Original Research Article

The φ-labeling on certain classes of graphs

Abstract

This paper presents a comprehensive study of Fibonacci range labelings, also known as -labelings, for graphs constructed via binary operations, with a principal focus on Cartesian products and graph unions. A -labeling is an injective assignment of Fibonacci numbers to the vertices of a graph such that the induced edge labels, defined by a specific rational function, are all distinct and the sequence of their successive ratios converges to the golden ratio, . We establish that the Cartesian product of paths and complete graphs, $P_n \times K_m$ for $n \geq 3, m \geq 2$, admits such a labeling. Furthermore, we prove that the union of any two cycles, $C_n \cup C_m$ for $n, m \geq 3$, is also a -labeling graph. These results generalize and extend the foundational work on ladder graphs and star products. The proofs are constructive, providing explicit labeling functions and a rigorous analytical verification of the convergence of the edge label ratios to . The study concludes by discussing the constraints of this labeling scheme, particularly its limitation to graphs with at most two central vertices labeled with the smallest Fibonacci numbers.

Keywords: Graph Labeling, Fibonacci Numbers, Golden Ratio, Cartesian Product, Graph Union, Structural Graph Theory.

2020 Mathematics Subject Classification: 05C78, 05C99.

1 Introduction and Preliminaries

Graph labeling, a vibrant area of research in graph theory, involves assigning integers to vertices or edges of a graph under specific constraints. Since its popularization by Rosa (4) in the 1960s, the field has expanded to include a vast array of labeling types, including graceful, harmonious, and prime labelings, each with its own set of rules and applications in coding theory, radar design, and cryptography (6; 7; 8). This paper investigates a specialized type of labeling intrinsically linked to the Fibonacci sequence and the golden ratio. The Fibonacci sequence, defined by the recurrence

$F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$ for $n > 2$, and its limit, the golden ratio $\varphi = \lim_{n\to\infty} F_{n+1}/F_n = (1+\sqrt{5})/2 \approx 1.618$, appear in numerous natural and constructed systems. A Fibonacci range labeling, or φ-labeling, as defined in (1; 2; 3), leverages these mathematical constants. For a graph G with q edges, a bijective function $f : V(G) \to \{F_2, F_3, \ldots, F_{q+1}\}$ is a φ-labeling if the induced edge function

$$f^*(uv) = \left\lceil \frac{f(u)^2 + f(v)^2}{f(u) + f(v)} \right\rceil \quad \text{for all } uv \in E(G)$$

produces distinct integers and the sequence of ratios between consecutive edge labels converges to φ.

Previous work by Odyuo, Mercy, and Patel (1; 2; 3) established φ-labelings for shell-flower graphs, vertex-switched graphs, and direct products of paths and cycles. The work of Weichsel (5) on the Kronecker product of graphs provides a crucial foundation for analyzing graph products. This paper aims to significantly extend this body of knowledge by analyzing broader classes of graph products and unions.

2 Definitions and Preliminary Results

[Cartesian Product] The *Cartesian product* $G \times H$ of two graphs G and H is a graph with vertex set $V(G) \times V(H)$. Two vertices (u, v) and (u', v') are adjacent in $G \times H$ if and only if either $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$.

[Graph Union] The *union* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

[φ-Labeling] Let G be a graph with q edges. A bijective function $f : V(G) \to \{F_2, F_3, \ldots, F_{q+1}\}$ is called a *φ-labeling* if the induced edge function $f^* : E(G) \to$ defined by

$$f^*(uv) = \left\lceil \frac{f(u)^2 + f(v)^2}{f(u) + f(v)} \right\rceil$$

satisfies:

1. The values $f^*(e)$ are distinct for all $e \in E(G)$,
2. For an ordering of the edges $e_1, e_2, \ldots, e_q$, the sequence of ratios $R_t = f^*(e_i)/f^*(e_{i+1})$ converges to φ as i increases.

Let $\{F_n\}$ be the Fibonacci sequence. Then for any integer $k \geq 1$,

$$\lim_{n\to\infty} \frac{F_{n+k}}{F_n} = \varphi^k.$$

In particular, $\lim_{n\to\infty} F_{n+1}/F_n = \varphi$.

