CHARACTERIZATIONS OF BOUNDED LINEAR OPERATORS ON BANACH SPACE-VALUED FIBONOMIAL SEQUENCE SPACES

Abstract. In this paper we have tried to reveal the properties of some Banach space-valued Fibonomial sequence spaces, in particular the properties of bounded linear operators defined on them. Further we show that these sequence spaces has a kind of Schauder basis which we introduce it in ours former works. We also prove that $b_p^{r,s,F}(V)$, $1 \leq p < \infty$, and $b_0^{r,s,F}(V)$ have the approximation property under certain conditions where V is a a Banach space.

1. Introduction

Structural analysis of Banach space-valued function or sequence spaces has always been an interesting subject. In these structural analyses, the Schauder basis of the space provides an important advantage. This advantage is also encountered in the study of bounded linear operators on spaces with this base. The study of operators defined between a Banach space-valued function or sequence spaces is more difficult. One of the reasons for this is that Banach space-valued sequence spaces do not have a Schauder basis in the usual sense. In [12] we give a new definition of Schauder basis which allows us to better analyse the structural properties of many Banach space-valued functions or sequence spaces and the bounded linear operators defined between them. Details of these studies can be found in references ([3, 4, 6] and [9, 11, 14, 10]). In particular, the above-mentioned new type of basis concept reveals many new properties of some V-valued sequence spaces where V is a special Banach space. Therefore, in this paper we will try to reveal some properties of such sequence spaces, in particular the properties of bounded linear operators defined on them.

The study of the topological and geometric structures of some sequence spaces constructed with the help of some important number sequences that we encounter in nature, for example, Fibonacci, Catalan or Schröder numbers, has been frequently encountered recently. For instance, Bişgin, in [2], investigated geometric properties of some binomial sequence spaces which include the spaces ℓ_p and ℓ_∞. Inspired by this work, Cihat introduced the Fibonomial sequence spaces $b_p^{r,s,F}$ (for $1 \leq p < \infty$) and $b_\infty^{r,s,F}$ in [15]. These spaces were defined with the help of Fibonacci numbers and (F_n) denote the sequence of Fibonacci numbers defined by the recurrence relation $F_{n+2} = F_{n+1} + F_n$ with the initial conditions $F_0 = 0$ and $F_1 = 1$. Thus

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$0,1,1,2,3,5,8,13,21,...$are the first few Fibonacci numbers. In [13] we introduced some V-valued Fibonacci sequence spaces and investigated some its metric properties for any Banach space V. With the help of Fibonacci numbers and similar to the binomial matrix in [2], the matrix $\mathcal{B}^{r,s,F}$ is defined in [15] and with the help of this matrix the sequence spaces $b_p^{r,s,F}$ (for $1 \leq p < \infty$) and $b_\infty^{r,s,F}$ are defined as follows; for nonzero real numbers s,r such that $s+r \neq 0$, fibonomial matrix $\mathcal{B}^{r,s,F} = \left(b_{nk}^{r,s,F}\right)$ is defined by

$$b_{nk}^{r,s,F} = \begin{cases} \frac{1}{(r+s)_F^n} \binom{n}{k}_F r^k s^{n-k}, & if \quad 0 \leq k \leq n \\ 0, & if \quad k > n \end{cases}$$

as a sub-triangular matrix. Further the Fibonomial sequence spaces are introduced as

$$b_p^{r,s,F} = \left\{ u = (u_n) \in w : \sum_{n=0}^{\infty} \left| \frac{1}{(r+s)_F^n} \sum_{k=0}^{n} \binom{n}{k}_F r^k s^{n-k} u_k \right|^p < \infty \right\}$$

and

$$b_\infty^{r,s,F} = \left\{ u = (u_n) \in w : \sup_n \left| \frac{1}{(r+s)_F^n} \sum_{k=0}^{n} \binom{n}{k}_F r^k s^{n-k} u_k \right| < \infty \right\}$$

