

Original Research Article

CENTER OF THE n^{th} DIHEDRAL GROUP AND CENTRALIZERS OF THE ELEMENTS OF THE n^{th} DIHEDRAL GROUP, D_n , WHERE $3 \leq n \leq 20$, $n \in \mathbb{Z}^+$

ABSTRACT

Aims: To derive general formulas for determining the center and centralizers of the elements of the n^{th} dihedral group, D_n , where $3 \leq n \leq 20$, and to evaluate their consistency and potential for generalization.

Study Design: This is a computational and theoretical study in group theory employing definition-based and pattern-recognition methodologies.

Place and Duration of Study: Department of Mathematics, College of Science, University of Eastern Philippines, Catarman, Northern Samar, Philippines, conducted between August 2023 and March 2024.

Methodology: The study applied foundational definitions from abstract algebra to compute the center and centralizers of D_n for values $3 \leq n \leq 20$. A systematic pattern recognition method was used to derive general formulas distinguishing between even and odd values of n . Verification was conducted through direct computation and formula-based prediction to validate the observed patterns.

Results: For odd values of n , the center $Z(D_n)$ consisted solely of the identity element, $\{e\}$, while for even n , it included both e and $a^{n/2}$. Centralizers of identity and specific elements like b , a^i , and $a^i b$ were derived and categorized, showing predictable, consistent patterns across tested values. For example, $C(e) = D_n$ and $C(a^i) = \{e, a, a^2, \dots, a^{n-1}\}$ for rotational elements, while reflections exhibited smaller centralizers. Formulas held for all tested cases, with random tests beyond $n = 20$ also showing alignment, supporting the conjecture of general applicability for all $n \geq 3$.

Conclusion: The study offers efficient, formula-based computation of the center and centralizers in D_n , significantly reducing time and effort compared to traditional methods. The derived expressions were validated for $3 \leq n \leq 20$ and are conjectured to be valid for all $n \geq 3$, providing a useful tool for further research in group theory and its applications.

Keywords: abstract algebra, dihedral group, center, centralizer, group theory

1. INTRODUCTION

Patterns are inherent in both tangible and abstract aspects of daily life, and their analysis is a fundamental pursuit in mathematics, particularly in the field of abstract algebra. One notable algebraic structure is the dihedral group, denoted by D_n , which represents the symmetries of a regular n -sided polygon, combining both rotational and reflectional transformations. Understanding the structural behavior of D_n has significant implications not only in theoretical mathematics but also in geometry, crystallography, and theoretical physics.

A fundamental challenge in group theory is identifying the **center** and **centralizers** of elements within a group. The center of a group, $Z(G)$, is defined as the set of elements that

commute with every element of the group, while the centralizer $C(a)$ of an element a is the set of elements that commute with a . Although these definitions are standard, computing these subsets manually, especially for complex groups like D_n , can be tedious and computationally intensive.

Previous works such as those by Fraleigh (2003), Rotman (1995), and Dummit and Foote (2004) have established the theoretical foundations of these group properties. However, limited literature offers direct computational strategies or generalized formulas specifically for the center and centralizers of the dihedral group across varying values of n . This gap poses difficulties for students and educators alike in exploring and teaching the internal structures of dihedral groups.

To address this, the present study proposes a pattern-based computational technique to derive generalized formulas for the center and centralizers of D_n for $3 \leq n \leq 20$. By analyzing and verifying observable patterns through definition-based computation, the study aims to streamline the process, saving time while maintaining mathematical rigor. The scope of this work includes validation of the derived formulas for all specified values of n , with preliminary testing suggesting their applicability beyond the range. This method is particularly beneficial for mathematics educators and students, providing a more intuitive and efficient approach to understanding core group theory concepts.

2. MATERIAL AND METHODS

2.1 Study Design and Approach

This study employed a computational and pattern-based approach to determine the center and centralizers of the n th dihedral group, D_n , for values $3 \leq n \leq 20$, $n \in \mathbb{Z}^+$. Using the foundational definitions of group theory as a framework, the researchers systematically analyzed elements within each D_n to identify consistent patterns and derive general formulas.