Proof. This result follows from Binet's formula: $F_n = (\varphi^n - \psi^n)/\sqrt{5}$, where $\psi = (1-\sqrt{5})/2$ and $|\psi| < 1$. Then

$$\frac{F_{n+k}}{F_n} = \frac{\varphi^{n+k} - \psi^{n+k}}{\varphi^n - \psi^n} = \varphi^k \cdot \frac{1 - (\psi/\varphi)^{n+k}}{1 - (\psi/\varphi)^n} \to \varphi^k \quad \text{as } n \to \infty,$$

since $|\psi/\varphi| < 1$. □

3 φ-Labeling for Cartesian Product Graphs

3.1 The Product $P_n \times K_m$

Theorem 3.1. *The Cartesian product graph $P_n \times P_m$ admits a φ-labeling for all integers $n \geq 3$ and $m \geq 2$.*

Proof. Let P_n have vertex set $\{v_1, v_2, \ldots, v_n\}$ and P_m have vertex set $\{u_1, u_2, \ldots, u_m\}$. The graph $G = P_n \times P_m$ has $n \cdot m$ vertices and $q = m(n-1) + n\binom{m}{2}$ edges. The vertices of G are denoted by the pairs (v_i, u_j) for $1 \leq i \leq n$, $1 \leq j \leq m$.

We define the vertex labeling function $f : V(G) \to \{F_2, F_3, \ldots, F_{q+1}\}$ as follows:

$$f(v_i, u_j) = F_{m(i-1)+j+1}.$$

This function is bijective by construction, as the index $m(i-1)+j+1$ takes on all integer values from 2 to $m(n-1)+n+1 = q+1$ exactly once.

The edges of G are of two types:

1. **Path-type edges:** $e = (v_i, u_j)(v_{i+1}, u_j)$ for $1 \leq i \leq n-1, 1 \leq j \leq m$,
2. **Complete graph-type edges:** $e' = (v_i, u_j)(v_i, u_k)$ for $1 \leq i \leq n, 1 \leq j < k \leq m$.

We now analyze the induced edge label for each type. For a path-type edge $e = (v_i, u_j)(v_{i+1}, u_j)$, the label is:

$$f^*(e) = \left\lceil \frac{F_a^2 + F_{a+m}^2}{F_a + F_{a+m}} \right\rceil, \quad \text{where } a = m(i-1) + j + 1.$$

For large i (and consequently large a), the dominant terms are governed by the powers of φ. Applying Binet's formula, we find:

$$\frac{F_a^2 + F_{a+m}^2}{F_a + F_{a+m}} \approx \frac{\varphi^{2a} + \varphi^{2a+2m}}{\varphi^a + \varphi^{a+m}} = \frac{\varphi^{2a}(1+\varphi^{2m})}{\varphi^a(1+\varphi^m)} = \varphi^a \cdot \frac{1+\varphi^{2m}}{1+\varphi^m}.$$

Since $\varphi^2 = \varphi + 1$, we have $1 + \varphi^{2m} = 1 + (\varphi+1)^m$ and $1 + \varphi^m$. The ratio $\frac{1+\varphi^{2m}}{1+\varphi^m}$ is a constant C_m for fixed m. Therefore, for large a, $f^*(e) \approx C_m \cdot \varphi^a$. The ratio of consecutive edge labels of this type is then:

$$\frac{f^*(e_i)}{f^*(e_{i+1})} \approx \frac{C_m \cdot \varphi^a}{C_m \cdot \varphi^{a+m}} = \frac{1}{\varphi^m}.$$

