Journal of Advances in Mathematics and Computer Science

xx(x): ....., 20yy; Article no.JAMCS.xxxxx

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

Restrained Semitotal Domination in Various Graph Structures

Original Research Article

Received: XX December 20XX
Accepted: XX December 20XX
Online Ready: XX December 20XX

Abstract

Restrained domination involves a vertex partitioning and considers the parameter domination on graph. Recently, another parameter called semitotal domination was introduced which strengthens domination while relaxing total and weakly connected domination constraints. In this paper, we introduced a new parameter of domination a combination of the aforementioned domination parameters called the restrained semitotal domination of graphs. Also, we have discussed some characterization type results with respect to certain family of graphs and graph operations.

Keywords: Restrained Domination; Semitotal Domination; Restrained Semitotal Domination

2010 Mathematics Subject Classification: 05C69; 05C76

1 Introduction

Graph theory is a branch of mathematics that began with the Konigsberg Bridge Problem, solved by Leonhard Euler in 1738. The famous Konigsberg Bridge Problem posed the question of whether it was possible to find a route that would cross each bridge exactly once, starting and ending at the same point. Since then, it has expanded to cover areas such as combinatorics, computational algebra, and engineering (Chartrand et al., 2015).

The concept of domination was formulated by Claude Berge and Oystein Ore (Haynes et al., 1998), and it has been a great interest in the field of graph theory (Gupta, 2013). As mentioned by Haynes et al., (1998) domination in graphs has numerous applications, including in network security and facility location problems such as hospital, school, prison, and etc. The concept of domination in graphs, with its many variations, is now well studied in graph and it has various domination models available in the literature. Some variations of domination include semitotal domination and restrained domination.

A new parameter is recently introduced by Goddard, Henning, and McPillan called semitotal dominating set (Goddard et al., 2014). A set S of vertices in a graph G with no isolated vertices is a semitotal dominating set of G if it is a dominating set of G and every vertex in S is within distance 2 of another

vertex in S (Henning & Marcon, 2014). The semitotal domination number, denoted by $\gamma_{t2}(G)$, is the minimum cardinality of a semitotal dominating set. It is a strengthening of domination but a relaxation of both total domination and weakly connected domination (Goddard et al., 2014). Another variation of domination which will be use as an extension of semitotal domination called restrained domination was introduced by Telle and Proskurowski indirectly as a vertex partitioning problem (Telle & Proskurowski, 1997) and its concepts well-known from the domination of graphs as one of its parameter variations. Accordingly, a restrained dominating set is a set S of vertices in a graph G where every vertex in S complement within $V(G)$ is adjacent to a vertex in S as well as to another vertex in S complement within $V(G)$. The restrained domination number of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set of G.
s Consider the roles of prisoners and guards in the prison settings. In this context, every vertex in the restrained dominating set corresponds to a guard's position, then each vertex not in the restrained dominating set corresponds to a prisoner's position. To ensure security each prisoner's position is observed by a guard. Guard monitor the prisoners to prevent unauthorized activities. Moreover, each prisoner's position is also seen by at least one other prisoner's position. This arrangement protects the rights of prisoners and ensures their well-being since other prisoners can observe their condition. On the other hand, if the prisoners are planning an escape, they might attempt to harass one of the guards at a specific location. To optimize cost-effectiveness, it is desirable to place as few guards as possible within a distance of 2. This way, if an incident occurs, other guards nearby can quickly respond and provide assistance. To address this problem, we introduced a new variation of domination, namely, the restrained semitotal domination.

2 Preliminaries

This section introduces key definitions and concepts essential for this study.

Definition 2.1. (Balakrishnan & Ranganathan, 2014) A **graph** G is an ordered pair (V, E) for which $V = V(G)$ and $E = E(G)$ where $V(G)$ is the vertex set and $E(G)$ is the edge set of G. If pair $e = \{u, v\}$ is in E, then e is an edge of G, and e is said to join u and v. In case, it is customary to write $e = uv$ and say that u and v are **adjacent**, while u and e are **incident**, as v and e are. The number of vertices in a graph G is the **order** of G, denoted by $|V(G)|$, and the number of edges is the **size** of G denoted by $|E(G)|$.

