Original Research Article

On Certain Relation of Sums of Squares and Quartic $\sum_{r=1}^{2k} a_r^4 + kd^4 = 2\sum_{r=1}^{2k-1}(a_r a_{r+1} + d^2)^2$

Abstract

Let k and d be a given positive integers and suppose that a_r is a given sequence. In this current study, we investiagate a diophantine identity relating the sums of squares and quartic from specific sequences to a variable d. In particular, the diophantine identity $\sum_{r=1}^{2k} a_r^4 + kd^4 = 2\sum_{r=1}^{2k-1}(a_r a_{r+1} + d^2)^2$ is developed and introduced. The objective of this research is to determine the conditions under which integer solutions for (a_r, d) exist within this diophantine equation. The methodology involves, decomposing polynomials, factorizing polynomials, and exploring the solution set of the given equation.

Keywords: Diophantine Equation; Sums of Squares, Integer Sequence, Quartic

2010 Mathematics Subject Classification: 53C25; 83C05; 57N16

1 Introduction

The study of polynomial identities seeking integer solutions, is a very significant area in the field of number theory. Historically, these equations have attracted the attention of number theorist due to their intrinsic challenge and significance in understanding the properties of integers. Despite the extensive studies of various Diophantine equations, including renowned challenges like Fermat's Last Theorem, Ramanujan Nagell equation, and Lebesgue Nagell equation, as well as studies focusing on polynomials of degree less than 5, the specific examination of the diophantine equation $\sum_{r=1}^{2k} a_r^4 + kd^4 = 2\sum_{r=1}^{2k-1}(a_r a_{r+1} + d^2)^2$ remains largely unknown. Recent research has mainly focussed into the intricacies of diophantine equation with degrees less than 5, as referenced in [1, 2,3,10,11,12]. For a comprehensive research on studies related to Fermat's Last Theorem and Ramanujan Nagell equations, readers are encouraged to explore[4,5,6,7,8,9,13,14,15,16]. Within the existing body of work, the literature concerning the diophantine equation $\sum_{r=1}^{2k} a_r^4 + kd^4 = 2\sum_{r=1}^{2k-1}(a_r a_{r+1} + d^2)^2$ remains largely unexplored. This study aims to contribute to this knowledge gap by introducing and developing the formula $\sum_{r=1}^{2k} a_r^4 + kd^4 = 2\sum_{r=1}^{2k-1}(a_r a_{r+1} + d^2)^2$, seeking to enhance our comprehension of this specific diophantine equation within the broader aspect of mathematical research.

2 Main Results

In the following sections, we begin by articulating our observations as a conjecture, and subsequently, we proceed to obtain solutions for particular instances of the aforementioned diophantine equation.

Conjecture 2.1. *For any integer $k \geq 1$, the diophantine equation*

$$\sum_{r=1}^{2k} a_r^4 + kd^4 = 2\sum_{r=1}^{2k}(a_r a_{r+1} + d^2)^2 \cdots (1)$$

admits solutions in integers if $a_n - a_{n-1} = a_{n-1} - a_{n-2} = \cdots = a_2 - a_1 = d$

In the subsequent sections, the focus of this investigation revolves around identifying the values of the variables $(k, a_1, a_2, \cdots, a_n, d)$ that fulfill the conditions of equation (1). Consequently, distinct cases have been established.

Theorem 2.2. *Consider equation (1) satisfying the condition $(k, a_1, a_2, d) = (1, a_1, a_2, d)$ Then, the diophantine equation*

$$a_1^4 + a_2^4 + d^4 = (a_1 a_2 + d^2)^2 + (a_1 a_2 + d^2)^2 = 2(a_1 a_2 + d^2)^2$$

has solution in integers if $a_2 - a_1 = d$.

Proof. Assume that $a_2 = a_1 + d$ and Consider the equation $a_1^4 + a_2^4 + d^2 = (a_1 a_2 + d^2)^2 + (a_1 a_2 + d^2)^2 = 2(a_1 a_2 + d^2)^2 \cdots (2.1)$. The, left hand side of equation (2.1) expressed as

$$a_1^4 + a_2^4 + d^4 = a_1^4 + (a_1 + d)^4 + d^4$$

simplifies to

$$2a_1^4 + 4a_1^3 d + 6a_1^2 d^2 + 4a_1 d^3 + 2d^4 \cdots (2.1.1).$$

Decomposing equation $(2.1.1)$ and modifying into double sum of a square, we obtain,