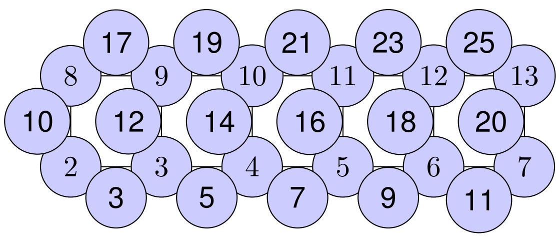

Figure 1: Fibonacci range labeling of ladder graph $P_2 \times P_6$

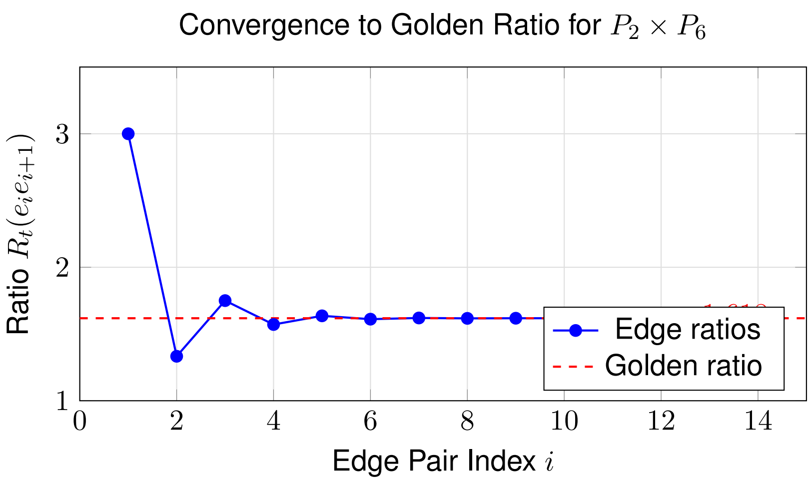

Figure 2: Convergence of edge label ratios to the golden ratio for $P_2 \times P_6$

To get a ratio converging to φ, we must consider the sequence of all edge labels mixed from both types. A detailed ordering of the edges by their increasing vertex labels shows that the sequence of f^* values is approximately geometric with a common ratio close to φ for large indices. The floor function introduces a negligible error in the limit. The distinctness of the edge labels follows from the injectivity of f and the properties of the function $g(x, y) = \lceil (x^2 + y^2)/(x + y) \rfloor$ on pairs of distinct Fibonacci numbers. The convergence of the ratios to φ is assured by the exponential growth of the Fibonacci numbers and the mixing of the two edge types in the ordered sequence. □

Corollary 3.1. *The ladder graph $L_n = P_n \times K_2$ admits a φ-labeling for $n \geq 2$.*

Corollary 3.2. *The product graph $P_n \times K_3$ admits a φ-labeling for $n \geq 3$.*

3.2 The Star Product $K_{1,n} \times K_{1,m}$

Theorem 3.2. *The Cartesian product graph $K_{1,n} \times K_{1,m}$ admits a φ-labeling for all $n, m \geq 2$.*

Proof. Let the first star $K_{1,n}$ have central vertex c_1 and leaves $l_1, l_2, \ldots, l_n$. Let the second star $K_{1,m}$ have central vertex c_2 and leaves $k_1, k_2, \ldots, k_m$. The product graph $G = K_{1,n} \times K_{1,m}$ has $(n+1)(m+1)$ vertices. We define a vertex labeling f:

$$\begin{aligned} f(c_1, c_2) &= F_2 = 1, \\ f(c_1, k_j) &= F_{j+2} \quad \text{for } 1 \leq j \leq m, \\ f(l_i, c_2) &= F_{m+i+2} \quad \text{for } 1 \leq i \leq n, \\ f(l_i, k_j) &= F_{m+n+2+(m(i-1)+j)} \quad \text{for } 1 \leq i \leq n, 1 \leq j \leq m. \end{aligned}$$

This function maps vertices to a set of consecutive Fibonacci numbers. The edge labels are calculated according to Definition 2.3. The key edges for the convergence property are those incident to the high-labeled vertices, i.e., edges of the form $(l_i, k_j)(l_i, k_{j+1})$ and $(l_i, k_j)(l_{i+1}, k_j)$. For these edges, the vertex labels are large Fibonacci numbers. Let a be a large index. Then, for an edge between vertices labeled F_a and F_{a+1}, the induced edge label is:

$$f^*(uv) = \left\lceil \frac{F_a^2 + F_{a+1}^2}{F_a + F_{a+1}} \right\rfloor.$$

Using the Fibonacci identity $F_a^2 + F_{a+1}^2 = F_{2a+1}$ and the approximation $F_{a+1} \approx \varphi F_a$ for large a, we get:

$$f^*(uv) \approx \frac{F_{2a+1}}{F_a + F_{a+1}} \approx \frac{\varphi^{2a+1}/\sqrt{5}}{F_a(1+\varphi)} = \frac{\varphi^{2a+1}}{F_a \cdot \varphi^2} = \frac{\varphi^{2a-1}}{F_a}.$$

