Compactness of pseudo-differential type operators involving fractional Fourier transform

Abstract

Some results on compactness for pseudo-differential type operators $A(x,\Delta')$ and $\mathscr{A}(x,\Delta')$ are investigated. Norms of pseudo-differential type operators modulo compact operators are discussed. **Keywords;Fractional Fourier transform, Generalized schwartz space, Sobolev type spaces, pseudo-differential type operators.**
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1 Introduction and motivation

The Fourier transform of a function $\varphi \in L_1(\mathbb{R})$ is defined by

$$\widehat{\varphi}(\eta) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\eta\zeta} \varphi(\zeta) d\zeta, \quad \forall \eta \in \mathbb{R}$$

and its inverse Fourier transform is as follows:

$$\varphi(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\eta\zeta} \widehat{\varphi}(\eta) d\eta.$$

The term "pseudo-differential operators"[1, 2, 3, 4] has a fairly broad definition and covers such topics as harmonic analysis, partial differential equation, geometry, mathematical physics, microlocal analysis, time-frequency analysis, imaging, computations, and quantum mechanics. In mathematics, natural sciences, medicine, scientific computing, and engineering, current trends and novel applications are highlighted. The emphasis is on contemporary developments in different branches of engineering, mathematical sciences, the natural sciences, medicine, scientific computers.

In reality, Kohn-Nirenberg and Hörmander [5] were the ones who first introduced the pseudo-differential calculus, and later authors expanded on it, primarily in a local context, to examine local regularity and local solvability of PDEs.

Pseudo-differential operators on $\mathbb{R}_+$ are standard or conventional generalizations of partial differential operators or ordinary differential operators and singular integrals.

Many faculties, scientists, Ph.D students and researchers of other field developed the theory of pseudo-differential operators with the help of different types of integral operators like Fourier transforms (see [6, 7]), Hankel transform (see [8, 9, 10]), Fourier Bessel Transform on $\mathbb{R}_+$ (see [11, 12]), Weinstein transform (see [13]), Laguerre hypergroups (see [14]) and Jacobi differential operators (see [15]).

From 19th century Fourier analysis is a most frequently used tools in signal processing and in any other scientific studies/streams [16, 17, 18, 19]. In the pure mathematics and apllied mathematics literature, a generalized concept of the Fourier transform well known as the fractional Fourier transform was considered in 1980-1987, by Mcbride, Kerr and Namias [20, 21], The Kernels of the two operators differ in a ratation by an angle θ in the time-frequency domain. From 1980s, The FrFt has been independently reinvented by a number of research workers and faculty members across the world.
Applications of these transforms are in the design of lens, analysis of laser cavity, study of wave propagation in quadratic refractive index medium when the system is axially symmetric.
The research area of many pure mathematicians / applied mathematicians has fractional Fourier transform. Fractional Fourier transform [22, 23, 24, 25] has been defined as follows:

$$(\mathscr{F}^{\theta}\varphi)(\xi) = \widehat{\varphi}^{\theta}(\xi) = \int_{\mathbb{R}} K^{\theta}(x,\xi)\varphi(x)dx \tag{1}$$

$$K^{\theta}(x,\xi)=\begin{cases} C^{\theta}e^{\frac{i(x^2+\xi^2)\cot\theta}{2}-ix\xi\csc\theta}, & \theta\neq n\pi, n\varepsilon\mathbb{Z}\\ \frac{1}{\sqrt{2\pi}}e^{-ix\xi}, & \theta=\frac{\pi}{2}\\ \delta(x-\xi), & \theta=2n\pi\\ \delta(x+\xi), & \theta=(2n+1)\pi, \end{cases}$$

$C^{\theta}=\sqrt{\frac{1-i\cot\theta}{2\pi}}$ and studied some properties of this transform.
The corresponding inversion formula of $(\mathscr{F}^{\theta}\varphi)(\xi)$ is defined in the following ways

$$\varphi(x)=\int_{\mathbb{R}}\overline{K^{\theta}(x,\xi)}(\mathscr{F}^{\theta}\varphi)(\xi)d\xi \tag{2}$$

$$\overline{K^{\theta}(x,\xi)}=\overline{C^{\theta}}e^{\frac{-i(x^2+\xi^2)\cot\theta}{2}+ix\xi\csc\theta}$$

and $\overline{C^{\theta}}=\sqrt{\frac{1+i\cot\theta}{2\pi}}=C^{-\theta}$.
Hence, $\overline{K^{\theta}(x,\xi)}=K^{-\theta}(x,\xi)$.
It implies that the inverse of a FrFT with the parameter θ is the FrFT with the parameter $-\theta$.

Definition 1. *A tempered distribution φ belongs to the Sobolev type space $H^s_\alpha(\mathbb{R})$, and $s\varepsilon\mathbb{R}$ if its fractional Fourier transform $\mathscr{F}_\alpha\varphi$ corresponding to a locally integrable function $(\mathscr{F}_\alpha\varphi)(\xi)$ over $\mathbb{R}$ such that*

$$^{\alpha}\|\varphi\|_s=\left(\int_{\mathbb{R}}\left|(1+|\xi|^2)^{\frac{s}{2}}(\mathscr{F}_\alpha\varphi)(\xi)\right|^2d\xi\right)^{\frac{1}{2}}<\infty. \tag{3}$$

This space is complete with respect to the norm $^{\alpha}\|\varphi\|_s$.

