A Study On Hyperbolic Generalized Edouard Numbers

Abstract. In this study, we present the generalized hyperbolic Edouard numbers a novel class of numerical sequences that broaden traditional recurrence relations by embedding them within a new mathematical framework.We provide a detailed exploration of several significant special cases, including the hyperbolic Edouard numbers and the hyperbolic Edouard-Lucas numbers, each demonstrating fascinating combinatorial patterns and noteworthy algebraic properties.

Explicit formulations for these sequences are established, including Binet-type expressions, generating functions, and summation identities, all of which provide analytical insight into their structural features and behavioral dynamics. Additionally, we investigate matrix representations linked to these sequences, offering a sophisticated algebraic framework that supports deeper theoretical advancements and opens avenues for practical applications.

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1. Introduction

In mathematical and geometric contexts, a hypercomplex system denotes a generalized framework that extends the foundational principles of complex numbers to encompass higher-dimensional algebraic structures. These systems exhibit intricate algebraic architectures and are actively investigated for their broad applications across physics and engineering disciplines. In the following section, we offer a succinct overview of the principal application domains of hypercomplex number systems within the fields of physics and engineering.

Unlike complex numbers, hypercomplex systems offer a more advanced algebraic framework for modeling transformations and capturing symmetries in higher dimensional spaces.As observed by Kantor in [20], hypercomplex systems may be interpreted as natural extensions of the real number line, furnishing algebraic structures specifically designed to facilitate analysis in multidimensional contexts. The primary categories of hypercomplex number systems include complex numbers, hyperbolic numbers, and dual numbers, each distinguished by its unique algebraic properties and geometric interpretations. Complex numbers, characterized by the combination of a real and an imaginary component, form the foundational basis from which more intricate hypercomplex systems are developed. Hyperbolic numbers build upon the complex number framework and are employed in diverse mathematical models, particularly those involving Lorentz transformations and spacetime geometries. Dual numbers, distinguished by the presence of a dual unit whose square is zero, are instrumental in various algebraic constructions, including automatic differentiation and kinematic analysis.

The following sections offer more detailed insights into the mathematical properties and application areas of these hypercomplex systems.

- Complex numbers are constructed by extending the real number system through the introduction of an imaginary unit, denoted as "i", which satisfies the identity $i^2 = -1$. A complex number is typically expressed in the form $z = a + bi$, where a and b are real numbers, and i represents the imaginary unit.
- Hyperbolic numbers also referred to as double numbers or split complex numbers extend the real number system by introducing a new unit element j, which satisfies the identity $j^2 = 1$ [30]. These numbers are distinct from real and complex numbers due to their unique algebraic properties. A hyperbolic number is defined as:

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\}.$$

  where a and b are real numbers and j is the hyperbolic unit. This structure enables the modeling of systems with split-signature metrics and has notable applications in areas such as special relativity and signal processing.
- Dual numbers [14] expand the real number system through the incorporation of a new element ε, which satisfies the identity $\varepsilon^2 = 0$. This infinitesimal unit distinguishes dual numbers from other hypercomplex systems and makes them especially valuable in modeling instantaneous rates of change. A dual number is defined as:

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

  where a and b are real numbers, and ε is the nilpotent unit. Dual numbers are commonly used in applications such as automatic differentiation, kinematics, and perturbation analysis, due to their ability to elegantly encode infinitesimal variations.

- Among the non-commutative examples of hypercomplex number systems are quaternions [17]. Quaternions generalize complex numbers by incorporating three distinct imaginary units, typically denoted as i,j, and k. A quaternion has the form as $a_0+ia_1+ja_2+ka_3$, where $a_0,a_1,a_2,a_3\in\mathbb{R}$.These multiplication rules result in a non-commutative structure, meaning the order of multiplication affects the result.The set of quaternion numbers is formally defined as:

$$\mathbb{H}_{\mathbb{Q}}=\{q=a_0+ia_1+ja_2+ka_3:a_0,a_1,a_2,a_3\in\mathbb{R},i^2=j^2=k^2=ijk=-1\},$$

- Additional hypercomplex systems include octonions and sedenions, which are discussed in [19] and [25]. The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions), and $\mathbb{S}$ (sedenions) are all constructed as real algebras derived from the real numbers $\mathbb{R}$ using a recursive procedure known as the Cayley–Dickson Process. This technique successively doubles the dimension of each algebra and continues beyond sedenions to produce what are collectively referred to as the 2^n-ions.The following table highlights selected publications from the literature that investigate the properties and applications of these extended number systems.

  For more information on hypercomplex algebra, see [22,16,24]

Table 1. Papers that have been published in the literature raleted to 2^n-ions.

Authors and Title of the paper↓		Papers↓
Biss, D.K., Dugger, D., Isaksen, D.C., Large annihilators in Cayley-Dickson algebras		[4]
Hamilton, W.R., Elements of Quaternions		[17]
Imaeda, K., Sedenions: algebra and analysis		[18]
Moreno, G., The zero divisors of the Cayley-Dickson algebras over the real numbers		[23]
Göcen, M., Soykan, Y., Horadam 2^k-Ions		[15]
Soykan,Y., Tribonacci and Tribonacci-Lucas Sedenions		[25]
On higher order Fibonacci hyper complex numbers		[21]

A dual hyperbolic number is a type of hypercomplex number, specifically a member of the hyperbolic number system. A dual hyperbolic number is defined as follows

$$q=(a_0+ja_1)+\varepsilon(a_2+ja_3)=a_0+ja_1+\varepsilon a_2+\varepsilon ja_3$$

where $a_0,a_1,a_2,a_3\in\mathbb{R}$.

$\mathbb{H}_{\mathbb{D}}$, the set of all dual hyperbolic numbers, are generally denoted by

$$\mathbb{H}_{\mathbb{D}}=\{a_0+ja_1+\varepsilon a_2+\varepsilon ja_3:a_0,a_1,a_2,a_3\in\mathbb{R},j^2=1,j\neq\pm1,\varepsilon^2=0,\varepsilon\neq0\}.$$

The $\{1,j,\varepsilon,\varepsilon j\}$ is linearly independent, and the algebra $\mathbb{H}_{\mathbb{D}}$ is generated by their span, i.e. $\mathbb{H}_{\mathbb{D}}=sp\{1,j,\varepsilon,\varepsilon j\}$

Therefore, $\{1,j,\varepsilon,\varepsilon j\}$ forms a basis for the dual hyperbolic algebra $\mathbb{H}_{\mathbb{D}}$. For more detail, see [1].

The next properties are holds for the base elements $\{1, j, \varepsilon, \varepsilon j\}$ of dual hyperbolic numbers (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j,\ \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0,\ j^2 = j.j = 1 \\ \varepsilon.j &= j.\varepsilon,\ \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0,\ j(\varepsilon j) = (\varepsilon j)j = \varepsilon \end{aligned}$$

where ε denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j denotes the hyperbolic unit ($j^2 = 1$), and εj denotes the dual hyperbolic unit ($(j\varepsilon)^2 = 0$).

We claim that p and q be two dual hyperbolic numbers that $q = a_0 + ja_1 + \varepsilon a_2 + j\varepsilon a_3$ and $p = b_0 + jb_1 + \varepsilon b_2 + j\varepsilon b_3$ and then we can write the product of p and q as

$$qp = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0) + \varepsilon(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + j\varepsilon(a_0b_3 + a_1b_2 + a_2b_1 + b_0a_3)$$

and we can write the sum dual hyperbolic numbers p and q as componentwise.