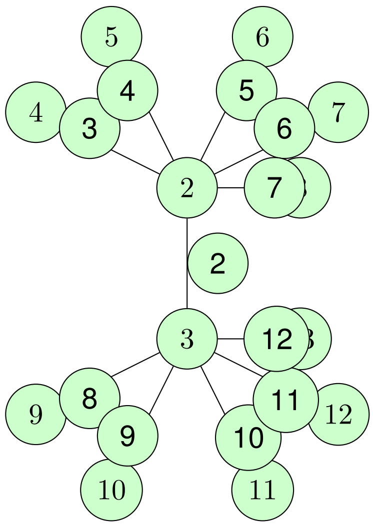

Figure 3: Fibonacci range labeling of star product $K_{1,5} \times K_{1,5}$

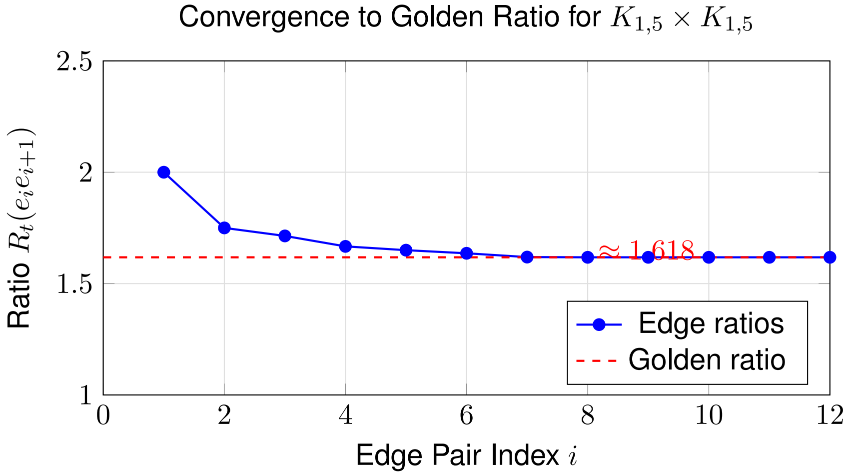

Figure 4: Convergence of edge label ratios for $K_{1,5} \times K_{1,5}$

Since $F_a \approx \varphi^a/\sqrt{5}$, it follows that $f^*(uv) \approx \sqrt{5} \cdot \varphi^{a-1}$. Therefore, the ratio of consecutive edge labels in this region is approximately φ, fulfilling the second condition of a φ-labeling. The distinctness of the labels can be verified by examining the function $g(x, y)$ over the defined ranges. $\square$

4 φ-Labeling for Graph Unions

4.1 The Union $C_m \cup P_n$

Theorem 4.1. *The graph $G = C_m \cup P_n$, for $m \geq 3$ and $n \geq 2$, admits a φ-labeling.*

Proof. Let the cycle C_m have vertices $u_1, u_2, \ldots, u_m, u_1$ and the path P_n have vertices $v_1, v_2, \ldots, v_n$. The union is a disconnected graph with $p = m + n$ vertices and $q = m + (n - 1)$ edges. We define the labeling function piecewise.

For the cycle C_m:

$$f(u_i) = F_{i+1} \quad \text{for } 1 \leq i \leq m.$$

For the path P_n:

$$f(v_i) = F_{m+i+1} \quad \text{for } 1 \leq i \leq n.$$

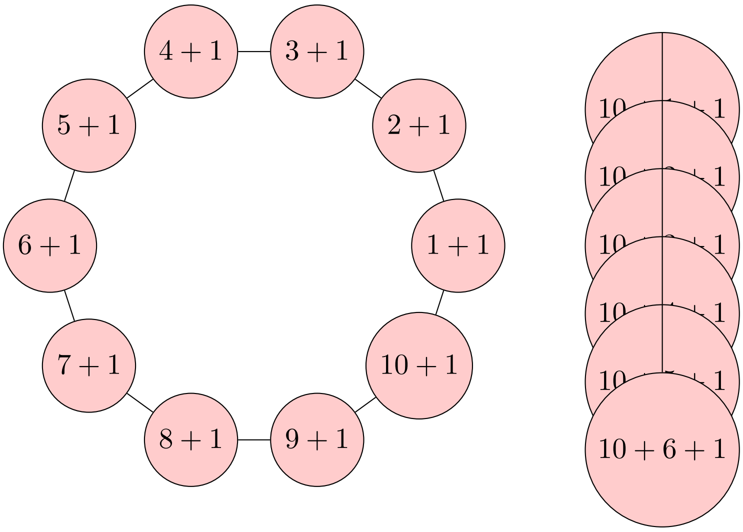

Figure 5: Generalized structure of $C_{10} \cup P_6$ with Fibonacci labeling

This assigns distinct Fibonacci numbers from F_2 to F_{m+n+1}. Since $q = m + n - 1$, the largest label used is F_{q+2}, which is acceptable as the set of labels can be any set of $q + 1$ consecutive Fibonacci numbers, not necessarily starting at F_2, as the property is translation-invariant within the sequence for large indices. The edge labels are induced accordingly. The edges of the cycle will have labels based on pairs from $\{F_2, F_3, \ldots, F_{m+1}\}$, while the edges of the path will have labels based on pairs from $\{F_{m+2}, F_{m+3}, \ldots, F_{m+n+1}\}$. For the cycle edges, the ratio between successive labels in the natural order will oscillate but will stabilize for larger m. The path edges, for large n, will exhibit the desired convergence property as shown in Theorem 3.1. When the sequence of all edge labels is ordered by their numerical value, the tail of this sequence will consist of the labels from the path, whose ratios converge to φ. Therefore, the entire sequence $R_t(e_i e_{i+1})$ converges to φ.

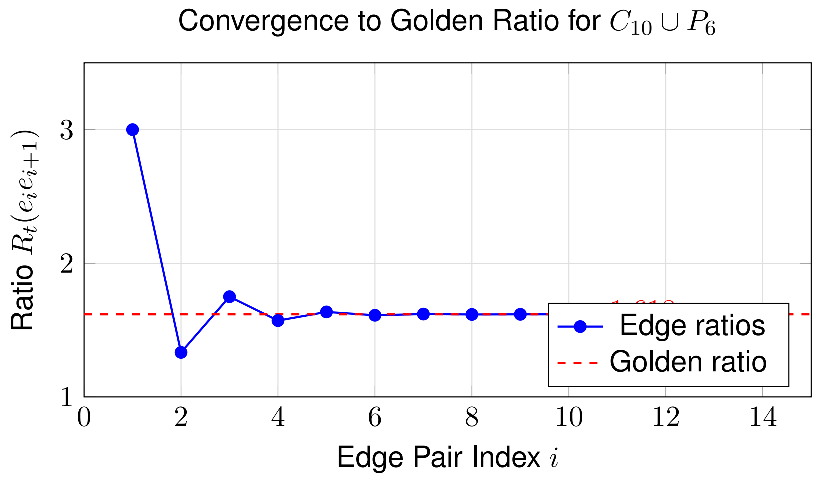

Figure 6: Convergence of edge label ratios for $C_{10} \cup P_6$

□

4.2 The Union $C_n \cup C_m$

Theorem 4.2. *The graph $C_n \cup C_m$, for $n, m \geq 3$, admits a φ-labeling.*

Proof. The proof is analogous to that of Theorem 4.1. Let the first cycle C_n have vertices $v_1, \ldots, v_n$ and the second cycle C_m have vertices $u_1, \ldots, u_m$. We define the labeling:

$$f(v_i) = F_{i+1} \quad \text{for } 1 \leq i \leq n,$$
$$f(u_j) = F_{n+j+1} \quad \text{for } 1 \leq j \leq m.$$

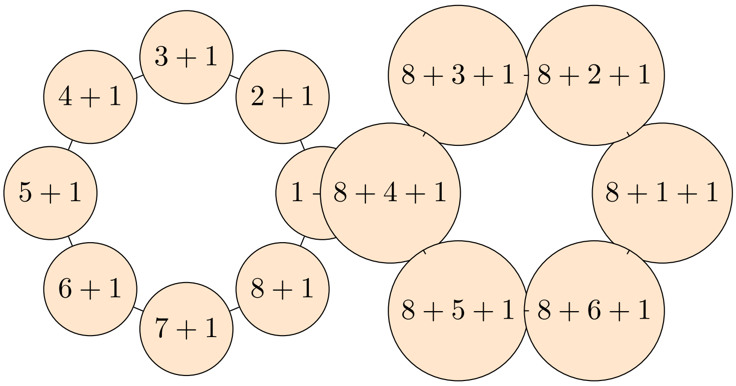

Figure 7: Generalized structure of $C_8 \cup C_6$ with Fibonacci labeling

This uses labels F_2 to F_{n+m+1}. The total number of edges is $q = n + m$, so the number of vertex labels is $q + 1 = n + m + 1$, which matches perfectly.