$$(a_1^4 + 2a_1^3 d + 3a_1^2 d^2 + 2a_1 d^3 + d^4) + (a_1^4 + 2a_1^3 d + 3a_1^2 d^2 + 2a_1 d^3 + d^4)$$
$$= (a_1^2 + a_1 d + d^2)^2 + (a_1^2 + a_1 d + d^2)^2 = ((a_1(a_1 + d) + d^2))^2 + ((a_1(a_1 + d) + d^2))^2$$
$$= (a_1 a_2 + d^2)^2 + (a_1 a_2 + d^2)^2 = 2(a_1 a_2 + d^2)^2.$$

This complete the proof.

□

Theorem 2.3. *Consider equation* (1) *satisfying the condition* $(k, a_1, a_2, a_3, a_4, d) = (2, a_1, a_2, a_3, a_4, d)$ *Then, the diophantine equation*

$$a_1^4 + a_2^4 + a_3^4 + a_4^4 + 2d^4 = 2((a_1a_2 + d^2)^2 + (a_3a_4 + d^2)^2)$$

has solution in integers if $a_2 - a_1 = a_3 - a_4 = d$.

Proof. Let $a_2 = a_1 + d, a_3 = a_1 + 2d, a_4 = a_1 + 3d$ and Consider the equation

$$a_1^4 + a_2^4 + a_3^4 + a_4^4 + 2d^4 = 2((a_1a_2 + d^2)^2 + (a_3a_4 + d^2)^2) \cdots (2.3).$$

The, left hand side written as

$$a_1^4 + a_2^4 + a_3^4 + a_4^4 + 2d^4 = a_1^4 + (a_1 + d)^4 + (a_1 + 2d)^4 + (a_1 + 3d)^4 + 2d^4$$

reduces to

$$4a_1^4 + 24a_1^3d + 84a_1^2d^2 + 144a_1d^3 + 100d^4 \cdots (2.3.1).$$

Breaking equation $(2.3.1)$ and rearranging into double sums of sums of two squares, we get,

$$(2a_1^4 + 4a_1^3d + 6a_1^2d^2 + 4a_1d^3 + 2d^4) + (2a_1^4 + 20a_1^3d + 78a_1^2d^2 + 140a_1d^3 + 98d^3)$$
$$= (a_1^4 + 2a_1^3d + 3a_1^2d^2 + 2a_1d^3 + d^4) + (a_1^4 + 2a_1^3d + 3a_1^2d^2 + 2a_1d^3 + d^4)$$
$$+(a_1^4 + 10a_1^3d + 39a_1^2d^2 + 70a_1d^3 + 49d^3) + (a_1^4 + 10a_1^3d + 39a_1^2d^2 + 70a_1d^3 + 49d^3)$$
$$= (a_1^2 + a_1d + d^2)^2 + (a_1^2 + a_1d + d^2)^2 + (a_1^2 + 5a_1d + 7d^2)^2 + (a_1^2 + 5a_1d + 7d^2)^2$$
$$= ((a_1(a_1 + d) + d^2))^2 + ((a_1(a_1 + d) + d^2))^2 + ((a_1 + 2d)(a_1 + 3d) + d^2)^2 + ((a_1 + 2d)(a_1 + 3d) + d^2)^2$$
$$= (a_1a_2 + d^2)^2 + (a_1a_2 + d^2)^2 + (a_3a_4 + d^2)^2 + (a_3a_4 + d^2)^2$$
$$= 2((a_1a_2 + d^2)^2 + (a_3a_4 + d^2)^2).$$

This complete the proof.

□

Theorem 2.4. *Consider equation* (1) *satisfying the condition* $(k, a_1, a_2, a_3, a_4, a_5, a_6, d) = (3, a_1, a_2, a_3, a_4, a_5, a_6, d)$ *Then, the diophantine equation*

$$a_1^4 + a_2^4 + a_3^4 + a_4^4 + a_5^4 + a_6^4 + 3d^4 = 2((a_1a_2 + d^2)^2 + (a_3a_4 + d^2)^2) + (a_5a_6 + d^2)^2)$$

has solution in integers if $a_2 - a_1 = a_3 - a_4 = a_5 - a_4 = a_6 - a_5 = d$.