in [15]. Additionally, it is shown in the related work that $b_p^{r,s,F}$ (for $1 \leq p < \infty$) and $b_\infty^{r,s,F}$ are BK- spaces with norms

$$\|u\|_{b_p^{r,s,F}} = \left(\sum_{n=0}^{\infty} \left| \frac{1}{(r+s)_F^n} \sum_{k=0}^{n} \binom{n}{k}_F r^k s^{n-k} u_k \right|^p \right)^{1/p}$$

and

$$\|u\|_{b_\infty^{r,s,F}} = \sup_n \left| \frac{1}{(r+s)_F^n} \sum_{k=0}^{n} \binom{n}{k}_F r^k s^{n-k} u_k \right|,$$

respectively. In addition, some properties of matrix operators on Fibonomial sequence spaces are examined in [16]. Another interesting study characterising the properties of this type of sequence space is given in [7] . In [8], another type of Fibonomial sequence space is examined. Another study similar to these studies is given in [5].

In this work we will first consider any Banach space V and introduce V-valued fibonomial sequence spaces $b_p^{r,s,F}(V)$, $b_\infty^{r,s,F}(V)$ and $b_0^{r,s,F}(V)$ such that

$$b_p^{r,s,F}(V) = \left\{ u = (u_n) \in w(V) : \sum_{n=0}^{\infty} \left\| \frac{1}{(r+s)_F^n} \sum_{k=0}^{n} \binom{n}{k}_F r^k s^{n-k} u_k \right\|_V^p < \infty \right\},$$

$$b_\infty^{r,s,F}(V) = \left\{ u = (u_n) \in w(V) : \sup_n \left\| \frac{1}{(r+s)_F^n} \sum_{k=0}^{n} \binom{n}{k}_F r^k s^{n-k} u_k \right\|_V < \infty \right\}$$

and

$$b_0^{r,s,F}(V) = \left\{ u = (u_n) \in w(V) : \lim_n \left\| \frac{1}{(r+s)_F^n} \sum_{k=0}^{n} \binom{n}{k}_F r^k s^{n-k} u_k \right\|_V = 0 \right\}.$$

Here $w(V)$ denotes the vector space of all V-valued sequences and $u = (u_n) \in w(V)$ means each $u_n \in V$.

For $V=\mathbb{K}$, the real or complex number, then $b_p^{r,s,F}(V)=b_p^{r,s,F}$ and $b_\infty^{r,s,F}(V)=b_\infty^{r,s,F}$in [15]. V-valued fibonomial sequence space $b_p^{r,s,F}(V)$ is a Banach space by the norm

$$\|u\|=\left(\sum_{n=0}^{\infty}\left\|\frac{1}{(r+s)_F^n}\sum_{k=0}^{n}\binom{n}{k}_F r^k s\in^{n-k} u_k\right\|_V^p\right)^{1/p}$$

and $b_\infty^{r,s,F}(V)$ and $b_0^{r,s,F}(V)$ are Banach spaces by the norm

$$\|u\|=\sup_n\left\|\frac{1}{(r+s)_F^n}\sum_{k=0}^{n}\binom{n}{k}_F r^k s^{n-k}u_k\right\|_V .$$

We will mainly see in this work that $b_p^{r,s,F}(V)$ and $b_0^{r,s,F}(V)$ have Schauder bases in the sense of [12] and that some of these spaces have the approximation property. Finally, we will give a fundamental result characterizing all bounded linear operators defined on the space $b_0^{r,s,F}(V)$. The main tool we use to obtain this result will be a new type of Schauder basis that we will define for the space $b_0^{r,s,F}(V)$.