Definition 2.2. (Chartrand & Zhang, 2009) The **distance** $d_G(u, v)$ between vertices u and v in a connected graph G is the length of a shortest path connecting u and v in G. Any $u - v$ path $P(u, v)$ of length $d_G(u, v)$ is called a $u - v$ geodesic.

Definition 2.3. (Chartrand et al., 2015) The **degree of a vetex** v in a graph G, denoted by $deg(v)$, is the number of vertices in G that are adjacent to v. A vertex of degree 0 is referred as an **isolated vertex**. The largest degree among the vertices of G is called the **maximum degree** of G, denoted by $\Delta(G)$, while the smallest degree among the vertices of G is called the **minimum degree** of G, denoted by $\delta(G)$.

Definition 2.4. (Chartrand et al., 2015) For an integer $n \geq 1$, the **path graph** P_n is a graph of order n and size $n - 1$ whose vertices can be labeled as $v_1, v_2, ..., v_n$ and whose edges are $v_i v_{i+1}$ for $i = 1, 2, ..., n - 1$.

Definition 2.5. (Balakrishnan & Ranganathan, 2014) The **cycle graph** denoted by C_n of order $n \geq 3$ and size n is a graph with n distinct vertices $V(C_n) = \{v_1, v_2, ..., v_n\}$ and n distinct edges $E(C_n) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1\}$.

Definition 2.6. (Chartrand et al., 2015) A **complete graph** K_n of order $n \geq 2$, is a graph with n vertices where every distinct vertices are adjacent.

Definition 2.7. (Agasthi et al., 2016) A **barbell graph** $B_{n,n}$, $n \geq 3$, have an order $2n$ and a simple graph obtained by joining two complete graphs K_n by an edge.

Definition 2.8. (Chartrand et al., 2015) The **join graph** $G = G_1 + G_2$ of G_1 and G_2 has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

Definition 2.9. (Haynes et al., 1998) Let $G = (V(G), E(G))$ be a graph. A subset S of $V(G)$ **dominating set** in G if for every $v \in V(G) \setminus S$, there exist $u \in S$ such that $uv \in E(G)$, that is, $N_G[S] = V(G)$. The **domination number** of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

Definition 2.10. (Aniversario et al., 2019) A set $S \subseteq V(G)$ is a **semitotal dominating set** in G if S is a dominating set in G such that for every $x \in S$ there exist $y \in S \setminus \{x\}$ such that $d_G(x, y) \leq 2$. The **semitotal domination number**, denoted by $\gamma_{t2}(G)$, is the minimum cardinality of a semitotal dominating set.

Definition 2.11. (Domke et al., 1999) A **restrained dominating set** is a set $S \subseteq V(G)$ where every vertex in $V(G) \setminus S$ is adjacent to a vertex in S as well as another vertex in $V(G) \setminus S$. The restrained domination number of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set of G.

We now formally define the restrained semitotal dominating sets in a graph G.

Definition 2.12. A nonempty subset S of $V(G)$ is a **restrained semitotal dominating set** if S is a semitotal dominating set and a restrained dominating set. The minimum cardinality of a restrained semitotal dominating set, denoted by $\gamma_{t2}^r(G)$, is called the **restrained semitotal domination number** of G.