Definition 2. *The space $\mathscr{G}$, the so-called space of smooth functions of rapid descent, is defined as follows: φ is a member of $\mathscr{G}$ iff it is a complex valued C^∞-function on $\mathbb{R}$ and for every choice of β and γ of non-negative integers, it satisfies*

$$\Gamma_{\beta,\gamma}(\varphi)=\sup_{x\in\mathbb{R}}\left|x^{\beta}\frac{d^{\gamma}\varphi(x)}{dx^{\gamma}}\right|<\infty. \tag{4}$$

Lemma 1. *(Peetre) For any real number s and for all $\xi,\eta\in\mathbb{R}$, the estimate*

$$\left(\frac{1+|\xi|^2}{1+|\eta|^2}\right)^{s}\leq 2^{s}(1+|\xi-\eta|^2)^{|s|} \tag{5}$$

is satisfied.

Proof. See [6]. □

2 Symbols

Let $a(x,\xi)$ be a complex valued function defined over $\mathbb{R}\times\mathbb{R}$ and the $a(x,\xi)\in C^{\infty}(\mathbb{R}\times\mathbb{R})$ is said to be an element of the class Λ if and only if $a(x,t\xi)=a(x,\xi)$ for $t>0$,

$$\lim_{|x|\to\infty} a(x,\xi)=a(\infty,\xi)$$

exists for $\xi\in\mathbb{R}-\{0\}$ and $a(\infty,\xi)$ is C^{∞}-function.
We define $a'(x,\xi)=a(x,\xi)-a(\infty,\xi)$ and assume the estimate as $\forall\ l,p,q\in\mathbb{N}$, there exits $D_{l,p,q}>0$ such that

$$(1+|x|^2)^l\left|D_x^p D_\xi^q a'(x,\xi)\right|<D_{l,p,q}\quad \text{for all } x\in\mathbb{R}. \tag{6}$$

3 The pseudo-differential type operator $A(x,\Delta')$

In 1965, the pseudo-differential operators were studied by Kohn and Nirenberg [5] and Hormander [26] by using the theory of Fourier transform and then seeing its importance in the theory of partial differential equations [27, 2, 7]. Motivated by the works of Zaidman [7] and Agrawal, Shekhar [23, 24, 25], we define the pseudo-differential type operator $\mathsf{A}(x,\Delta')$ involving the fractional Fourier transform for any function $\varphi\in\mathscr{G}$ and $a(x,\xi)\in\Lambda$ [23, 24]as follows.
Let $a(x,\xi)=a(\infty,\xi)+e^{\frac{i}{2}x^2\cot\alpha}a'(x,\xi)$ be a symbol and as previously, $\forall\ \lambda,\ \xi\in\mathbb{R}$

$$\hat{a}'_\alpha(\lambda,\xi)=\int_{\mathbb{R}}K_\alpha(x,\lambda)a'(x,\xi)dx. \tag{7}$$

Since the fractional Fourier transform is a continous linear map of $\mathscr{G}$ onto itself. This implies that $\hat{a}'_\alpha(\lambda,\xi)\in\mathscr{G}$ uniformly for $\xi\in\mathbb{R}$. Let us define, for any $\varphi\in\mathscr{G}$ and x $\in\mathbb{R}$, a function $\vartheta(x)=[\mathsf{A}(x,\Delta')\varphi](x)$, by

$$[A(x,\Delta')\varphi](x)=\int_{\mathbb{R}}\overline{K_\alpha(x,\xi)}G_\alpha(\xi)d\xi\quad\forall\quad\xi\in\mathbb{R}. \tag{8}$$

The function $G_\alpha(\xi)$ is given by

$$G_\alpha(\xi)=a(\infty,\xi)\hat{\varphi}_\alpha(\xi)+C'_\alpha\int_{\mathbb{R}}e^{-i(\eta^2-\xi\eta)\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\xi)\hat{\varphi}_\alpha(\eta)d\eta. \tag{9}$$

Evidently, it has to be proved that $G_\alpha(\xi)$ is fractional Fourier transformable, in fact, we have $G_\alpha(\xi)\in L_1(\mathbb{R})$ as

$$|a(\infty,\xi)\widehat{\varphi}_\alpha(\xi)|\le\max_{|\xi|=1}|a(\infty,\xi)||\widehat{\varphi}_\alpha(\xi)|$$

then obviously, it is sufficient to show that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\widehat{a}'_\alpha(\xi-\eta,\xi)\widehat{\varphi}_\alpha(\eta)|d\eta d\xi < \infty.$$

Since

$$\hat{a}'_\alpha(\xi-\eta,\xi) = \int_{\mathbb{R}} K_\alpha(x,\xi-\eta)a'(x,\xi)dx,$$

in fact,

$$\int_{\mathbb{R}} |\hat{a}'_\alpha(\xi-\eta,\xi)\widehat{\varphi}_\alpha(\eta)|d\eta \leq D_l \int_{\mathbb{R}} (1+|\xi-\eta|^2\csc^2\alpha)^{-\frac{l}{2}}|\widehat{\varphi}_\alpha(\eta)|d\eta.$$