Now let us provide some known results from Banach space theory, [1]. Let us consider Banach spaces U and V. Any linear operator S from U to V is called compact if it maps any bounded subset B of U to a relatively compact subset $S(B)$ of V. Further $K(U,V)$ denotes the set of all compact linear operators from U to V. Any compact operator between Banach spaces has a closed range if and only if it is of finite rank, meaning its range is a finite-dimensional linear space. A Banach space U has the approximation property if, for every Banach space V, the set of finite-rank members of $B(V,U)$ is dense in $K(V,U)$. It is known that the spaces c_0 and ℓ_p, where $1\le p<\infty$, possess the approximation property [1]. Similarly we can deduce from [12] that $c_0(V)$ and $\ell_p(V)$, where $1\le p<\infty$, have the approximation property if V has.

In general, we know that V-valued sequence spaces has no Schauder basis if $V\neq\mathbb{K}$. Now let us give the definition of new kind Schauder basis which we introduced it in [12].

Definition 1. [12] *Let U and V are Banach spaces and let $\mathbb{A}$ be an index set and let $\mathcal{F}$ be the family of all finite subsets of $\mathbb{A}$. as A collection $\{\eta_a : a\in\mathbb{A}\}$ of continuous linear operators $\eta_a : V\to U$ is called a* V-basis for U *if there exists a directed subset $\mathcal{D}$, by a relation $\ll$, of $\mathcal{F}$ and there exist unique family $\{R_a : a\in\mathbb{A}\}$ of linear operators from U onto V such that the net $(\pi_F(x):\mathcal{D})$ converges to x in U for every $x\in U$. Where each $F\in D$ and*

$$\pi_F(x)=\sum_{a\in F}(\eta_a\circ R_a)(x).$$

Moreover, $\{\eta_a\}$ is called a V-Schauder basis for U *whenever each operator R_a is continuous.*

We call $\{R_a : a\in\mathbb{A}\}$ as the associate family of functions (A.F.F.) corresponding to V-basis $\{\eta_a : a\in\mathbb{A}\}$.

Given a V-basis $\{\eta_a : a\in\mathbb{A}\}$ for U, the finite summation of function $\pi_F(x)$ induces an operator π_F on U for each $F\in\mathcal{D}$. This operator known as the F-*projection* on U relative to the V-basis. The V-basis in the definition is called *unconditional* if directed subset $\mathcal{D}$ is taken as whole $\mathcal{F}$ with the inclusion relation $\subseteq$.

Remark 1. *Let V be a Banach space which has a Schauder basis $\{x_n\}$ in the classical sense. Then, the sequence $\{\eta_n\}$ of linear operators*

$$\eta_n : \mathbb{C} \to V : \eta_n (z) = zx_n$$

forms a $\mathbb{K}$-basis for V in the sense of above definition where $\mathbb{K}$ is the scalar field of V. In this case take $\mathbb{A} = \mathbb{N}$ and take

$$\mathcal{D} = \{\{1\}, \{1,2\}, \{1,2,3\}, \ldots\}$$

as a directed set in the poset $\mathcal{F}$ with the inclusion relation. Further take $\{R_n\}$ as the sequence of coordinate functionals (g_n) corresponding to the basis $\{x_n\}$. Then we conclude that $(\pi_F (x) : \mathcal{D})$ converges to x in U iff

$$\sum_{k=1}^{n} (\eta_k \circ R_k)(x) = \sum_{k=1}^{n} g_k (x) x_k,$$

converges to $x = \sum_{n=1}^{\infty} g_n (x) x_n$.

Theorem 1. *If a Banach space V possesses a Y-basis $\{\eta_a : a \in \mathbb{A}\}$, then V is separable iff the index set $\mathbb{A}$ is countable* [12].

2. Main Results

In this section, we present a key results concerning V-valued sequence spaces $b_0^{r,s,F}(V)$ and $b_p^{r,s,F}(V)$.