Example 2.1. *Consider the cycle graph C_5 in Figure 1. By the definition of a semitotal dominating set and restrained dominating set, it is visible that the set $S = \{v_1, v_2, v_5\}$ is a semitotal dominating set and at the same time a restrained dominating set. Thus, S is a restrained semitotal dominating set. Other possible restrained semitotal dominating set in $V(C_5)$ are $\{v_1, v_2, v_3\}$, $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_5\}$, and $\{v_4, v_5, v_1\}$. Thus, $\gamma_{t2}^r(C_5) = 3$.*

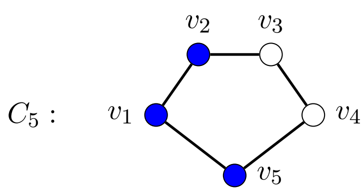

Figure 1: A graph C_5 and its restrained semitotal dominating set

3 Results

This section presents the results of characterization of the restrained semitotal dominating set in path graphs, cycle graph, complete graph, barbell graph, and join of graphs. Moreover, the restrained semitotal domination number is also presented.

Proposition 3.1. *For any connected graph G, $V(G)$ with $|V(G)| \geq 2$ is a restrained semitotal dominating set.*

Proof. Let $G = (V(G), E(G))$ be a connected graph with $|V(G)| \geq 2$. Let $S = V(G)$, it follows trivially that every vertex in $V(G)$ is adjacent to a vertex in S, that is, S is a dominating set. Since $V(G) \geq 2$ and G connected, for every vertex $v \in V(G)$, there is at least one other vertex $u \in V(G)$ such that $d_G(v, u) = 1$. In particular, for any $v \in V(G)$, there exist another vertex $u \in V(G)$ such that $u \neq v$ and $d(v, u) \leq 2$. Hence, every vertex in S has another vertex in S within distance at most 2. Moreover, since $S = V(G)$, we have $\langle V(G) \setminus S \rangle = \varnothing$. Since S do not have isolated vertices, S is a restrained dominating set. Therefore, $S = V(G)$ is a restrained semitotal dominating set. ☐

Remark 1.1. For any connected graph G, a restrained semitotal dominating set is also a semitotal dominating set such that

$$2 \leq \gamma_{t2}(G) \leq \gamma^r_{t2}(G) \leq |V(G)|.$$

Theorem 3.1. *Let $G = P_n$ be a path graph with $n \geq 2$. Then $S \subsetneq V(P_n)$ is restrained semitotal dominating set if and only if the following holds:*

(i) *$S = |V(P_n)|$ where $2 \leq n \leq 5$;*

(ii) *the leaf and support vertices of P_n are in S;*

(iii) *the components of $\langle V(P_n) \setminus S \rangle$ are P_2; and*

(iv) *the components of $\langle S \rangle$ have no isolated vertices.*

Proof. Suppose S is a restrained semitotal dominating set of P_n. Consider a path P_n for $2 \leq n \leq 5$. For the $|V(P_5) \setminus S| = 1$, say $\{v_1\}$, then v_1 is isolated in the induced subgraph, a contradiction. For $|V(P_5) \setminus S| \geq 1$, if $V(P_5) \setminus S$ induces a subgraph with no isolated vertices, the elements of $V(P_5) \setminus S$ must form a P_2, this will always lead to a vertex in $V(P_5) \setminus S$ not having a neighbor in S or will have an isolated vertex, v_i, such that $d_G(v_i, v_j) \geq 2, i \neq j$. Hence, we have $S = \{v_1, v_2, v_3, v_4, v_5\}$. Similar arguments hold for $n = 4$, $n = 3$, and $n = 2$. In these cases, any proper subset S will leave at least one vertex without a neighbor in S, violates restrained semitotal dominating set condition. Thus, (i) holds.

Suppose the vertices of P_n be $v_1, v_2, v_3, ..., v_n$, such that v_1 and v_n are the leaves and v_2 and v_{n-1} are the support vertices. Assume on the contrary, that a leaf vertex say $v_1 \notin S$. Since S is a dominating set, v_1 must be adjacent to a vertex S, which can only be v_2. Now, consider $v_1 \in V(P_n) \setminus S$. Since S is a restrained semitotal dominating set S must have no isolated vertices, that is v_1 must have a neighbor in $V(P_n) \setminus S$. The only neighbor of v_1 is v_2, but $v_2 \in S$, which is a contradiction. Thus, the leaf vertices v_1 and v_n must be in S. Let $v_2 \in V(P_n)$ be the support vertex of P_n. Then all leaf vertices are in S, since S is dominating set. Suppose $v_2 \notin S$, this means that $v_3 \in S$, since S is semitotal dominating set. But v_2 will be an isolated vertex, a contradiction to the fact that S is restrained semitotal dominating set, v_2 and $v_{n-1} \in S$ and (ii) holds.