2.1.1 Definition-Based Analysis

The analysis began by referencing standard definitions from abstract algebra. The center of a group, denoted $Z(G)$, consists of all elements that commute with every element in the group:

$$Z(G) = \{a \in G \mid ax = xa, \forall x \in G\}$$

The centralizer of an element $a \in G$, denoted $C(a)$, includes all elements $g \in G$ that commute with a :

$$C(a) = \{g \in G \mid ga = ag\}$$

These definitions, as described in Fraleigh (2003) and Clifton (2010), were the primary basis for computation.

2.1.1.1 Manual Computation for Element Identification

Each element of D_n was expressed in the form:

$$D_n = \{e, a, a^2, \dots, a^{n-1}, b, ab, a^2b, \dots, a^{n-1}b\}$$

Using substitution and direct calculation, the researchers identified elements that met the conditions for centers and centralizers. For each n , the computations were manually performed, recorded, and cross-verified.

2.2 Pattern Recognition and Formula Derivation

2.2.1 Observational Analysis

The researchers analyzed the manually computed results across all tested values of n , distinguishing between odd and even values. The following patterns were observed:

For **odd** n :

- $Z(D_n) = \{e\}$
- $C(a^i) = \{e, a, a^2, \dots, a^{n-1}\}$
- $C(b) = C(a^i b) = \{e, a^i b\}$

For **even** n :

- $Z(D_n) = \{e, a^{n/2}\}$
- $C(e) = C(a^{n/2}) = D_n$
- $C(b) = C(a^{n/2}b) = \{e, a^{n/2}, b, a^{n/2}b\}$
- For other $a^i b$:
 - If $i < n/2$, $C(a^i b) = \{e, a^{n/2}, a^i b, a^{(n/2+i)}b\}$
 - If $i > n/2$, $C(a^i b) = \{e, a^{n/2}, a^{(n/2+i) \bmod n} b, a^i b\}$

2.2.2 Derivation of General Formulas

Using these observations, the researchers developed general expressions for:

- $Z(D_n)$, based on parity of n
- $C(e), C(a^i), C(b), C(a^i b)$

These formulas were tested across all n from 3 to 20.

2.3 Validation of Results

2.3.1 Cross-Verification

Each derived formula was validated through direct substitution and computation using the definitions. Discrepancies, if any, were reviewed and corrected to maintain accuracy.

2.3.2 Testing Beyond the Range

To test generalizability, the formulas were applied to select values of $n > 20$. These tests indicated similar consistency, supporting the conjecture that the derived formulas may hold for all $n \geq 3$.

2.4 Mathematical Notation and Conventions

- D_n – The n th dihedral group
- $Z(D_n)$ – The center of D_n
- $C(a)$, $C(b)$, $C(e)$ – Centralizers of the respective elements in D_n
- n – A positive integer

All computations were done manually without the use of symbolic algebra software, ensuring replicability through basic group theory knowledge.

3. RESULTS AND DISCUSSION

3.1 General Results and Observations

The study successfully derived formulas to determine the center and centralizers of the elements of the dihedral group D_n for values $3 \leq n \leq 20$, confirming predictable structural behaviors based on whether n is even or odd. The results were obtained through the application of fundamental definitions and the identification of recurring patterns across tested cases. The findings enabled the formulation of efficient computation strategies for various algebraic structures within the group.

3.1.1 Center of the Dihedral Group

For **odd** values of n , such as $n = 5$ and $n = 7$, the center of the dihedral group $Z(D_n)$ was found to include only the identity element:

$$Z(D_n) = \{e\}$$

This means that only the identity element commutes with all other elements in the group.

In contrast, for **even** values of n , such as $n = 4, 6, 8$, the center was found to include both the identity and $a^{n/2}$:

$$Z(D_n) = \{e, a^{n/2}\}$$

This structural difference in the center supports known group theory literature (Fraleigh, 2003; Dummit & Foote, 2004), which states that even-order dihedral groups contain more commuting elements due to their symmetrical structure.