Proof. Let $a_2 = a_1 + d, a_3 = a_1 + 2d, a_4 = a_1 + 3d, a_5 = a_1 + 4d, a_6 = a_1 + 5d$ and Consider the equation

$$a_1^4 + a_2^4 + a_3^4 + a_4^4 + a_5^4 + a_6^4 + 3d^4 = 2((a_1a_2 + d^2)^2 + (a_3a_4 + d^2)^2) + (a_5a_6 + d^2)^2) \cdots (2.4).$$

The, left hand side written as

$$a_1^4 + a_2^4 + a_3^4 + a_4^4 + a_5^4 + a_6^4 + 3d^4 = a_1^4 + (a_1 + d)^4 + (a_1 + 2d)^4 + (a_1 + 3d)^4 + (a_1 + 4d)^4 + (a_1 + 5d)^4 + 3d^4$$

reduces to

$$6a_1^4 + 60a_1^3d + 330a_1^2d^2 + 900a_1d^3 + 982d^4 \cdots (2.4.1).$$

Breaking equation $(2.4.1)$ and rearranging into double sums of sums of two squares, we get,

$$(2a_1^4 + 4a_1^3d + 6a_1^2d^2 + 4a_1d^3 + 2d^4) + (2a_1^4 + 20a_1^3d + 78a_1^2d^2 + 140a_1d^3 + 98d^4) + (2a_1^4 + 36a_1^3d + 246a_1^2d^2 + 756a_1d^3 + 882d^4)$$
$$= (a_1^4 + 2a_1^3d + 3a_1^2d^2 + 2a_1d^3 + d^4) + (a_1^4 + 2a_1^3d + 3a_1^2d^2 + 2a_1d^3 + d^4)$$
$$+(a_1^4 + 10a_1^3d + 39a_1^2d^2 + 70a_1d^3 + 49d^3) + (a_1^4 + 10a_1^3d + 39a_1^2d^2 + 70a_1d^3 + 49d^3)$$

$$+(a_1^4 + 18a_1^3d + 123a_1^2d^2 + 378a_1d^3 + 441d^4) + (a_1^4 + 18a_1^3d + 123a_1^2d^2 + 378a_1d^3 + 441d^4)$$
$$= (a_1^2+a_1d+d^2)^2+(a_1^2+a_1d+d^2)^2+(a_1^2+5a_1d+7d^2)^2+(a_1^2+5a_1d+7d^2)^2+(a_1^2+9a_1d+21d^2)^2+(a_1^2+9a_1d+21d^2)^2$$
$$= ((a_1(a_1 + d) + d^2))^2 + ((a_1(a_1 + d) + d^2))^2 + ((a_1 + 2d)(a_1 + 3d) + d^2)^2$$
$$+((a_1 + 2d)(a_1 + 3d) + d^2)^2((a_1 + 4d)(a_1 + 5d) + d^2)^2((a_1 + 4d)(a_1 + 5d) + d^2)^2$$
$$= (a_1a_2 + d^2)^2 + (a_1a_2 + d^2)^2 + (a_3a_4 + d^2)^2 + (a_3a_4 + d^2)^2 + (a_4a_5 + d^2)^2 + (a_4a_5 + d^2)^2$$
$$= 2((a_1a_2 + d^2)^2 + (a_3a_4 + d^2)^2 + (a_4a_5 + d^2)^2).$$

This complete the proof.

□

Theorem 2.5. *Consider equation* (1) *satisfying the condition* $(k, a_1, a_2, a_3, a_4, a_5, a_6, d) = (4, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, d)$ *Then, the diophantine equation*

$$a_1^4+a_2^4+a_3^4+a_4^4+a_5^4+a_6^4+a_5^4+a_8^4+4d^4 = 2((a_1a_2+d^2)^2+(a_3a_4+d^2)^2)+(a_5a_6+d^2)^2)+(a_6a_7+d^2)^2)$$

has solution in integers if $a_2 - a_1 = a_3 - a_4 = a_5 - a_4 = a_6 - a_5 = a_7 - a_6 = d.$

Proof. Let $a_2 = a_1+d, a_3 = a_1+2d, a_4 = a_1+3d, a_5 = a_1+4d, a_6 = a_1+5d, a_7 = a_1+6d, a_8 = a_1+7d$ and Consider the equation

$$a_1^4+a_2^4+a_3^4+a_4^4+a_5^4+a_6^4+a_5^4+a_8^4+4d^4 = 2((a_1a_2+d^2)^2+(a_3a_4+d^2)^2)+(a_5a_6+d^2)^2)+(a_6a_7+d^2)^2).$$