Suppose $v \in V(P_n) \setminus S$. Since S is a restrained semitotal dominating set, for all $v \in V(P_n)$, $|N(v) \cap (V(P_n) \setminus S)| = 1$. Moreover, $|N(v) \cap (V(P_n) \setminus S)| \neq 2$, since this would imply that v will not be dominated by S, a contradiction. Thus, (iii) holds.

Suppose the induced subgraph, $\langle S \rangle$ has an isolated vertex, say w. By (iii), this implies that there exist $u \in S$ such that $d_G(u, w) \geq 2$ of S. This is a contradiction since S is a restrained semitotal dominating set impying that $\langle S \rangle$ is not an isolated vertex. Thus, (iv) holds.

Conversely, suppose $(i)-(iv)$ hold. Clearly, if $2 \leq n \leq 5$, $S = |V(P_n)|$ is a restrained semitotal dominating set. Now, suppose that $n \geq 6$. By (ii) and (iv) there exist $(x,y) \in S$ such that $d_G(x,y) \leq 2$. Thus, S is semitotal dominating set. Moreover, (iii) implies that $\langle V(P_n) \setminus S \rangle$ is not an isolated vertex. Hence, S is a restrained dominating set. Therefore, S is a restrained semitotal dominating set. □

Corollary 3.2. *For a path graph P_n of order $n \geq 2$,*

$$\gamma_{t2}^r(P_n) = \begin{cases} n, & \text{if } 2 \leq n \leq 5 \\ \frac{n+4}{2} & \text{if } n \equiv 0 (\text{mod } 4) \text{ and } n \geq 8 \\ \frac{n+5}{2} & \text{if } n \equiv 1 (\text{mod } 4) \text{ and } n \geq 9 \\ \frac{n+2}{2} & \text{if } n \equiv 2 (\text{mod } 4) \text{ and } n \geq 6 \\ \frac{n+3}{2} & \text{if } n \equiv 3 (\text{mod } 4) \text{ and } n \geq 7. \end{cases}$$

Proof. Let $P_n = \{v_1, v_2, ..., v_{n-1}\}$ be a path graph. Clearly, $\gamma_{t2}^r(P_n) = n$ if $2 \leq n \leq 5$. Let $n \geq 6$. Assume $S \subseteq V(P_n)$ is a restrained semitotal dominating set of P_n.

Case 1: $n \equiv 0$(mod 4) and $n \geq 8$

Let $S = \{v_1, v_2, v_5, v_6, ..., v_{n-3}, v_{n-2}, v_{n-1}, v_n\}$. By the Theorem 3.1, S is a restrained semitotal dominating set. Now, suppose that $S' = S \setminus \{v_i\}$ where $i = \{1, 2, 5, 6, ..., n-1, n\}$ is a minimum restrained semitotal dominating set. Note that $S \setminus \{v_i\}$ is not a restrained semitotal dominating set, since there exists $\langle V(P_n) \setminus (S \setminus \{v_i\}) \rangle = P_3$ or P_1, a contradiction to Theorem 3.1 (iii). Hence, S is the minimum restrained semitotal dominating set. Thus, $\gamma_{t2}^r(P_n) = |S| = \frac{n+4}{2}$.

Case 2: $n \equiv 0$(mod 4) and $n \geq 9$

Let $S = \{v_1, v_2, v_5, v_6, ..., v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n\}$. By the Theorem 3.1, S is a restrained semitotal dominating set. Similarly in Case 1, $S' = S \setminus \{v_i\}$ where $i = \{1, 2, 5, 6, ..., n-4, n-3, n-2, n-1, n\}$ is not a restrained semitotal dominating set. Thus, S is a minimum restrained semitotal dominating set. Now, $\gamma_{t2}^r(P_n) = |S| = \frac{n+5}{2}$.