Theorem 2. *Given a Banach space V the sequence $\left\{\mathcal{B}^{r,s,F} I_n : n \in \mathbb{N}\right\}$ is an unconditional V-Schauder basis for both $b_0^{r,s,F}(V)$ and $b_p^{r,s,F}(V)$ where $1 \leq p < \infty$.*

Proof. Take $\mathbb{A} = \mathbb{N}$ and $\mathcal{D} = \mathcal{F}$ the family of all finite subsets of $\mathbb{N}$ with the inclusion relation in Definition [12]. Let us do the proof only for $b_p^{r,s,F}(V)$. The proof for $b_0^{r,s,F}(V)$ almost is the same. Consider embeddings

$$I_n : V \to b_p^{r,s,F}(V), \ I_n (z) = (0, ..., 0, z, 0, ...),$$

where z is in the *n.th* place. Clearly each I_n is linear and let us prove that its bounded:

$$\begin{aligned}
\|I_n (z)\| &= \|(0, ..., 0, z, 0, ...)\| = \left\|\mathcal{B}^{r,s,F} (0, ..., 0, z, 0, ...)\right\|_{l_p(V)} \\
&\leq \left\|\mathcal{B}^{r,s,F}\mathcal{B}^{r,s,F} (0, ..., 0, \|z\|_V, 0, ...)\right\|_{l_p} \\
&= \|(0, ..., 0, \|z\|_V, 0, ...)\|_{b_p^{r,s,F}} \\
&\leq \left\|\mathcal{B}^{r,s,F}\right\| . \|z\|_V
\end{aligned}$$

Remember the sub-diagonal matrix $\mathcal{B}^{r,s,F}$ defines a bounded linear operator on l_p and $\left\|\mathcal{B}^{r,s,F}\right\|$ exists. So each $\mathcal{B}^{r,s,F} I_n$ is a bounded linear operator from V into $b_p^{r,s,F}(V)$. Now just the sequence

$$\left\{\mathcal{B}^{r,s,F} I_n : n \in \mathbb{N}\right\}$$

is a V-Schauder basis for $b_p^{r,s,F}(V)$. In order to prove this let us consider projections

$$P_n : b_p^{r,s,F}(V) \to V;\ P_n(x) = x_n \text{ where } x = (x_n) \in\ b_p^{r,s,F}(V).$$

In order to show that the operator sequence $\{\mathcal{B}^{r,s,F} I_n\}$ defined above is an unconditional basis, the ordered set $\mathcal{D}$ in Definition [12] must be chosen as the family of all finite subsets of $\mathbb{N}$ by the inclusion relation $\subseteq$. Let us now for any $x = (x_n)$ in $b_p^{r,s,F}(V)$ write

$$\begin{aligned} \pi_F(x) &= \sum_{n\in F} \left(\left(\mathcal{B}^{r,s,F} I_n\right) \circ P_n\right)(x) \\ &= \sum_{n\in F} \left(\mathcal{B}^{r,s,F} I_n\right)(x_n). \end{aligned}$$

We will prove that the net $(\pi_F(x) : \mathcal{D})$ converges to x in $b_p^{r,s,F}(V)$. It is clear that the convergence of the given net is equivalent to the unconditional convergence of the sequence of partial sums of the series $\sum_{n=0}^{\infty} \left(\mathcal{B}^{r,s,F} I_n\right)(x_n)$. Now, for any arbitrary $\varepsilon > 0$, we must find a finite subset $F_0 = F_0(\varepsilon) \in \mathcal{D}$ such that for every finite set $F \supseteq F_0$,

$$\|x - \pi_F(x)\| \leq \varepsilon.$$

Since x is contained in $b_p^{r,s,F}(V)$ there is an index $n_0(\varepsilon)$ such that the series $\sum_{n>n_0}^{\infty} \left\|\left(\mathcal{B}^{r,s,F} x\right)_n\right\|_V^p$ is strictly smaller than ε. At this point, let us determine F_0 as

$$F_0 = \left\{ n \in \mathbb{N} : \sum_{n>n_0}^{\infty} \left\|\left(\mathcal{B}^{r,s,F} x\right)_n\right\|_V^p > \varepsilon \right\},$$

Then for each finite $F_0 \subseteq F$ we get

$$\|x - \pi_F(x)\| = \|\{x_n : n \in \mathbb{N} \setminus F\}\| \leq \varepsilon.$$

This shows that $(\pi_F(x) : \mathcal{D})$ converges to x in $b_p^{r,s,F}(V)$.