The dual hyperbolic numbers form a commutative ring, real vector space and an algebra. $\mathbb{H}_{\mathbb{D}}$ is not field since every dual hyperbolic numbers doesn't have an inverse. For more detail about dual hyperbolic numbers, see [1].

It's known that many author studied the generalized (r, s, t) sequence. One of these sequences is generalized Edouard numbers. Soykan, [26] defined generalized Edouard numbers. Before we present our original study , we recall some proporities related to generalized Edouard numbers such as reccurance relations, Binet's formula, generating function .

A generalized Edouard sequence , with the initial values W_0, W_1, W_2 not all being zero, $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 7W_{n-1} - 7W_{n-2} + W_{n-3};\ W_0, W_1, W_2 \quad (n \geq 3) \tag{1.1}$$

Moreover, we define generalized Edouard sequence given to negative subscripts as follows,

$$W_{-n} = 7W_{-(n-1)} - 7W_{-(n-2)} + W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Thus, recurrence (1.1) is true for all integer n.

In the Table 2 we give the first some generalized Edouardnumbers with positive subscript and negative subscript

Table 2. A few generalized Edouard numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$7W_0 - 7W_1 + W_2$
2	W_2	$42W_0 - 48W_1 + 7W_2$
3	$W_0 - 7W_1 + 7W_2$	$246W_0 - 287W_1 + 42W_2$
4	$7W_0 - 48W_1 + 42W_2$	$1435W_0 - 1680W_1 + 246W_2$
5	$42W_0 - 287W_1 + 246W_2$	$8365W_0 - 9799W_1 + 1435W_2$
6	$246W_0 - 1680W_1 + 1435W_2$	$48756W_0 - 57120W_1 + 8365W_2$

If we obtain,respectively, $W_0 = 0, W_1 = 1, W_2 = 7$ then$\{W_n\} = \{E_n\}$ is called the Edouard sequence, $W_0 = 3, W_1 = 7, W_2 = 35$ then $\{W_n\} = \{K_n\}$ is called the Edouard-Lucas sequence. Alternatively, Edouard sequence $\{E_n\}_{n\geq 0}$, Edouard-Lucas sequence $\{K_n\}_{n\geq 0}$ are given by the third-order recurrence relations as

$$E_n = 7E_{n-1} - 7E_{n-2} + E_{n-3}, \qquad E_0 = 0, E_1 = 1, E_2 = 7, \tag{1.2}$$

$$K_n = 7K_{n-1} - 7K_{n-2} + K_{n-3}, \qquad K_0 = 3, K_1 = 7, K_2 = 35, \tag{1.3}$$

The sequences given above can be extended to negative subscripts by defining, respectively,

$$\begin{aligned} E_{-n} &= 7E_{-(n-1)} - 7E_{-(n-2)} + E_{-(n-3)}, \\ K_{-n} &= 7K_{-(n-1)} - 7K_{-(n-2)} + K_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, \ldots$. As a consequence, recurrences (1.2)-(1.3) hold for all integer n.

We can list some important properties of generalized Edouard numbers that are needed.

Binet formula of generalized Edouard sequence can be calculated using its characteristic equation written as

$$x^3 - 7x^2 + 7x - 1 = \left(x^2 - 6x + 1\right)(x - 1) = 0.$$

The roots of the characteristic equation are

$$\begin{aligned} \alpha &= 3 + 2\sqrt{2}, \\ \beta &= 3 - 2\sqrt{2}, \\ \gamma &= 1, \end{aligned}$$

By using these roots and the recurrence relation, Binet formula are written below

$$\begin{aligned} W_n &= \frac{z_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{z_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{z_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \\ &= \frac{z_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{z_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} - \frac{z_3}{4} \end{aligned}$$

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where

$$\begin{aligned} z_1 &= W_2-(\beta+1)W_1+\beta W_0, \\ z_2 &= W_2-(\alpha+1)W_1+\alpha W_0, \\ z_3 &= W_2-6W_1+W_0. \end{aligned}$$

and

$$\begin{aligned} A_1 &= \frac{W_2-(\beta+1)W_1+\beta W_0}{(\alpha-\beta)(\alpha-\gamma)}, \\ A_2 &= \frac{W_2-(\alpha+1)W_1+\alpha W_0}{(\beta-\alpha)(\beta-\gamma)}, \\ A_3 &= \frac{W_2-6W_1+W_0}{(\gamma-\alpha)(\gamma-\beta)}. \end{aligned} \tag{1.4}$$

Then we present Binet formula of Edouard sequences and Edouard-Lucas sequences, respectively, given below

$$\begin{aligned} E_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-1)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-1)}-\frac{1}{4}, \\ K_n &= \alpha^n+\beta^n+1. \end{aligned}$$

After then we can write the generating function of generalized Edouard numbers,

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0+(W_1-7W_0)x+(W_2-7W_1+7W_0)x^2}{1-7x+7x^2-x^3}. \tag{1.5}$$

Next, we give the exponential generating function of $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence W_n.

Lemma 1. *[3, Lemma 1.4]. Suppose that $f_{GW_n}(x)=\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is the exponential generating function of the generalized Edouard sequence $\{W_n\}$. Then*

$$\sum_{n=0}^{\infty} W_n \frac{x^n}{n!} = \frac{(W_2-(\beta+1)W_1+\beta W_0)}{(\alpha-\beta)(\alpha-1)}e^{\alpha x}+\frac{(W_2-(\alpha+1)W_1+\alpha W_0)}{(\beta-\alpha)(\beta-1)}e^{\beta x}-\frac{(W_2-6W_1+W_0)}{4}e^{x}.$$

The previous Lemma gives the following results as particular examples.

Corollary 2. *Exponential generating function of Edouard and Edouard-Lucas numbers are*

a):

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-1)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-1)}-\frac{1}{4}\right)\frac{x^n}{n!} = \frac{\alpha e^{\alpha x}}{(\alpha-\beta)(\alpha-1)}+\frac{\beta e^{\beta x}}{(\beta-\alpha)(\beta-1)}-\frac{1}{4}e^{x}.$$

b):

$$\sum_{n=0}^{\infty} K_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (\alpha^n+\beta^n+1)\frac{x^n}{n!} = e^{\alpha x}+e^{\beta x}+e^{x}.$$

For more details, see [26].

We now present an overview of specific numerical systems, focusing particularly on the hypercomplex framework, which encompasses complex numbers, hyperbolic numbers, and dual numbers. It is worth noting that hyperbolic numbers will serve as a central component of our investigation, providing essential analytical tools throughout the development of our theoretical framework. Moreover, hyperbolic functions and numbers find widespread application across numerous engineering disciplines, including electrical engineering (e.g., transmission line modeling), control systems (e.g., dynamic system behavior), and signal processing (e.g., advanced filter design). They also play a significant role in various domains of engineering physics, such as special relativity, wave propagation, fluid dynamics, optics, and thermal conduction. It is important to recognize that, despite the intriguing mathematical characteristics of hyperbolic numbers, their practical utility is inherently problem-dependent and contingent upon the extent to which they offer computational or analytical advantages over alternative numerical systems within a given context.