Case 3: $n \equiv 0$(mod 4) and $n \geq 6$

Let $S = \{v_1, v_2, v_5, v_6, ..., v_{n-5}, v_{n-4}, v_{n-1}, v_n\}$. By Theorem 3.1, S is a restrained semitotal dominating set. Similary to Case 1, S is a minimum restrained semitotal dominating set. Thus, $\gamma_{t2}^r(P_n) = \frac{n+2}{2}$.

Case 4: $n \equiv 0$(mod 4) and $n \geq 7$

Let $S = \{v_1, v_2, v_5, v_6, ..., v_{n-6}, v_{n-5}, v_{n-2}, v_{n-1}, v_n\}$. By Theorem 3.1, S is restrained semitotal dominating set. Similarly to Case 1 , now $S' = S \setminus \{v_i\}$, $i = \{1, 2, 5, 6, ..., , n-2, n-1, n\}$ is not a restrained semitotal dominating set. Thus, S is a minimum restrained semitotal dominating set. Now, $\gamma_{t2}^r(P_n) = |S| = \frac{n+3}{2}$. □

Theorem 3.3. *Let $G = C_n$ be a cycle graph, $n \geq 4$. Then $S \subsetneq V(C_n)$ is restrained semitotal dominating set if and only if the following holds:*

(i) $1 \leq |N_G(v) \cap S| \leq 2$ *where $v \in S$; and*

(ii) *the components of $\langle V(C_n) \setminus S \rangle$ are P_2.*

Proof. Let $S \subseteq V(C_n)$. Suppose that S is a restrained semitotal dominating set. Since S is a semitotal dominating set, for any $u, v \in S$, $d_G(u,v) \leq 2$ and so, $1 \leq |N_G(v) \cap S| \leq 2$. Thus, (i) holds.

Suppose $w \in V(C_n) \setminus S$. Since S is restrained dominating set, for all

$|N_G(v) \cap (V(C_n) \setminus S)| \geq 1$. Moreover, since S is a dominating set and $\Delta(C_n) = 2$, $|N_G(w) \cap (V(C_n) \setminus S)| \neq 2$. Thus, $\langle V(C_n) \setminus S \rangle = P_2$. Thus, (ii) is satisfied.

For the converse, suppose that S satisfies (i) and (ii). By (i), $1 \leq |N_G(v) \cap S| \leq 2$, for all $v \in S$ implies that for every $u, v \in S$, $d_G(u, v) \leq 2$. Thus, S is semitotal dominating set. Moreover, by (ii), every vertex $u \in V(G) \setminus S$ is adjacent to one vertex in $V(C_n) \setminus S$ and is adjacent to a vertex in S. Thus, S is a restrained semitotal dominating set. □

Corollary 3.4. *For a cycle graph C_n of order $n \geq 3$,*

$$\gamma^r_{t2}(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \text{(mod 4)} \\ \frac{n+1}{2} & \text{if } n \equiv 1 \text{(mod 4)} \\ \frac{n+2}{2} & \text{if } n \equiv 2 \text{(mod 4)} \\ \frac{n+3}{2} & \text{if } n \equiv 3 \text{(mod 4).} \end{cases}$$

Proof. Let $C_n = \{v_1, v_2, ..., v_n\}$. Clearly, $\gamma^r_{t2}(C_3) = 3$. Let $n \geq 4$, consider the following cases:
Case 1: $n = 4r$ or $n \equiv 0$(mod 4)

Group the first $4r$ vertices of C_n into r disjoint subsets:

$$\begin{aligned} &S_1 = \{v_1, v_2, v_3, v_4\} \\ &S_2 = \{v_5, v_6, v_7, v_8\} \\ &\vdots \\ &S_{r-1} = \{v_{4r-7}, v_{4r-6}, v_{4r-5}, v_{4r-4}\} \\ &S_r = \{v_{4r-3}, v_{4r-2}, v_{4r-1}, v_{4r}\} \end{aligned}$$