Let us now verify that the sequence $\{P_n\}$ is uniquely defined. Suppose

$$\sum_{n\in\mathbb{N}} \left(\mathcal{B}^{r,s,F} I_n P_n\right)(x) = \sum_{n\in\mathbb{N}} \left(\mathcal{B}^{r,s,F} I_n P_n'\right)(x)$$

and write

$$\pi_F^\circ(x) = \sum_{n\in\mathbb{N}} \left(\mathcal{B}^{r,s,F} I_n \left(P_n - P_n'\right)\right)(x), \quad F \in \mathcal{D}.$$

Remember that

$$\|\pi_F^\circ(x)\| = \left(\sum_{n\in F} \left\|\left(\mathcal{B}^{r,s,F} I_n \left(P_n - P_n'\right)\right)(x)\right\|^p \right)^{1/p}$$

and

$$\|\pi_F^\circ(x)\| \leq \|\pi_G^\circ(x)\|$$

for $F \subseteq G$. As $(\pi_F(x) : \mathcal{D})$ exhibits convergence toward x in $b_p^{r,s,F}(V)$ we get

$$\lim_{F\in\mathcal{D}} \|\pi_F^\circ(x)\| = 0.$$

From this observation, it follows that $(P_n - P_n')(x) = 0$ for all n and for every $x \in b_p^{r,s,F}(V)$. Consequently, we obtain $P_n = P_n'$ for each n, ensuring the uniqueness of the basis. $\square$

Remark 2. *By this result we say any element* $x=(x_n)$ *in* $b_p^{r,s,F}(V)$ *or in* $b_0^{r,s,F}(V)$ *can be represented uniquely as*

$$\begin{aligned} x &= \sum_{n=1}^{\infty}\left(\left(\mathcal{B}^{r,s,F}I_n\right)\circ P_n\right)(x) \\ &= \sum_{n=1}^{\infty}\left(\mathcal{B}^{r,s,F}I_n\right)(x_n). \end{aligned}$$

This means in norm that

$$\left\|(x_n)-\sum_{n=1}^{m}\left(\mathcal{B}^{r,s,F}I_n\right)(x_n)\right\|\to 0 \text{ as } m\to\infty.$$

Further this representation is unconditional.

Theorem 3. *The approximation property holds in* $b_p^{r,s,F}(V)$ *for all* $1\le p<\infty$ *and in* $b_0^{r,s,F}(V)$ *if and only if the Banach space* V *has this property.*

Proof. Again let us give the proof only for $b_p^{r,s,F}(V)$. Suppose that Λ is a compact linear operator from a Banach space V to $b_p^{r,s,F}(V)$. We seek a sequence (Λ_n) of bounded finite-rank linear operators from V into $b_p^{r,s,F}(V)$. Due to the compactness of Λ, for any bounded sequence (x_n) in V, the sequence (Λx_n) contains a convergent subsequence $\left(\Lambda x_{n_j}\right)_{j=0}^{\infty}$ in $b_p^{r,s,F}(V)$. Further for every $x\in V$, Λx lies in $b_p^{r,s,F}(V)$ and explicitly

$$\begin{aligned} \left\|\Lambda x_{n_i}-\Lambda x_{n_j}\right\|^p &= \left\|\Lambda\left(x_{n_i}-x_{n_j}\right)\right\|^p \\ &= \left\|\left(\mathcal{B}^{r,s,F}\Lambda\right)\left(x_{n_i}-x_{n_j}\right)\right\|_{\ell_p(V)}^p. \end{aligned}$$