We begin by examining hypercomplex number systems, which serve as extensions of the real number line. For a more comprehensive discussion, refer to [20]. In addition, several commutative special cases of hypercomplex number systems such as complex numbers, hyperbolic numbers, and dual numbers are widely utilized across diverse branches of mathematics and physics. We now proceed to present these number systems in a sequential manner, as outlined in the following sections.

- Complex numbers simplest form of hypercomplex numbers. Complex numbers defined as $z = a+ib$, where a and b real numbers and i imaginary unit that satisfy $i^2 = -1$. In addition that a and b named, respectively, $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ Consequently, the definition of complex numbers given by,

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\}.$$

- Hyperbolic (double, split-complex) numbers, for more detail see [30], Split-complex numbers, commonly recognized as hyperbolic numbers, defined as $h = a + jb$ where a and b real numbers and j hyperbolic unit that satisfy $j^2 = 1$. In addition that a and b named, respectively, $Re(h)$ and $Hyp(h)$. Thus, the definition of hyperbolic numbers given by,

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\},$$

- Dual numbers, see [14], defined as $d = a + \varepsilon b$ where a and b real numbers and ε dual unit that satisfy $\varepsilon^2 = 0$. Furthermore, a and b called, respectively, $Re(d)$ and $Du(d)$. Thus, defination of dual numbers given by,

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

- A dual hyperbolic number, specifically within the hyperbolic number system, constitutes a distinct type of hypercomplex number. A dual hyperbolic number is defined by,

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon j a_3$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$ and the set of all dual hyperbolic numbers are defined by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon j a_3 : a_0, a_1, a_2, a_3 \in \mathbb{R},\ j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The $\{1, j, \varepsilon, \varepsilon j\}$ is linear independent and $\mathbb{H}_{\mathbb{D}} = sp\{1, j, \varepsilon, \varepsilon j\}$ so that $\{1, j, \varepsilon, \varepsilon j\}$ is a basis of $\mathbb{H}_{\mathbb{D}}$. For more detail see, [1]

The next properties are true for the base elements $\{1, j, \varepsilon, \varepsilon j\}$ (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j,\ \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0,\ j^2 = j.j = 1 \\ \varepsilon.j &= j.\varepsilon,\ \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0,\ j(\varepsilon j) = (\varepsilon j)j = \varepsilon \end{aligned}$$

where ε satisfy the pure dual unit $(\varepsilon^2 = 0, \varepsilon \neq 0)$, j satisfy the hyperbolic unit $(j^2 = 1)$, and εj satisfy the dual hyperbolic unit $((j\varepsilon)^2 = 0)$.

In addition that the other number sytems are quarternions, octonions and sedenions given below, respectively,

- Quaternion numbers, non-commutative examples of hypercomplex number systems, are a four-dimensional extension of complex numbers. They are expressed as $a_0 + ia_1 + ja_2 + ka_3$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$, and i, j, and k are the quaternion units that satisfy specific multiplication rules. For more detail see [17]. Quaternion numbers are defined by

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\},$$

- Octonions is a set, every element of the set linear combinations of unit octonions $\{e_i : i = 0, 1.2, ..., 7\}$, doneted as $\mathbb{O}$. Octonions are defined by,

$$\mathbb{O} = \{\sum_{i=0}^{7} a_i e_i : a_i \in \mathbb{R}, e_0 e_i = e_i e_0 = e_i,\ e_i e_j = -\delta_{ij} e_0 + \varepsilon_{ijk} e_k\ \}$$

where $e_e = 1$, δ_{ij} is Kroneker delta (equal to 1 if and only if $i = j$), ε_{ijk} is anti-symetric tensor. For more detail see [19, 34]

- Sedenions is a set, every element of the set linear combinations of unit sedenions $\{e_i : i = 0, 1.2, ..., 15\}$, denoted by $\mathbb{S}$. It can be seen from here that ever sedenion can be written as

$$\sum_{i=0}^{15} a_i e_i$$

where a_i is real number. For more detail see, [25, 34].

Next we give some proporties on two hyperbolic numbers, $h_1 = a + jb$ and $h_2 = c + jd$, as

$$
\begin{aligned}
h_1 + h_2 &= (a+b) + j(c+d), \\
h_1 . h_2 &= (ac+bd) + j(ad+bc), \\
\overline{h_1} &= a - jb \\
\frac{h_1}{h_2} &= \frac{(ac-bd)+j(cb-ad)}{c^2-d^2}, \\
h_1 &= h_2 \text{ if only if } a = c \text{ and } b = d, \\
\langle h_1, h_2 \rangle &= (ac+bd) + j(bc+ad), \\
\|h_1\| &= \sqrt{|a^2-b^2|}, \text{ called norm of } h_1, \\
\text{if } \left|a^2-b^2\right| &> 0, \ h_1 \text{ is named spacelike vector}, \\
\text{if } \left|a^2-b^2\right| &< 0, \ h_1 \text{ is named timelike vector}, \\
\text{if } \left|a^2-b^2\right| &= 0, \ h_1 \text{ is named null(light-like) vector}.
\end{aligned}
$$

Note that $\{\mathbb{R}^2, H, \langle , \rangle\}$ is called Lorentz plane and denoted as $\mathbb{R}^2_1$. There is an isomorphism relationship between the Lorentz plane and hyperbolic numbers. For more detail, see [34].

Hence the algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are real algebras attained from the real numbers $\mathbb{R}$ by a doubling procedure known as the Cayley-Dickson Process. This doubling process can be extended beyond the sedenions to form what are known as the 2^n-ions (see for example [4, 17, 18, 23, 15].

Some authors have conducted studies about the dual, hyperbolic, dual hyperbolic and other special numbers. Now we give some information published papers in litarature.

- Cockle [9] explored hyperbolic numbers with complex coefficients, contributing to the early development of hypercomplex algebra.
- Eren and Soykan [13] studied the generalized Generalized Woodall Numbers.
- Cheng and Thompson [8] introduced dual numbers with complex coefficients, expanding the algebraic versatility of dual number systems for applications in polynomial equations and transformation theory.
- Akar at al [1] introduced the concept of dual hyperbolic numbers, combining characteristics of dual and hyperbolic systems into a unified algebraic structure.

Next, we present some information on hyperbolic numbers presented in literature.

- Aydın [2] presented hyperbolic Fibonacci numbers given by

$$\widetilde{F}_n = F_n + hF_{n+1},$$

where Fibonacci numbers are given by $F_{n+2} = F_{n+1} + F_n$, with the initial conditation $F_0 = 0$, $F_1 = 1$.

- Soykan and Taşdemir [28] studied hyperbolic generalized Jacobsthal numbers given by

$$\widetilde{V}_n = V_n + hV_{n+1}$$

where generalized Jacobsthal numbers are $V_{n+2} = V_{n+1} + 2V_n$ with the initial conditation $V_0 = a$, $V_1 = b$.

- Taş [33] studied hyperbolic Jacobsthal-Lucas sequence written by

$$HJ_n = J_n + hJ_{n+1}$$

where Jacobsthal-Lucas numbers given by $J_{n+2} = J_{n+1} + 2J_n$ with the inintial conditation $J_0 = 2$, $J_1 = 1$.