For every individual subgraph, $\langle v_i, v_{i+1}, v_{i+2}, v_{i+3} \rangle$, where $i = 1, 5, ..., 4r-7, 4r-3$, the vertices v_{i+1} and v_{i+2} form a restrained semitotal dominating set. Thus, $T = \{v_2, v_3, v_6, v_7, ..., v_{4r-2}, v_{4r-1}\}$ is a restrained semitotal dominating set of C_n. Since $|T_1| = 2r$, $\gamma^r_{t2}(C_n) \leq 2r$. Note that every 4 adjacent vertices of C_n can be dominated by 2 vertices. Thus, every restrained semitotal domninating set of C_n contains at least $\frac{n}{2}$ vertices. Hence, $\gamma^r_{t2}(C_n) \geq \frac{n}{2} = 2r$. Thus, $\gamma^r_{t2}(C_n) = \frac{n}{2}$.
Case 2: $n = 4r + 1$ or $n \equiv 1$(mod 4)

Consider the groupings in Case 1. The set

$$T_2 = \{v_2, v_3, v_6, v_7, ..., v_{4r-2}, v_{4r-1}\} \cup \{v_{4r}\}$$

is a restrained semitotal dominating set of C_n. Thus, $\gamma^r_{t2} \leq |T_1| + 1 = 2r + 1 = \frac{n+1}{2}$. Note that each of the first $r - 1$ subgraph $\langle v_i, v_{i+1}, v_{i+2}, v_{i+3} \rangle$ can be dominated by $\langle v_{i+1}, v_{i+2} \rangle$ and the induced subgraph $\langle v_{4r-3}, v_{4r-2}, v_{4r-1}, v_{4r}, v_{4r+1} \rangle$ can be dominated by $\langle v_{4r-2}, v_{4r-1}, v_{4r} \rangle$. Thus, every restrained semitotal dominating set of C_n contains at least $2(r-1) + 3 = 2r + 1 = \frac{n+1}{2}$. Thus, $\gamma^r_{t2}(C_n) \geq \frac{n+1}{2}$. Therefore, $\gamma^r_{t2}(C_n) = \frac{n+1}{2}$.
Case 3: $n = 4r + 2$ or $n \equiv 2$(mod 4)

In Case 1, the set $T_1 = \{v_2, v_3, v_6, v_7, ..., v_{4r-2}, v_{4r-1}\}$ is a restrained semitotal dominating set. By Case 2, $T_2 = T_1 \cup \{v_{4r}\}$ is a restrained semitotal dominating set. Thus, $T_3 = T_2 \cup \{v_{4r+1}\}$ is a restrained semitotal dominating set. Therefore, $T_3 = |T_2| \cup \{v_{4r+1}\}$ implies

$$|T_3| = |T_2| + 1 = |2r + 1| + 1 = 2r + 2 = \frac{n+2}{2}.$$

Thus, $\gamma^r_{t2}(C_n) = \frac{n+2}{2}$.

Case 4: $n = 4r + 3$ or $n \equiv 3(\text{mod } 4)$

By Case 1 to 3, $T_3 = \{v_2, v_3, ..., v_{4r-2}, v_{4r-1}, v_{4r}, v_{4r+1}\}$ is a restrained semitotal dominating set. Similarly by Case 2, $T_4 = T_3 \cup \{v_{4r+2}\}$ is a restrained semitotal dominating set. Thus,

$$|T_4| = |T_3| + 1 = |2r + 2| + 1 = 2\Big(\frac{n-3}{4}\Big) + 3 = \frac{n-3+6}{2} = \frac{n+3}{2}.$$

Thus, $\gamma^r_{t2}(C_n) = \frac{n+3}{2}$. □

Theorem 3.5. *Let $G = K_n$ be a complete graph with $n \geq 3$. Then $S \subsetneq V(K_n)$ is a restrained semitotal dominating set if and only if S satisfies one of the following:*