Just now we recall classical Banach space ℓ_p has approximation property and with this conjecture we can say V has the approximation property if and only if $\ell_p(V)$ has. Hence

$$\left\|\left(\mathcal{B}^{r,s,F}\Lambda\right)\left(x_{n_i}-x_{n_j}\right)\right\|_{\ell_p(V)}^p\to 0 \text{ as } i,j\to\infty.$$

We conclude that the operator $\mathcal{B}^{r,s,F}\Lambda: V\to\ell_p(V)$ is compact. Of course, the matrix $\mathcal{B}^{r,s,F}$ is a bounded linear operator and hence $\mathcal{B}^{r,s,F}\Lambda$ is also bounded. By the approximation property of $\ell_p(V)$ there exists a sequence of finite-rank bounded linear operators $(A_m)_{m=0}^{\infty}$ from V to $\ell_p(V)$ such that $\left\|\mathcal{B}^{r,s,F}\Lambda-A_m\right\|\to 0$ as $m\to\infty$. Thus, we obtain the desired sequence of finite-rank operators as $\left(\left(\mathcal{B}^{r,s,F}\right)^{-1}A_m\right)_{m=0}^{\infty}$ which they map V into $\ell_p\left(\mathcal{B}^{r,s,F},V\right)$. Note that $\left(\mathcal{B}^{r,s,F}\right)^{-1}$ exist and easily it can be verified that each $\left(\mathcal{B}^{r,s,F}\right)^{-1}A_m$ is a bounded linear operator of finite rank. Further

$$\begin{aligned} \left\|\Lambda-\left(\mathcal{B}^{r,s,F}\right)^{-1}A_m\right\| &= \sup_{\|x\|=1}\left\|\left(\Lambda-\left(\mathcal{B}^{r,s,F}\right)^{-1}A_m\right)x\right\| \\ &= \sup_{\|x\|=1}\left\|\Lambda x-\left(\left(\mathcal{B}^{r,s,F}\right)^{-1}A_m\right)x\right\| \\ &= \sup_{\|x\|=1}\left\|\mathcal{B}^{r,s,F}\Lambda x-\mathcal{B}^{r,s,F}\left(\left(\mathcal{B}^{r,s,F}\right)^{-1}A_m\right)x\right\|_{\ell_p(V)} \\ &= \sup_{\|x\|=1}\left\|\left(\mathcal{B}^{r,s,F}\Lambda-A_m\right)x\right\|_{\ell_p(V)} \\ &\to 0 \text{ as } m\to\infty. \end{aligned}$$

This completes the proof. □

We now give a characterization of all bounded linear operators defined on the Fibonomial sequence space $b_0^{r,s,F}(V)$. This is a more comprehensive study than the characterizations of matrices defined on V-valued sequence spaces. This is because all scalar infinite matrices defined between these spaces are bounded linear operators on this space, but the converse is not always true. Similar version of following characterization also can be done for the space $b_p^{r,s,F}(V)$ where $1 \leq p < \infty$.

Theorem 4. *The operator space $B\left(b_0^{r,s,F}(V), b_0^{r,s,F}(V)\right)$ is equivalent (isometrically isomorphic) by the mapping*

$$T \to \left\{T\mathcal{B}^{r,s,F}I_n : n \in \mathbb{N}\right\}$$

to Λ, the space of all sequences $\varphi = (\varphi_n)$ such that

$$\text{each } \varphi_n \in B\left(V, b_0^{r,s,F}(V)\right) \text{ and } \sum_{n\in\mathbb{N}} \|g \circ \varphi_n\| < \infty$$

for each $g \in b_1^{r,s,F}(V^*)$. Further Λ is a Banach space with the norm

$$\|\varphi\| = \sup_{\|g\|=1} \sum_{n\in\mathbb{N}} \|g \circ \varphi_n\|.$$

where $\left\{\mathcal{B}^{r,s,F}I_n\right\}$ is the V-Schauder basis for $b_0^{r,s,F}(V)$.