- Dikmen and Altınsoy, [11] studied On Third Order Hyperbolic Jacobsthal Numbers given by

$$\begin{aligned} \widehat{J}_n^{(3)} &= J_n^{(3)} + hJ_{n+1}^{(3)}, \\ \widehat{j}_n^{(3)} &= j_n^{(3)} + hj_{n+1}^{(3)} \end{aligned}$$

  where Jacobsthal numbers, respectively, given by $J_n^{(3)} = J_{n-1}^{(3)} + J_{n-2}^{(3)} + 2J_{n-3}^{(3)}$, $J_0^{(3)} = 0$, $J_1^{(3)} = 1$, $J_2^{(3)} = 1$, $j_n^{(3)} = j_{n-1}^{(3)} + j_{n-2}^{(3)} + 2j_{n-3}^{(3)}$, $j_0^{(3)} = 2$, $j_1^{(3)} = 1$, $j_2^{(3)} = 5$.

Following this, we provide details on dual hyperbolic sequences as they are presented in literature.

- Soykan at al [27] presented dual hyperbolic generalized Pell numbers given by

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}$$

where generalized Pell numbers, with the initial values V_0, V_1 not all being zero, are given by $V_n = 2V_{n-1} + V_{n-2}$, $V_0 = a$, $V_1 = b$ $(n \geq 2)$.

- Cihan at al [7] studied dual hyperbolic Fibonacci and Lucas numbers given by, respectively,

$$\begin{aligned} DHF_n &= F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3}, \\ DHL_n &= L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3} \end{aligned}$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$, $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$.

- Soykan at al [28] studied dual hyperbolic generalized Jacopsthal numbers given by

$$\widehat{J}_n = J_n + jJ_{n+1} + \varepsilon J_{n+2} + j\varepsilon J_{n+3}$$

where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = a$, $J_1 = b$.

- Bród at al [6] studied dual hyperbolic generalized balancing numbers as

$$DHB_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}$$

where $B_n = 6B_{n-1} - B_{n-2}$, $B_0 = 0$, $B_1 = 1$.

- Yılmaz and Soykan [36] introduced dual hyperbolic generalized Guglielmo numbers are

$$\widehat{T}_0 = T_0 + jT_1 + \varepsilon T_2 + j\varepsilon T_3$$

where $T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, T_0 = 0, T_1 = 1, T_2 = 3$.

Next section, we define the hyperbolic generalized Edouard numbers and some special properties, generating function and Binet's formula , of these numbers.

2. Hyperbolic Generalized Edouard Numbers and their Generating Functions and Binet's Formulas

In this section, we define hyperbolic generalized Edouard numbers then using this definition, we present generating functions and Binet's formula of hyperbolic generalized Edouard numbers.

We now examine hyperbolic generalized Edouard numbers within the algebra $\mathbb{H}$. The nth such number is defined as

$$HW_n = W_n + jW_{n+1} \tag{2.1}$$

with the initial values HW_0, HW_1, HW_2. (2.1)The hyperbolic Edouard numbers ,which is defined above, can be written to negative subscripts by defining,

$$HW_{-n} = W_{-n} + jW_{-n+1} \tag{2.2}$$

so identity (2.1) holds for all integers n.

Now, we define some special cases of hyperbolic generalized Edouard numbers. The nth hyperbolic Edouard numbers, the nth hyperbolic Edouard-Lucas numbers, respectively, are given as the nth hyperbolic Edouard numbers is given $HE_n = E_n + jE_{n+1}$, with the initial values

$$\begin{aligned} HE_0 &= E_0 + jE_1, \\ HE_1 &= E_1 + jE_2, \\ HE_2 &= E_2 + jE_3, \end{aligned}$$

the nth hyperbolic Edouard-Lucas numbers is given $HK_n = K_n + jK_{n+1}$ with the initial values

$$\begin{aligned} HK_0 &= K_0 + jK_1, \\ HK_1 &= K_1 + jK_2, \\ HK_2 &= K_2 + jK_3, \end{aligned}$$

Note that,for hyperbolic Edouard numbers (by using $W_n = E_n$, $E_0 = 0$, $E_1 = 1$, $E_2 = 7$) we get

$$\begin{aligned} HE_0 &= j \\ HE_1 &= 1 + 7j \\ HE_2 &= 7 + 42j, \end{aligned}$$

for hyperbolic Edouard-Lucas numbers (bu using $W_n = K_n$, $K_0 = 3$, $K_1 = 7$, $K_2 = 35$) we obtain

$$\begin{aligned} HK_0 &= 3 + 7j, \\ HK_1 &= 7 + 35j, \\ HK_2 &= 35 + 199j. \end{aligned}$$

So, using (2.1), we can write the following identity for non negative integers n,

$$HW_n = 7HW_{n-1} - 7HW_{n-2} + HW_{n-3}. \tag{2.3}$$

and the sequence $\{HW_n\}_{n\geq 0}$ can be given as

$$HW_{-n} = 7HW_{-(n-1)} - 7HW_{-(n-2)} + HW_{-(n-3)},$$

for $n = 1, 2, 3, ...$ by using (2.2).As a result, recurrence (2.3) holds for all integer n.

Table 3 presents the initial values of the hyperbolic generalized Edouard numbers HW_n, showcasing terms with both positive and negative subscripts for a comprehensive view of the sequence's symmetric structure.

Table 3. A few hyperbolic generalized Edouard numbers

n	HW_n	HW_{-n}
0	HW_0	HW_0
1	HW_1	$7HW_0 - 7HW_1 + HW_2$
2	HW_2	$42HW_0 - 48HW_1 + 7HW_2$
3	$HW_0 - 7HW_1 + 7HW_2$	$246HW_0 - 287HW_1 + 42HW_2$
4	$7HW_0 - 48HW_1 + 42HW_2$	$1435HW_0 - 1680HW_1 + 246HW_2$
5	$42HW_0 - 287HW_1 + 246HW_2$	$8365HW_0 - 9799HW_1 + 1435HW_2$
6	$246HW_0 - 1680HW_1 + 1435HW_2$	$48756HW_0 - 57120HW_1 + 8365HW_2$

Note that

$$\begin{aligned} HW_0 &= W_0 + jW_1, \\ HW_1 &= W_1 + jW_2, \\ HW_2 &= W_2 + jW_3. \end{aligned}$$

A few hyperbolic Edouard numbers, hyperbolic Edouard-Lucas numbers with positive subscript and negative subscript are given in the following Table 3, Table 4.