The, left hand side written as

$$a_1^4+a_2^4+a_3^4+a_4^4+a_5^4+a_6^4+a_7^4+a_8^4+4d^4 = a_1^4+(a_1+d)^4+(a_1+2d)^4+(a_1+3d)^4+(a_1+4d)^4+(a_1+5d)^4+(a_1+6d)^4+(a_1+7d)^4+4d^4$$

reduces to

$$8a_1^4 + 112a_1^3d + 840a_1^2d^2 + 3136a_1d^3 + 4680d^4 \cdots (2.5.1).$$

Breaking equation $(2.5.1)$ and rearranging into double sums of sums of two squares, we get,

$$(2a_1^4 + 4a_1^3d + 6a_1^2d^2 + 4a_1d^3 + 2d^4) + (2a_1^4 + 20a_1^3d + 78a_1^2d^2 + 140a_1d^3 + 98d^4)$$
$$+(2a_1^4 + 36a_1^3d + 246a_1^2d^2 + 756a_1d^3 + 882d^4) + (2a_1^4 + 56a_1^3d + 510a_1^2d^2 + 2236a_1d^3 + 3698d^4)$$
$$= (a_1^4+2a_1^3d+3a_1^2d^2+2a_1d^3+d^4)+(a_1^4+2a_1^3d+3a_1^2d^2+2a_1d^3+d^4)+(a_1^4+10a_1^3d+39a_1^2d^2+70a_1d^3+49d^3)$$
$$+(a_1^4+10a_1^3d+39a_1^2d^2+70a_1d^3+49d^3)+(a_1^4+28a_1^3d+255a_1^2d^2+1118a_1d^3+1849d^3)+(a_1^4+28a_1^3d+255a_1^2d^2+1118a_1d^3+1849d^3)$$
$$+(a_1^4 + 18a_1^3d + 123a_1^2d^2 + 378a_1d^3 + 441d^4) + (a_1^4 + 18a_1^3d + 123a_1^2d^2 + 378a_1d^3 + 441d^4)$$
$$= (a_1^2 + a_1d + d^2)^2 + (a_1^2 + a_1d + d^2)^2 + (a_1^2 + 5a_1d + 7d^2)^2 + (a_1^2 + 5a_1d + 7d^2)^2$$
$$+(a_1^2 + 9a_1d + 21d^2)^2 + (a_1^2 + 9a_1d + 21d^2)^2 + (a_1^2 + 13a_1d + 43d^2)^2 + (a_1^2 + 13a_1d + 43d^2)^2$$
$$= ((a_1(a_1 + d) + d^2))^2 + ((a_1(a_1 + d) + d^2))^2 + ((a_1 + 2d)(a_1 + 3d) + d^2)^2 + ((a_1 + 2d)(a_1 + 3d) + d^2)^2$$
$$+((a_1+4d)(a_1+5d)+d^2)^2+((a_1+4d)(a_1+5d)+d^2)^2+((a_1+5d)(a_1+6d)+d^2)^2+((a_1+5d)(a_1+6d)+d^2)^2$$
$$= (a_1a_2+d^2)^2+(a_1a_2+d^2)^2+(a_3a_4+d^2)^2+(a_3a_4+d^2)^2+(a_4a_5+d^2)^2+(a_4a_5+d^2)^2+(a_6a_7+d^2)^2))+(a_6a_7+d^2)^2))$$
$$= 2((a_1a_2 + d^2)^2 + (a_3a_4 + d^2)^2 + (a_4a_5 + d^2)^2) + (a_6a_7 + d^2)^2).$$

This complete the proof.

□

3 Conclusion

In conclusion, the integer solution of the diophantine equation $\sum_{r=1}^{2k} a_r^4 + kd^4 = 2\sum_{r=1}^{2k-1}(a_r a_{r+1} + d^2)^2$ under the specified conditions of a common difference d between consecutive terms $a_n, a_{n-1}, \cdots, a_2, a_1$ where $a_n - a_{n-1} = a_{n-1} - a_{n-2} = \cdots = a_2 - a_1 = d$ has been achieved for $k \leq 4$. This solution provides valuable insights into the relation among the sequence terms, enhancing our understanding of the inherent patterns and structures within the equation. For future investigations, it is recommended to explore extensions of this diophantine equation by proving the general case for conjecture (1).

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