Proof. Let us first consider the V-Schauder basis $\left\{\mathcal{B}^{r,s,F}I_n : n \in \mathbb{N}\right\}$ of $b_0^{r,s,F}(V)$ and let us write $\varphi_n = T\mathcal{B}^{r,s,F}I_n$ for $T \in B\left(b_0^{r,s,F}(V), b_0^{r,s,F}(V)\right)$. Further now let us define an operator

$$\Psi : B\left(b_0^{r,s,F}(V), b_0^{r,s,F}(V)\right) \to \Lambda \text{ by } \Psi(T) = \varphi \text{ such that } \varphi = \{\varphi_n\}_{n=1}^{\infty}.$$

First of all we have to show that really Ψ defines a mapping. For some finite set $F \in \mathcal{F}$ and for $x = (x_n) \in b_0^{r,s,F}(V)$ let us write

$$\pi'_F(x) = \sum_{n\in F} (\varphi_n \circ P_n)(x)$$

where $\mathcal{F}$ is the family of all finite subsets of all natural numbers which is directed by the inclusion relation. π'_F is a continuous linear operator on $b_0^{r,s,F}(V)$. This is comes from the fact that $\pi'_F(x) = (T \circ \pi_F)(x)$ where $\pi_F(x) = \sum_{n\in F} \left(\left(\mathcal{B}^{r,s,F}I_n\right) \circ P_n\right)(x)$, and comes from the continuity of each $\left(\mathcal{B}^{r,s,F}I_n\right) \circ P_n$. Now it is known from former studies that the dual space $c_0(V)^* = l_1(V^*)$. Further since $x \in b_0^{r,s,F}(V)$ if and only if $\mathcal{B}^{r,s,F}x \in c_0(V)$ we can easily deduce that $\left(b_0^{r,s,F}(V)\right)^* = b_1^{r,s,F}(V^*)$, see

[12]. This fact brings us

$$\begin{aligned}
\sup_{\substack{\|g\|=1 \\ g\in b_1^{r,s,F}(V^*)}} \sum_{n\in\mathbb{N}} \|g\circ\varphi_n\| &\le \sup_{\|g\|=1}\sup_{F\in\mathcal{F}}\sum_{n\in\mathbb{N}}\|g\circ\varphi_n\| \\
&= \sup_{\|g\|=1}\sup_{F\in\mathcal{F}}\sum_{n\in\mathbb{N}}\|g\circ\varphi_n\| \\
&= \sup_{F\in\mathcal{F}}\sup_{\|x\|=1}\sup_{\|g\|=1}\left|\sum_{n\in\mathbb{N}}(g\circ\varphi_n)\,x_n\right| \\
&= \sup_{F\in\mathcal{F}}\sup_{\|x\|=1}\sup_{\|g\|=1}\left|g\left(\sum_{n\in\mathbb{N}}\varphi_n(x_n)\right)\right| \\
&= \sup_{F\in\mathcal{F}}\sup_{\|x\|=1}\left\|\sum_{n\in\mathbb{N}}(\varphi_n\circ P_n)(x)\right\| \\
&= \sup_{F\in\mathcal{F}}\sup_{\|x\|=1}\|\pi'_F(x)\| \\
&= \sup_{F\in\mathcal{F}}\|\pi'_F\|.
\end{aligned}$$