Table 4. hyperbolic Edouard numbers

n	HE_n	HE_{-n}
0	j	
1	$1+7j$	0
2	$7+42j$	1
3	$42+246j$	$7+j$
4	$246+1435j$	$42+7j$
5	$1435+8365j$	$246+42j$

Table 5. hyperbolic Edouard-Lucas numbers

n	HK_n	HK_{-n}
0	$3+7j$	
1	$7+35j$	$7+3j$
2	$35+199j$	$35+7j$
3	$199+1155j$	$199+35j$
4	$1155+6727j$	$1155+199j$
5	$6727+39203j$	$6727+1155j$

Now, we will give some expressions that we will use in the rest of the paper and then we define Binet's formula for the hyperbolic generalized Edouard numbers.

$$\widetilde{\alpha} = 1 + j\alpha, \tag{2.4}$$

$$\widetilde{\beta} = 1 + j\beta, \tag{2.5}$$

$$\widetilde{\gamma} = 1 + j. \tag{2.6}$$

Note that using above equalities we can write the following identities:

$$\begin{aligned}
\widetilde{\alpha}^2 &= 1 + 2\alpha j, \\
\widetilde{\beta}^2 &= 1 + 2\beta j, \\
\widetilde{\gamma}^2 &= 1 + 2j, \\
\widetilde{\alpha}\widetilde{\beta} &= 1 + j(\alpha+\beta), \\
\widetilde{\alpha}\widetilde{\gamma} &= 1 + j(\alpha+\gamma), \\
\widetilde{\gamma}\widetilde{\beta} &= 1 + j(\gamma+\beta).
\end{aligned}$$

THEOREM 3. *(Binet's Formula) For any integer n, the nth hyperbolic generalized Edouard number is*

$$HW_n = \widetilde{\alpha}A_1\alpha^n + \widetilde{\beta}A_2\beta^n + \widetilde{\gamma}A_3. \tag{2.7}$$

where $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\gamma}$ are given as (2.4),(2.5),(2.6).

Proof. Using Binet's formula of the generalized Edouard numbers given below

$$W_n = A_1\alpha^n + A_2\beta^n + A_3$$

where A_1, A_2, A_3 are given (1.4) we get

$$\begin{aligned}
HW_n &= W_n + jW_{n+1}, \\
&= A_1\alpha^n + A_2\beta^n + A_3 + (A_1\alpha^{n+1} + A_2\beta^{n+1} + A_3)j \\
&= \widetilde{\alpha}A_1\alpha^n + \widetilde{\beta}A_2\beta^n + \widetilde{\gamma}A_3.
\end{aligned}$$

This proves (2.7). □

In particular, for any integer n, the Binet's Formula of nth hyperbolic Edouard number, Edouard-Lucas numbers, respectively, provided by

$$HE_n = \frac{\widetilde{\alpha}\alpha^{n+1}}{(\alpha-\beta)(\alpha-1)} + \frac{\widetilde{\beta}\beta^{n+1}}{(\beta-\alpha)(\beta-1)} - \frac{\widetilde{\gamma}}{4}, \quad (2.8)$$

$$HK_n = \widetilde{\alpha}\alpha^n + \widetilde{\beta}\beta^n + \widetilde{\gamma}, \quad (2.9)$$

In the following Theorem, we now derive the generating function for the sequence of hyperbolic generalized Edouard numbers, providing a compact analytical representation of their structure and recursive behavior.

THEOREM 4. *The generating function for the hyperbolic generalized Edouard numbers is*

$$f_{HW}(x) = \frac{HW_0 + (HW_1 - 7HW_0)x + (HW_2 - 7HW_1 + 7HW_0)x^2}{(1-7x+3x^2-x^3)}. \quad (2.10)$$

Proof. We assume that $f_{HW}(x)$ is the generating function of the hyperbolic generalized Edouard numbers and then we can write

$$f_{HW}(x) = \sum_{n=0}^{\infty} HWx^n$$

Then, in light of the definition of the hyperbolic generalized Edouard numbers, and substracting $7xg(x)$ and $-7x^2g(x)$ from $x^3g(x)$, we get

$$\begin{aligned}
(1-7x+7x^2-x^3)f_{HW}(x) &= \sum_{n=0}^{\infty} HWx^n - 7x\sum_{n=0}^{\infty} HWx^n + 7x^2\sum_{n=0}^{\infty} HWx^n - x^3\sum_{n=0}^{\infty} HWx^n, \\
&= \sum_{n=0}^{\infty} HWx^n - 7\sum_{n=0}^{\infty} HWx^{n+1} + 7\sum_{n=0}^{\infty} HWx^{n+2} - \sum_{n=0}^{\infty} HWx^{n+3}, \\
&= \sum_{n=0}^{\infty} HWx^n - 7\sum_{n=1}^{\infty} HWx^n + 7\sum_{n=2}^{\infty} HWx^n - \sum_{n=3}^{\infty} HWx^n, \\
&= (HW_0 + HW_1x + HW_2x^2) - 7(HWx + HW_1x^2) + 7HW_0x^2 \\
&\quad + \sum_{n=3}^{\infty}(HW_n - 7HW_{n-1} + 7HW_{n-2} - HW_{n-3})x^n, \\
&= HW_0 + HW_1x + HW_2x^2 - 7HW_0x - 7HW_1x^2 + 7HW_0x^2, \\
&= HW_0 + (HW_1 - 7HW_0)x + (HW_2 - 7HW_1 + 7HW_0)x^2.
\end{aligned}$$

Note that , using the recurrence relation $\widehat{W}_n = 7\widehat{W}_{n-1} - 7\widehat{W}_{n-2} + \widehat{W}_{n-3}$ and rearranging above equation, the (2.10) has been obtained. □

Now we can write the generating functions of the hyperbolic Edouard, Edouard-Lucas numbers as

$$\begin{aligned}
f_{HE_n}(x) &= \frac{j+x}{(1-7x+7x^2-x^3)}, \\
f_{HK_n}(x) &= \frac{7j+3+(-14j-14)x+(3j+7)x^2}{(1-7x+7x^2-x^3)}. \ \square
\end{aligned}$$

Next, we give the exponential generating function of $\sum\limits_{n=0}^{\infty} HW_n \frac{x^n}{n!}$ of the sequence HW_n.

Lemma 5. *Suppose that $f_{HW_n}(x) = \sum\limits_{n=0}^{\infty} HW_n \frac{x^n}{n!}$ is the exponential generating function of the hyperbolic generalized Edouard sequence $\{HW_n\}$.*

Then $\sum\limits_{n=0}^{\infty} HW_n \frac{x^n}{n!}$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty} HW_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty} HW_n \frac{x^n}{n!} + j\sum_{n=0}^{\infty} HW_{n+1} \frac{x^n}{n!} \\ &= \frac{(W_2-(\beta+1)W_1+\beta W_0)}{(\alpha-\beta)(\alpha-1)}e^{\alpha x} + \frac{(W_2-(\alpha+1)W_1+\alpha W_0)}{(\beta-\alpha)(\beta-1)}e^{\beta x} - \frac{(W_2-6W_1+W_0)}{4}e^{x} \\ &\quad + j\Big(\frac{(W_2-(\beta+1)W_1+\beta W_0)\alpha}{(\alpha-\beta)(\alpha-1)}e^{\alpha x} + \frac{(W_2-(\alpha+1)W_1+\alpha W_0)\beta}{(\beta-\alpha)(\beta-1)}e^{\beta x} - \frac{(W_2-6W_1+W_0)}{4}e^{x}\Big).\end{aligned}$$

Proof: Note that we have

$$\sum_{n=0}^{\infty} HW_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (HW_n + \varepsilon HW_{n+1}) \frac{x^n}{n!}.$$

Then using the Binet's formula of hiperbolic generalized Edouard numbers or exponential generating function of the generalized Edouard sequence we get the required identy.

The previous Lemma gives the following results as particular examples.