3.1.2 Centralizers of Elements

The centralizer of the identity element $C(e)$ was confirmed to be the full set of group elements D_n , since the identity commutes with all elements by definition. The centralizers of the rotational elements a, a^2, \dots, a^{n-1} were observed to contain only the cyclic subgroup:

$$C(a^i) = \{e, a, a^2, \dots, a^{n-1}\}$$

Reflections, such as b and $a^i b$, exhibited smaller centralizers. For **odd** n , it was observed that:

$$C(b) = C(a^i b) = \{e, a^i b\}$$

For **even** n , however, reflection elements displayed more structure:

$$C(b) = C(a^{n/2}b) = \{e, a^{n/2}, b, a^{n/2}b\}$$

And for $a^i b$, where $i < n/2$,

$$C(a^i b) = \{e, a^{n/2}, a^i b, a^{n/2+i} b\}$$

For $i > n/2$, the centralizer was computed using modulo arithmetic:

$$C(a^i b) = \{e, a^{n/2}, a^i b, a^{(n/2+i) \bmod n} b\}$$

These results were consistent across all tested values, confirming that the derived formulas accurately represent the structure of D_n . The observed patterns validate the method's potential for generalization to larger values of n .

3.1.3 Significance and Interpretation

The findings are aligned with theoretical results found in the literature. For example, Gallian (2017) and Rotman (1995) note that in non-abelian groups such as D_n , centers and centralizers provide valuable insights into internal group symmetry. The distinction between even and odd cases reflects deeper structural implications, particularly in how reflectional and rotational elements interact.

Moreover, the derived formulas reduce the computational complexity often associated with calculating commutative properties in dihedral groups. This is especially beneficial for educators and students, making complex algebraic topics more accessible in instructional settings.

3.1.4 Generalization and Validation Beyond the Range

Although the study focused on $3 \leq n \leq 20$, the derived expressions were applied to random cases where $n > 20$, and consistent results were observed. This supports a conjecture that the patterns and formulas may hold true for all $n \geq 3$. However, further formal proof is recommended to validate this hypothesis rigorously.

4. CONCLUSION

This study derived and validated general formulas for determining the center and centralizers of elements in the n th dihedral group, D_n , for $3 \leq n \leq 20$. The results showed that for odd values of n , the center consists solely of the identity element, while for even n , it includes both the identity and $a^{n/2}$. The structure of centralizers varied predictably between rotation and reflection elements, with consistent patterns observed across all tested cases. These findings confirm that the center and centralizers of D_n can be efficiently computed through the derived expressions, significantly reducing computational effort. Preliminary tests beyond $n = 20$ indicate that the formulas may also apply to higher values of n , supporting the conjecture that they hold for all $n \geq 3$.

COMPETING INTERESTS DISCLAIMER:

Authors have declared that they have no known competing financial interests OR non-financial interests OR personal relationships that could have appeared to influence the work reported in this paper.

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DEFINITIONS, ACRONYMS, ABBREVIATIONS

The terms used in this study were defined operationally so as to facilitate easy understanding of their meanings:

Dihedral Group - The n^{th} **dihedral group** is a group of symmetries of a regular n-gon (Lepetic, 2016). $D_n = \{a^i b^j \mid a^n = e, b^2 = e, a'b = ba, 1 \leq i \leq n-1, 0 \leq j \leq 1\}$.

Group – A **Group** $(G, *)$ is a set G , closed under a binary operation $*$, such that the following axioms are satisfied (Fraleigh, 2003):

1. For all $a, b, c, \in G$ we have,
$$a * (b * c) = (a * b) * c \quad \text{Associativity of } *$$
2. There is an element e in G such that for all $x \in G$,
$$x * e = e * x = x \quad \text{Identity element } e \text{ for } *$$
3. Corresponding to each $a \in G$, there is an element a' in G such that,
$$a * a' = a' * a = e \quad \text{Inverse } a' \text{ of } a$$

Center – The **center** of a group G , $Z(G)$, is the set of all those elements of G that commute with every element of G , that is, $Z(G) = \{a \in G \mid ax = xa \forall x \in G\}$ (Lepetic, 2016).

Centralizer - The **centralizer of a** in G , $C(a)$, is the set of all elements in G that commute with a . In symbols, $C(a) = \{g \in G \mid ga = ag\}$ (Clifton, 2010).