4).

We use the following lemma to prove our result.

**Lemma 2.1:** Let the function given by is convex in . Suppose also that the function given by is holomorphic in . If , then .

3. coefficient bounds of the function class

This part of the paper is devoted to the estimation of coefficient bounds for the function class .

**Theorem 3.1:** Let be a function belongs to the class which is in the form given by equation (1) and then following inequalities hold.

, (13)

 (14)

where

**Proof:** Let . Then by definition of and the subordination principle, we get,

 (15)

 (16)

where and and having Taylor-Maclaurin series expansions

 , (17)

 (18)

respectively.

Now, by using the expressions of and given by equations (17) and (18) in equations (15) and (16) and equating coefficients of like powers of and , we get

, (19)

 , (20)

 , (21)

 . (22)

 Now, from (19) and (21), we get

 (23)

and

 . (24)

It gives

 (25)

Now, by using (20) and (22), we find that

 . (26)

It gives

 . (27)

Here Then by using Lemma 2.1, we have

 (28)

 (29)

By using (28) and (29) for the coefficients and , from the inequalities (25) and (27) we obtain

 (30)

and

 (31)

respectively. Inequalities (30) and (31) give the desire estimate on the coefficient bound of as stated in (13).

 Next, to obtain the coefficient bound on , we subtract (22) from (20), we get

 . (32)

This can be written as

 . (33)

Substituting the value of obtained from (25) in (33), we get

 . (34)

By using (28) and (29) in (34), we get

 (35)

On the other hand, substituting the value of obtained from (27) in (33), we get

 . (36)

By using (28) and (29) in (36), we get

 . (37)

By comparing (35) and (37), we get the desire coefficient bound of stated in (14).

Hence the theorem.

4. Corollaries and consequences

By setting in theorem 3.1, we get the following corollary.

**Corollary 4.1:** Let be a function belongs to the class which is in the form given by equation (1) then following inequalities hold.

, (38)

 (39)

where

By setting in theorem 3.1, we get the following corollary.

**Corollary 4.2:** Let be a function belongs to the class which is in the form given by equation (1) then following inequalities hold.

, (40)

 (41)

where

By setting in theorem 3.1, we get the following corollary.

**Corollary 4.3:** Let be a function belongs to the class which is in the form given by equation (1) then following inequalities hold.

, (42)

 (43)

where

If we put in the corollary 4.3 then we get improvement of estimates obtained by Frasin B.A. and Aouf M.K. (2011).

By setting in theorem 3.1, we get the following corollary.

**Corollary 4.4:** Let be a function belongs to the class which is in the form given by equation (1) then following inequalities hold.

, (44)

 (45)

where

If we put in the corollary 4.4 then we get improvement of estimates obtained by Srivastava H.M., Mishra A.K. & Gochhayat, P. (2010).

5. Conclusion

In this paper, using integral operator , we defined a subclass of bi-univalent functions and later on, we obtained the coefficient bounds for and . We observed that many interesting results and well-known findings are corollaries of our result. The study of coefficient estimation of a subclass of bi-univalent functions can be further extended to the Fekete–Szegö inequality.

Disclaimer (Artificial Intelligence)

Author hereby declares that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during writing or editing of this manuscript.

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conflict of interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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