Corollary 6. *Exponential generating function of hyperbolic Edouard and hiperbolic Edouard-Lucas numbers are*

a):

$$\sum_{n=0}^{\infty} HE_n \frac{x^n}{n!} = \frac{\alpha e^{\alpha x}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta e^{\beta x}}{(\beta-\alpha)(\beta-1)} - \frac{1}{4}e^{x} + j\Big(\frac{\alpha^2 e^{\alpha x}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta^2 e^{\beta x}}{(\beta-\alpha)(\beta-1)} - \frac{1}{4}e^{x}\Big).$$

b):

$$\sum_{n=0}^{\infty} HK_n \frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^{x} + j(\alpha e^{\alpha x} + \beta e^{\beta x} + e^{x}).$$

3. Getting the Binet's Formula from the generating function.

Next ,by using generating function $f_{HW}(x)$, we investigate Binet formula of $\{HW_n\}$.

Theorem 7. *(Binet formula of hyperbolic generalized Edouard numbers)*

$$HW_n = \widetilde{\alpha} A_1 \alpha^n + \widetilde{\beta} A_2 \beta^n + \widetilde{\gamma} A_3. \tag{3.1}$$

Proof. Using the $\sum_{n=0}^{\infty} HWx^n$ we can write

$$\sum_{n=0}^{\infty} HWx^n = \frac{HW_0 + (HW_1 - 7HW_0)x + (HW_2 - 7HW_1 + 7HW_0)x^2}{(1-7x+7x^2-x^3)} = \frac{d_1}{(1-\alpha x)} + \frac{d_2}{(1-\beta x)} + \frac{d_3}{(1-x)}, \tag{3.2}$$

so that

$$\begin{aligned}\sum_{n=0}^{\infty} HWx^n &= \frac{d_1}{(1-\alpha x)} + \frac{d_2}{(1-\beta x)} + \frac{d_3}{(1-x)}, \\ &= \frac{d_1(1-x)(1-\beta x) + d_2(1-\alpha x)(1-x) + d_3(1-\alpha x)(1-\beta x)}{(x^2-6x+1)(1-x)},\end{aligned}$$

thus, we obtain

$$HW_0+(HW_1-7HW_0)x+(HW_2-7HW_1+7HW_0)x^2 = d_1+d_2+d_3+(-d_2-\alpha d_2-\beta d_1-\alpha d_3-\beta d_3)x+(\alpha d_2+\beta d_1+\alpha\beta d_3)x^2.$$

By equation the coefficients of corresponding powers of x in the above equation, we get

$$\begin{aligned} HW_0 &= d_1+d_2+d_3, \\ HW_1-7HW_0 &= -d_2-\alpha d_2-\beta d_1-\alpha d_3-\beta d_3, \\ HW_2-7HW_1+7HW_0 &= \alpha d_2+\beta d_1+\alpha\beta d_3. \end{aligned} \tag{3.3}$$

If we solve (3.3) we obtain

$$\begin{aligned} d_1 &= \frac{HW_0\alpha^2+(HW_1-7HW_0)\alpha+(HW_2-7HW_1+7HW_0)}{(\alpha-\beta)(\alpha-\gamma)}, \\ d_2 &= \frac{HW_0\beta^2+(HW_1-7HW_0)\beta+(HW_2-7HW_1+7HW_0)}{(\beta-\alpha)(\beta-\gamma)}, \\ d_3 &= \frac{HW_0+(HW_1-7HW_0)+(HW_2-7HW_1+7HW_0)}{(\gamma-\alpha)(\gamma-\beta)}, \end{aligned}$$

Thus (3.2) can be given as

$$\begin{aligned}\sum_{n=0}^{\infty} HW_nx^n &= d_1\sum_{n=0}^{\infty}\alpha^n x^n + d_2\sum_{n=0}^{\infty}\beta^n x^n + d_3\sum_{n=0}^{\infty}x^n, \\ &= \sum_{n=0}^{\infty}(d_1\alpha^n+d_2\beta^n+d_3)x^n, \\ &= \sum_{n=0}^{\infty}\Big(\frac{HW_2-(\beta+1)HW_1+\beta HW_0}{(\alpha-\beta)(\alpha-\gamma)}\alpha^n + \frac{HW_2-(\alpha+1)HW_1+\alpha HW_0}{(\beta-\alpha)(\beta-\gamma)}\beta^n + \frac{HW_2-6HW_1+HW_0}{(\gamma-\alpha)(\gamma-\beta)}\Big)x^n.\end{aligned}$$

Hence, we get

$$HW_n = \widetilde{\alpha}A_1\alpha^n + \widetilde{\beta}A_2\beta^n + \widetilde{\gamma}A_3. \ \square$$

4. Some Identities

We now introduce distinctive identities pertaining to the sequence $\{HW_n\}$ of hyperbolic generalized Edouard numbers. The forthcoming theorem introduces a Simpson type formula within this framework, delineating the structural relationships between consecutive terms of the sequence.

THEOREM 8. *(Simpson's formula for hyperbolic generalized Edouard numbers) For all integers n we have,*

$$\begin{vmatrix} HW_{n+2} & HW_{n+1} & HW_{n} \\ HW_{n+1} & HW_{n} & HW_{n-1} \\ HW_{n} & HW_{n-1} & HW_{n-2} \end{vmatrix} = \begin{vmatrix} HW_{2} & HW_{1} & HW_{0} \\ HW_{1} & HW_{0} & HW_{-1} \\ HW_{0} & HW_{-1} & HW_{-2} \end{vmatrix}. \tag{4.1}$$

Proof. For the proof, we use mathematical induction on $n \geq 0$. For $n = 0$ identity (4.1) is true. Now we assume that (4.1) is true for $n = k$. Hence, the identity given below can be written

$$\begin{vmatrix} HW_{k+2} & HW_{k+1} & HW_{k} \\ HW_{k+1} & HW_{k} & HW_{k-1} \\ HW_{k} & HW_{k-1} & HW_{k-2} \end{vmatrix} = \begin{vmatrix} HW_{2} & HW_{1} & HW_{0} \\ HW_{1} & HW_{0} & HW_{-1} \\ HW_{0} & HW_{-1} & HW_{-2} \end{vmatrix}.$$

For $n = k + 1$, we obtain

$$\begin{aligned} \begin{vmatrix} HW_{k+3} & HW_{k+2} & HW_{k+1} \\ HW_{k+2} & HW_{k+1} & HW_{k} \\ HW_{k+1} & HW_{k} & HW_{k-1} \end{vmatrix} &= \begin{vmatrix} 7HW_{k+2} - 7HW_{k+1} + HW_{k} & HW_{k+2} & HW_{k+1} \\ 7HW_{k+1} - 7HW_{k} + HW_{k-1} & HW_{k+1} & HW_{k} \\ 7HW_{k} - 7HW_{k-1} + HW_{k-2} & HW_{k} & HW_{k-1} \end{vmatrix} \\ &= 7\begin{vmatrix} HW_{k+2} & HW_{k+2} & HW_{k+1} \\ HW_{k+1} & HW_{k+1} & HW_{k} \\ HW_{k} & HW_{k} & HW_{k-1} \end{vmatrix} - 7\begin{vmatrix} HW_{k+1} & HW_{k+2} & HW_{k+1} \\ HW_{k} & HW_{k+1} & HW_{k} \\ HW_{k-1} & HW_{k} & HW_{k-1} \end{vmatrix} \\ &\quad + \begin{vmatrix} HW_{k} & HW_{k+2} & HW_{k+1} \\ HW_{k-1} & HW_{k+1} & HW_{k} \\ HW_{k-2} & HW_{k} & HW_{k-1} \end{vmatrix} \\ &= \begin{vmatrix} HW_{k+2} & HW_{k+1} & HW_{k} \\ HW_{k+1} & HW_{k} & HW_{k-1} \\ HW_{k} & HW_{k-1} & HW_{k-2} \end{vmatrix}. \end{aligned}$$

For the case $n < 0$ the proof has been seen similarly. Thus, the proof is completed. □

From Theorem 4.1 we get following corollary.