(i) $S = V(K_n)$ *if $n = 3$; and*

(ii) S *is a nonsingleton dominating set in $n \geq 4$.*

Proof. Suppose that $S \subsetneq V(K_n)$ is a restrained semitotal dominating set. Let $V(K_3) = \{v_1, v_2, v_3\}$. Assume further that $S = \{v_1, v_2\}$. Then $V(K_3) \setminus S = \{v_3\}$ an isolated vertex, a contradiction since S is a restrained dominating set. Thus, $S = V(K_n)$ if $n = 3$, and (i) holds.

Let $n \geq 4$. Now, suppose that $S \subseteq V(K_n)$ is a singleton dominating set, that is $v \in S \cap V(K_n)$. Since S is a semitotal dominating set, there exist $u \in S$ such that $d_G(u, v) \leq 2$. A contradiction, thus S is a nonsingleton dominating set.

Conversely, suppose (i) and (ii) holds. If $n = 3$, clearly, $S = V(K_3)$ is a restrained semitotal dominating set. Now, suppose $n \geq 4$. By (ii), there exist $a, b \in S \cap V(K_n)$ such that $d_G(a, b) = 1$, hence a semitotal dominating set. Further, since $n \geq 4$ and by the definition of K_n, for any $x, y \in V(G) \setminus S$, vertices x and y are not isolated vertices. Thus, S is a restrained semitotal dominating set. □

Corollary 3.6. *For a complete graph K_n of $n \geq 3$,*

$$\gamma^r_{t2}(K_n) = \begin{cases} 3 & \text{if } n = 3 \\ \\ 2 & \text{if } n \geq 4. \end{cases}$$

Proof. Proof follows from Theorem 3.5. □

Theorem 3.7. *Let $G = B_{n,n}$ be a barbell graph where $n \geq 3$. Then $S \subsetneq V(B_{n,n})$ is a restrained semitotal dominating set if and only if S satisfies one of the following:*

(i) $S = S_1 \cup S_2$ *where $S_1 \subseteq V(K_{n'})$ and $S_2 \subseteq V(K_n)$ are nonsingleton dominating set in $K_{n'}$ and K_n, respectively;*

(ii) $\{v_1\} \cup S_1$, $S_1 \subseteq V(K_{n'})$ *is a dominating set and v_1 is a cut-vertex in K_n;*

(iii) $\{v_1^{'}\} \cup S_2$, $S_2 \subseteq V(K_n)$ *is a dominating set and $v_1^{'}$ is a cut-vertex in $K_{n'}$; and*

(iv) $S = \{v_1, v_1^{'}\}$ *where $v_1, v_1^{'}$ are the cut-vertices in K_n and $K_{n'}$, respectively.*

Proof. Suppose that $S \subsetneq V(B_{n,n})$ is a restrained semitotal dominating set. Since S is a semitotal dominating set, $|S| \geq 2$. Let $v_1, v_1^{'}$ be a cut-vertices of K_n and $K_n^{'}$, respectively. Let $S = S_1 \cup S_2$ where $S_1 \subseteq V(K_{n'})$ and $S_2 \subseteq V(K_n)$. Since S is a restrained semitotal dominating set, S for any $x, y \in V(K_{n'})$ such that $d_G(x, y) = 1$, implies that $|S_1| \geq 2$. Thus, S_1 is a nonsingleton dominating set. Similarly, S_2 is a nonsingleton dominating set. Thus, (i) holds. Now, suppose that $v_1 \in S$. Since S is a semitotal dominating set, there exists $v^{'} \in S_1 \subseteq V(K_{n'})$ such that $d_G(v^{'}, v_1) = 2$. Moreover, since $K_{n'}$ is complete graph, S_1 is dominating set. Thus, (ii) holds. Similary, if $v_1 \in S$, then $S_2 \subseteq V(K_n)$ is a dominating set. Thus, (iii) holds. Now, let $S = \{v_1^{'}, v_1\}$ since $v_1^{'}$ and v_1 are adjacent, $d_G(v_1^{'}, v_1) = 1$. Moreover, $n^{'} > 2$ and $n > 2$ impying that $V(G) \setminus S$ are not isolated vertices. Hence, $S = \{v_1^{'}, v_1\}$ is a restrained semitotal dominating set. Thus, (iv) holds.