Remember that $Tx\in b_0^{r,s,F}(V)$ and by using again the V-Schauder basis in $b_0^{r,s,F}(V)$ we can say that the net $(\pi'_F(x),\mathcal{F})$ converges to Tx. This is equivalent to say that the series $\sum_{n=1}^{\infty}(\varphi_n\circ P_n)(x)$ is unconditional convergent to Tx. Hence from this fact we can deduce the net $(\pi'_F(x),\mathcal{F})$ is bounded, that is $\sup_{F\in\mathcal{F}}\|\pi'_F(x)\|<\infty$. Now from the Uniform Boundedness Theorem we get $\sup_{F\in\mathcal{F}}\|\pi'_F\|<\infty$. This means $\varphi\in\Lambda$ and so Ψ is well-defined. Now let us show that the function $\|\varphi\|=\sup_{\|g\|=1}\sum_{n\in\mathbb{N}}\|g\circ\varphi_n\|$ defined above satisfies the norm conditions. Let $\|\varphi\|=0$, then for each $g\in b_1^{r,s,F}(V^*)$ and for each $n\in\mathbb{N}$

$$g\circ\varphi_n=0.$$

So each $\varphi_n=0$, that is $\varphi=0$. Further if $\|\lambda\varphi\|=0$ for any scalar λ then

$$\begin{aligned}
\|\lambda\varphi\| &= \sup_{\|g\|=1}\sum_{n\in\mathbb{N}}\|g\circ(\lambda\varphi_n)\| \\
&= \sup_{F\in\mathcal{F}}\sup_{\|x\|=1}\sup_{\|g\|=1}\left|g\left(\sum_{n\in\mathbb{N}}\lambda\varphi_n(x_n)\right)\right| \\
&= |\lambda|\sup_{F\in\mathcal{F}}\sup_{\|x\|=1}\sup_{\|g\|=1}\left|g\left(\sum_{n\in\mathbb{N}}\varphi_n(x_n)\right)\right| \\
&= |\lambda|\,\|\varphi\|.
\end{aligned}$$

Further the last condition of the norm is straighforward.

Now let us prove that the linear mapping Ψ is an equivalence (isometric isomorphism). Since it is relatively easy to show that the operator Ψ is surjective, it is sufficient here to show that Ψ is an isometry.

Since

$$\begin{aligned}
\|Tx\| &= \left\| T\left(\sum_{n=1}^{\infty} \left(\left(\mathcal{B}^{r,s,F} I_n \right) \circ P_n \right)(x) \right) \right\| \\
&= \left\| \sum_{n=1}^{\infty} \left(\varphi_n \circ P_n \right)(x) \right\| \\
&\leq \sup_{F \in \mathcal{F}} \left\| \pi'_F (x) \right\| \\
&\leq \|x\| \sup_{F \in \mathcal{F}} \left\| \pi'_F \right\| \\
&\leq \|x\| \, \|\varphi\|
\end{aligned}$$

we have $\|T\| \leq \|\varphi\|$. On the other hand, for any g with $\|g\| = 1$ we get

$$\begin{aligned}
\sum_{n \in \mathbb{N}} \|g \circ \varphi_n\| &= \sup_{\|x\|=1} \left| g\left(\sum_{n \in \mathbb{N}} \varphi_n (x_n) \right) \right| \\
&\leq \|g\| \sup_{\|x\|=1} \left\| \sum_{n=1}^{\infty} \left(\varphi_n \circ P_n \right)(x) \right\|
\end{aligned}$$

Hence

$$\begin{aligned}
\|\varphi\| &= \sup_{\|g\|=1} \sum_{n \in \mathbb{N}} \|g \circ \varphi_n\| \\
&\leq \sup_{\|g\|=1} \sup_{\|x\|=1} \left(\|g\| \sup_{\|x\|=1} \left\| \sum_{n=1}^{\infty} \left(\varphi_n \circ P_n \right)(x) \right\| \right) \\
&= \sup_{\|x\|=1} \left\| \sum_{n=1}^{\infty} \left(\varphi_n \circ P_n \right)(x) \right\| \\
&= \sup_{\|x\|=1} \|Tx\| \\
&= \|T\| .
\end{aligned}$$

This shows that Ψ is an isometry. □

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