COROLLARY 9.

(a): $\begin{vmatrix} HE_{n+2} & HE_{n+1} & HE_{n} \\ HE_{n+1} & HE_{n} & HE_{n-1} \\ HE_{n} & HE_{n-1} & HE_{n-2} \end{vmatrix} = -8(j+1).$

(b): $\begin{vmatrix} HK_{n+2} & HK_{n+1} & HK_{n} \\ HK_{n+1} & HK_{n} & HK_{n-1} \\ HK_{n} & HK_{n-1} & HK_{n-2} \end{vmatrix} = 4096(j+1).$

Theorem 10. *Let n and m be integers, E_n is Edouard numbers, the following identity is true:*

$$HW_{m+n} = E_{m-1}HW_{n+2} + (E_{m-3} - 7E_{m-2})HW_{n+1} + E_{m-2}HW_n. \tag{4.2}$$

Proof. For $n, m \geqslant 0$ the identity (10) can be proved by mathematical induction on m. If $m = 0$ we get

$$HW_n = E_{-1}HW_{n+2} + (E_{-3} - 7E_{-2})HW_{n+1} + E_{-2}HW_n$$

which is true by seeing that $E_{-1} = 0, E_{-2} = 1, E_{-3} = 3$. We assume that the identity given holds for $m = k$. For $m = k + 1$, we get

$$\begin{aligned}
HW_{(k+1)+n} &= 7HW_{n+k} - 7HW_{n+k-1} + HW_{n+k-2} \\
&= 7(E_{k-1}HW_{n+2} + (E_{k-3} - 7E_{k-2})HW_{n+1} + E_{k-2}HW_n) \\
&\quad -7(T_{k-2}HW_{n+2} + (E_{k-4} - 7E_{k-3})HW_{n+1} + E_{k-3}HW_n) \\
&\quad +(E_{k-3}HW_{n+2} + (E_{k-5} - 7E_{k-4})HW_{n+1} + E_{k-4}HW_n) \\
&= (7E_{k-1} - 7E_{k-2} + E_{k-3})HW_{n+2} + ((7E_{k-3} - 7E_{k-4} + E_{k-5}) \\
&\quad -7(7E_{k-2} - 7E_{k-3} + E_{k-4}))HW_{n+1} + (7E_{k-2} - 7E_{k-3} + E_{k-4})HW_n \\
&= E_kHW_{n+2} + (E_{k-2} - 7E_{k-1})HW_{n+1} + E_{k-1}HW_n \\
&= E_{(k+1)-1}HW_{n+2} + (E_{(k+1)-3} - 7E_{(k+1)-2})HW_{n+1} + E_{(k+1)-2}HW_n.
\end{aligned}$$

Consequently, by mathematical induction on m, this proves (10). For the other case, the proof can be done similarly. □

5. Linear Sums

In this section, we give the summation formulas of the hyperbolic generalized Edouard numbers with subscripts.

Proposition 11. *For the generalized Edouard numbers, we have the following formulas:*

- **(a):** $\sum_{k=0}^{n} W_k = \frac{1}{4}(-(n+3)W_n + (n+2)(7W_{n+1} - W_{n+2}) - (n+1)W_{n+1} + 2W_2 - 13W_1 + 7W_0)$.
- **(b):** $\sum_{k=0}^{n} W_{2k} = \frac{1}{32}(-(n+3)W_{2n} + (n+2)(-7W_{2n+2} + 48W_{2n+1} - 7W_{2n}) - (n+1)W_{2n+2} + 15W_2 - 96W_1 + 49W_0)$.
- **(c):** $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{32}(-(n+3)W_{2n+1} + (n+2)(-W_{2n+2} + 42W_{2n+1} - 7W_{2n}) - (n+1)(7W_{2n+2} - 7W_{2n+1} + W_{2n}) + 9W_2 - 56W_1 + 15W_0)$.

Proof. It is given in Soykan [32, Theorem 3.3]. □

Now, we will give the formulas of the sum of hyperbolic generalized Edouard numbers.

Theorem 12. *For $n \geq 0$, hyperbolic generalized Edouard numbers have the following formulas:*

(a): $\sum_{k=0}^{n} HW_k = \frac{1}{4}(-(n+3)HW_n+(n+2)(7HW_{n+1}-HW_{n+2})-(n+1)HW_{n+1}+2HW_2-13HW_1+7HW_0)$.

(b): $\sum_{k=0}^{n} HW_{2k} = \frac{1}{32}(-(n+3)HW_{2n}+(n+2)(-7HW_{2n+2}+48HW_{2n+1}-7HW_{2n})-(n+1)HW_{2n+2}+15HW_2-96HW_1+49HW_0)$.

(c): $\sum_{k=0}^{n} HW_{2k+1} = \frac{1}{32}(-(n+3)HW_{2n+1}+(n+2)(-HW_{2n+2}+42HW_{2n+1}-7HW_{2n})-(n+1)(7HW_{2n+2}-7HW_{2n+1}+HW_{2n})+9HW_2-56HW_1+15HW_0)$.

Proof.

(a): Note that using (2.1), we get

$$\sum_{k=0}^{n} HW_k = \sum_{k=0}^{n} W_k + j\sum_{k=0}^{n} W_{k+1}$$

and using Proposition 11 the proof completed.

(b): Note that using (2.1), we get

$$\sum_{k=0}^{n} HW_{2k} = \sum_{k=0}^{n} W_{2k} + j\sum_{k=0}^{n} W_{2k+1}$$

and using Proposition 11 the proof completed.

(c): Note that using (2.1), we get

$$\sum_{k=0}^{n} HW_{2k+1} = \sum_{k=0}^{n} W_{2k+1} + j\sum_{k=0}^{n} W_{2k+2}$$

and using Proposition 11 the proof completed. □

As a special case of the Theorem 12 (a), we present the following corollary.

Corollary 13.

(a): $\sum_{k=0}^{n} HE_k = \frac{1}{4}(-(n+3)HE_n+(n+2)(7HE_{n+1}-HE_{n+2})-(n+1)HE_{n+1}+1)$.

(b): $\sum_{k=0}^{n} HK_k = \frac{1}{4}(-(n+3)HK_n+(n+2)(7HK_{n+1}-HK_{n+2})-(n+1)HK_{n+1}-8j)$.

As a special case of the Theorem 12 (b), we present the following corollary.

Corollary 14.

(a): $\sum_{k=0}^{n} HE_{2k} = \frac{1}{32}(-(n+3)HE_{2n}+(n+2)(-7HE_{2n+2}+48HE_{2n+1}-7HE_{2n})-(n+1)HE_{2n+2}+15(7+42j)-96(1+7j)+49j)$.

(b): $\sum_{k=0}^{n} HK_{2k} = \frac{1}{32}(-(n+3)HK_{2n}+(n+2)(-7HK_{2n+2}+48HK_{2n+1}-7HK_{2n})-(n+1)HK_{2n+2}+15(35+199j)-96(7+35j)+49(3+7j))$.