Conversely, $(i) - (iv)$ holds. Since $v_1^{'}$ and v_1 are the cut-vertices by (iv), S is a dominating set. Since S_1 and S_2 are dominating set, by $(i), (ii), (iii)$, there exists $x, y \in S$ such that $d_G(x, y) \leq 2$ and so S is a semitotal dominating set. Lastly, since $K_{n'}$ and K_n are complete graph, $K_{n'}$ and K_n with $n' > 2$ and $n > 2$, $V(G) \setminus S$ do not have isolated vertices. Thus, S is a restrained dominating set. Therefore, S is a restrained semitotal dominating set. ☐

Corollary 3.8. *For a barbell graph $B_{n,n}$ of $n \geq 3$, $\gamma_{t2}^r(B_{n,n}) = 2$.*

Proof. Proof follows from Theorem 3.7. ☐

The next results, Theorem 3.9 and Corollary 3.10, were used to prove the characterization of the restrained semitotal dominating set in the join of graphs.

Theorem 3.9. *(Aniversario et al., 2019) Let G and H be nontrivial graphs, and $S \subseteq V(G+H)$. Then S is a semitotal dominating set in $G + H$ if and only if one of the following holds:*

- (i) *$S \subseteq V(G)$ is a nonsingleton dominating set in G;*
- (ii) *$S \subseteq V(H)$ is a nonsingleton dominating set in H;*
- (iii) *$S \cap V(G) \neq \varnothing$ and $S \cap V(H) \neq \varnothing$.*

Corollary 3.10. *(Aniversario et al., 2019) For all graphs G and H, $\gamma_{t2}(G + H) = 2$.*

Theorem 3.11. *Let G and H be nontrivial connected graph. Then $S \subsetneq V(G + H)$ is a restrained semitotal dominating set if and only if:*

- (i) *S is a semitotal dominating set in $V(G + H)$; and*
- (ii) *the components of $\langle V(G + H) \setminus S\rangle$ are not an isolated vertex.*

Proof. Assume S is a restrained semitotal dominating set in $V(G + H)$. Since S is a restrained semitotal dominating set in $V(G+H)$, S is a semitotal dominating set in $V(G+H)$. By Theorem 3.9, $S \subseteq V(G)$, $S \subseteq V(H)$, $S \cap V(G) \neq \varnothing$ and $S \cap V(G) \neq \varnothing$ is a semitotal dominating set. Thus, (i) holds. Moreover, S is a restrained dominating set implies that the components of $\langle V(G + H) \setminus S\rangle$ do not have an isolated vertex. Thus, (ii) holds. Therefore, conditions (i) and (ii) are satisfied by the definition of restrained semitotal dominating set.

Conversely, suppose (i) and (ii) holds. By (i), S is a semitotal dominating set. By (ii), S is a restrained dominating set. Therefore, S is a restrained semitotal dominating set.

☐

Corollary 3.12. *Let G and H be nontrivial connected graph. Then $S \subseteq V(G + H)$ is a restrained semitotal dominating set if and only if*

$$\gamma_{t2}^r(G + H) = 2.$$

Proof. Immediately follows from Corollary 3.10 and Theorem 3.11. ☐

4 CONCLUSION

This study introduced and investigated the concept of restrained semitotal dominating set. The restrained semitotal dominating set in the path, cycle, complete, barbell, and join of graphs have been characterized and the restrained semitotal domination number of these graph have been determined. Further, the concept can be studied for other graphs not considered in this study.

Acknowledgment

The authors want to express their profound gratitude to their family and to everyone who contributed to the completion of this study.

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