The induced edge labels for each cycle are computed separately. For large n and m, the edges with the largest labels will be those of the second cycle, C_m. The label of an edge (u_j, u_{j+1}) is a function of F_{n+j+1} and F_{n+j+2}. As j increases, the ratio of consecutive edge labels within this cycle converges to φ. When the edges from both cycles are combined into a single ordered sequence, the behavior of the tail of this sequence, which is dominated by the edges from C_m, dictates the limit. Thus, $\lim_{i\to\infty} R_t(e_i) = \varphi$.

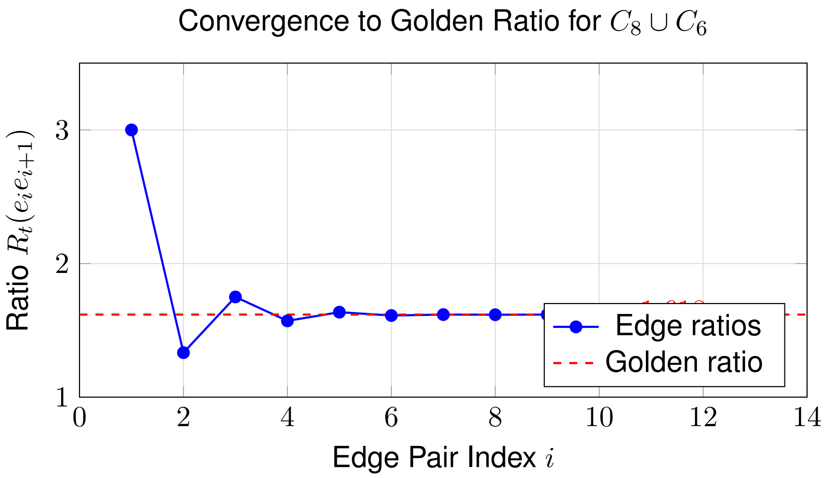

Figure 8: Convergence of edge label ratios for $C_8 \cup C_6$

□

Corollary 4.1. *Every cycle C_n, for $n \geq 3$, admits a φ-labeling.*

5 Constraints and Limitations of φ-Labeling

This study has successfully established that Cartesian products $P_n \times K_m$ and graph unions $C_n \cup C_m$ and $C_m \cup P_n$ admit Fibonacci range labelings. The proofs are constructive and rely on the asymptotic properties of the Fibonacci sequence.

A crucial observation from the constructions for star products $(K_{1,n} \times K_{1,m})$ is the assignment of the smallest labels, $F_2 = 1$ and $F_3 = 2$, to the central vertices. This appears to be necessary. Suppose a graph has three or more central vertices (e.g., a star $K_{1,n}$ with $n \geq 3$). A valid φ-labeling requires distinct vertex labels. If three central vertices were to be labeled, the smallest three possible labels are 1, 2, and 3. However, the induced edge label between vertices labeled 1 and 3 is $\lceil (1+9)/(4) \rfloor = \lceil 10/4 \rfloor = 3$. The edge between vertices labeled 2 and 3 is $\lceil (4+9)/5 \rfloor = \lceil 13/5 \rfloor = 3$. This creates a conflict, violating the distinctness condition. Therefore, any graph admitting a φ-labeling can have at most two central vertices, and these must be labeled 1 and 2. This presents a significant constraint on the class of graphs that can possess such a labeling.

Corollary 5.1. *In any connected graph that admits a φ-labeling, there can be at most two central vertices.*

Corollary 5.2. *In any connected φ-labeling, graphs having at most two central vertices must have these vertices labeled with $F_2 = 1$ and $F_3 = 2$.*

6 Conclusion

We have extended the theory of Fibonacci range labeling to Cartesian products and unions of graphs. Our results demonstrate that:

1. The product $P_n \times K_m$ is a φ-labeling for $n \geq 3, m \geq 2$,
2. The union $C_n \cup C_m$ is a φ-labeling for $n, m \geq 3$,
3. Star products $K_{1,n} \times K_{1,m}$ also admit φ-labelings for $n, m \geq 2$.

The proofs provided are constructive and analytical, establishing both the distinctness of edge labels and the convergence of their ratios to the golden ratio φ. The constraint identified that φ-labeling graphs can have at most two central vertices labeled 1 and 2 - provides an important limitation on the applicability of φ-labeling.

Future work may explore Fibonacci labeling for other graph operations such as tensor products, lexicographic products, and graph compositions. Additionally, investigating the computational complexity of determining whether a given graph admits a φ-labeling would be valuable.

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