As a special case of the Theorem 12 (c), we present the following corollary.

Corollary 15.

(a): $\sum_{k=0}^{n} HE_{2k+1} = \frac{1}{32}(-(n+3)HE_{2n+1}+(n+2)(-HE_{2n+2}+42HE_{2n+1}-7HE_{2n})-(n+1)(7HE_{2n+2}-7HE_{2n+1}+HE_{2n})+j+7)$.

(b): $\sum_{k=0}^{n} HK_{2k+1} = \frac{1}{32}(-(n+3)HK_{2n+1} + (n+2)(-HK_{2n+2} + 42HK_{2n+1} - 7HK_{2n}) - (n+1)(7HK_{2n+2} - 7HK_{2n+1} + HK_{2n}) + -64j - 32)$.

6. Matrices linked to Hyperbolic Generalized Edouard Numbers

In this section of our study, we present several algebraic identities pertaining to matrices associated with hyperbolic Edouard numbers..By using the $\{E_n\}$ which is defined by the third-order recurrence relation as follows

$$E_n = 7E_{n-1} - 7E_{n-2} + E_{n-3}$$

with the initial conditions $E_0 = 0$, $E_1 = 1$, $E_2 = 7$ we present the square matrix A of order 3 as

$$A = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 1$. Then, we give the following Lemma.

Lemma 16. *For all integers n the following identity is true*

$$\begin{pmatrix} HW_{n+2} \\ HW_{n+1} \\ HW_n \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}. \tag{6.1}$$

Proof. First, for the proof we assume that $n \geq 0$. Lemma 16 can be given by mathematical induction on n. If $n = 0$ we get

$$\begin{pmatrix} HW_2 \\ HW_1 \\ HW_0 \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}$$

which is true. We assume that the identity(6.1) given holds for $n = k$. Thus the following identity is true.

$$\begin{pmatrix} HW_{k+2} \\ HW_{k+1} \\ HW_k \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}.$$

For $n = k + 1$, we get

$$\begin{aligned}
\begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} HW_2 \\ HW_1 \\ HW_0 \end{pmatrix} &= \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} HW_2 \\ HW_1 \\ HW_0 \end{pmatrix} \\
&= \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} HW_{k+2} \\ HW_{k+1} \\ HW_k \end{pmatrix} \\
&= \begin{pmatrix} 7HW_{k+2} - 7HW_{k+1} + HW_k \\ HW_{k+2} \\ HW_{k+1} \end{pmatrix} \\
&= \begin{pmatrix} HW_{k+3} \\ HW_{k+2} \\ HW_{k+1} \end{pmatrix}.
\end{aligned}$$

For the other case $n < 0$ the proof is easily attainable. Consequently, using mathematical induction on n, the proof is completed.

Note that

$$A^n = \begin{pmatrix} E_{n+1} & -7E_n + E_{n-1} & E_n \\ E_n & -7E_{n-1} + E_{n-2} & E_{n-1} \\ E_{n-1} & -7E_{n-2} + E_{n-3} & E_{n-2} \end{pmatrix}.$$

For the proof see [31].

THEOREM 17. *If we define the matrices N_{HW} and S_{HW} as follow*

$$\begin{aligned}
N_{HW} &= \begin{pmatrix} HW_2 & HW_1 & HW_0 \\ HW_1 & HW_0 & HW_{-1} \\ HW_0 & HW_{-1} & HW_{-2} \end{pmatrix}, \\
S_{HW} &= \begin{pmatrix} HW_{n+2} & HW_{n+1} & HW_n \\ HW_{n+1} & HW_n & HW_{n-1} \\ HW_n & HW_{n-1} & HW_{n-2} \end{pmatrix}.
\end{aligned}$$

then the following identity is true:

$$A^n N_{HW} = S_{HW}.$$

Proof. For the proof, we can use the following identities

$$\begin{aligned} A^n N_{HW} &= \begin{pmatrix} E_{n+1} & -7E_n + E_{n-1} & E_n \\ E_n & -7E_{n-1} + E_{n-2} & E_{n-1} \\ E_{n-1} & -7E_{n-2} + E_{n-3} & E_{n-2} \end{pmatrix} \begin{pmatrix} HW_2 & HW_1 & HW_0 \\ HW_1 & HW_0 & HW_{-1} \\ HW_0 & HW_{-1} & HW_{-2} \end{pmatrix}, \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} a_{11} &= HW_2 E_{n+1} + HW_1 (E_{n-1} - 7E_n) + HW_0 E_n, \\ a_{12} &= HW_1 E_{n+1} + HW_0 (E_{n-1} - 7E_n) + HW_{-1} E_n, \\ a_{13} &= HW_0 E_{n+1} + HW_{-1} (E_{n-1} - 7E_n) + HW_{-2} E_n, \\ a_{21} &= HW_2 E_n + HW_1 (E_{n-2} - 7E_{n-1}) + HW_0 E_{n-1}, \\ a_{22} &= HW_1 E_n + HW_0 (E_{n-2} - 7E_{n-1}) + HW_{-1} E_{n-1}, \\ a_{23} &= HW_0 E_n + HW_{-1} (E_{n-2} - 7E_{n-1}) + HW_{-2} E_{n-1}, \\ a_{31} &= HW_2 E_{n-1} + HW_1 (E_{n-3} - 7E_{n-2}) + HW_0 E_{n-2}, \\ a_{32} &= HW_1 E_{n-1} + HW_0 (E_{n-3} - 7E_{n-2}) + HW_{-1} E_{n-2}, \\ a_{33} &= HW_0 E_{n-1} + HW_{-1} (E_{n-3} - 7E_{n-2}) + HW_{-2} E_{n-2}. \end{aligned}$$

Using the Theorem 10 the proof is done. □

From Theorem 17, we have the following corollary.

Corollary 18.

(a): *Let the matrices N_{HE} and S_{HE} are defined as the following*

$$N_{HT} = \begin{pmatrix} HE_2 & HE_1 & HE_0 \\ HE_1 & HE_0 & HE_{-1} \\ HE_0 & HE_{-1} & HE_{-2} \end{pmatrix},$$

$$S_{HT} = \begin{pmatrix} HE_{n+2} & HE_{n+1} & HE_n \\ HE_{n+1} & HE_n & HE_{n-1} \\ HE_n & HE_{n-1} & HE_{n-2} \end{pmatrix},$$

so that the identity given below is true for A^n, N_{HE}, S_{HE}

$$A^n N_{HE} = S_{HE},$$

(b): *Let the matrices N_{HK} and S_{HK} are defined as the following*

$$N_{HO} = \begin{pmatrix} HK_2 & HK_1 & HK_0 \\ HK_1 & HK_0 & HK_{-1} \\ HK_0 & HK_{-1} & HK_{-2} \end{pmatrix},$$

$$S_{HO} = \begin{pmatrix} HK_{n+2} & HK_{n+1} & HK_n \\ HK_{n+1} & HK_n & HK_{n-1} \\ HK_n & HK_{n-1} & HK_{n-2} \end{pmatrix},$$

so that the identity given below is true for A^n, N_{HK}, S_{HK}

$$A^n N_{HK} = S_{HK}.$$

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