This last expression is the convolution between $(1+|\xi|^2\csc^2\alpha)^{-\frac{l}{2}}$ and $\hat{\varphi}_\alpha(\xi)$ both integrable for l sufficiently large.
Hence

$$\int_{\mathbb{R}} |\hat{a}'_\alpha(\xi-\eta,\eta)||\widehat{\varphi}_\alpha(\eta)|d\eta < \infty.$$

Thus $\hat{A}_\alpha(x,\Delta')\varphi$ is continous and bounded on $\mathbb{R}$. Hence we can say that

$$[\mathscr{F}_\alpha(A(x,\Delta'))\varphi](\xi) = a(\infty,\xi)\hat{\varphi}_\alpha(\xi) + C'_\alpha \int_{\mathbb{R}} e^{-i(\eta^2-\xi\eta)\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\eta)\widehat{\varphi}_\alpha(\eta)d\eta$$

is verified the fractional Fourier transform being in $\mathscr{G}'$.

4 The pseudo-differential type operator $\mathscr{A}(x,\Delta')$

In this section, let $a(x,\xi)\in\Lambda$ be a symbol. We define the pseudo-differential type operator $\mathscr{A}(x,\Delta')$ associated with the symbol $a(x,\xi)$ of $\mathscr{G}$ in $\mathscr{G}'$ by means of the formula [23, 24]

$$\left[\mathscr{A}(x,\Delta')\varphi\right](x) = \int_{\mathbb{R}} \overline{K_\alpha(x,\xi)}H_\alpha(\xi)d\xi. \tag{10}$$

Where, for $\varphi\in\mathscr{G}$, the function $H_\alpha(\xi)$ is defined by the relation

$$H_\alpha(\xi) = a(\infty,\xi)\widehat{\varphi}_\alpha(\xi) + C'_\alpha \int_{\mathbb{R}} e^{-i(\eta^2-\xi\eta)\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\eta)\widehat{\varphi}_\alpha(\eta)d\eta \tag{11}$$

$\forall\ \varphi\in\mathscr{G}$ and $\xi\in\mathbb{R}$.

With the same proof used for $A(x,\Delta')$ we have; the function $\mathscr{A}(x,\Delta')\varphi$ is continous and bounded for $x\in\mathbb{R}$. Besides, we see that if the symbol $a(x,\xi)$ does not depend on x, we have $A(\Delta') = \mathscr{A}(\Delta')$.

5 Compactness of Pseudo-differential type Operator

Let $a(x,\xi)$, $b(x,\xi)$ and $c(x,\xi)=a(x,\xi)b(x,\xi)$ be three symbols, and $A(x,\Delta')$, $B(x,\Delta'),C(x,\Delta')$ the associated pseudo-differential type operators.
Note 1.: Let $a(x,\xi),b(x,\xi)$ be two symbols such that $a(x,\xi)b(x,\xi)=0 \ \ \forall\, x,\xi \neq 0 \in \mathbb{R}$. Then the $A(x,\Delta')B(x,\Delta')$ is compact in $L^2(\mathbb{R})$.
In fact, $A(x,\Delta')B(x,\Delta')-C(x,\Delta')$ is compact, where $C(x,\Delta')$ is associated to $a(x,\xi)b(x,\xi)\equiv 0$. So, $C(x,\Delta')$ is the null operator and the result follows.
Note 2.: Let $\varphi(x),\psi(x)$ be $C^\infty - functions$ with disjoint supports, and $a(x,\xi)$ be a symbol. Then the operator $\varphi(x)A(x,\Delta')\psi(x) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a compact operator.

In the present chapter, we use the following criterion of compactness. Let $\mathscr{S} \subset L^2(\mathbb{R})$ be a set, such that
(i) ${}^{\alpha}\|\varphi\|_1 = \left(\int_{\mathbb{R}}(1+|\xi|^2)|\widehat{\varphi}_\alpha(\xi)|^2 d\xi\right)^{\frac{1}{2}} \leq C$ for $\varphi \in \mathscr{G}$ and
(ii)

$$\lim_{\upsilon\to 0} \int\limits_{|\xi|\leq\sigma} |\widehat{\varphi}_\alpha(\xi+\upsilon)-\widehat{\varphi}_\alpha(\xi)|^2 d\xi = 0,$$

uniformly for $\varphi \in \mathscr{G}$, for any fixed $\sigma > 0$.
The $\mathscr{G}$ is precompact in $L^2(\mathbb{R})$, and therefore a subsequence of every sequence in $\mathscr{G}$ is convergent in $L^2(\mathbb{R})$.
As a set $\mathscr{G}$ is precompact in $L^2(\mathbb{R})$ if and only if the set $\widehat{\mathscr{G}}$ of fractional Fourier's transform is precompact in $L^2(\mathbb{R})$, it will be sufficient to prove that: From criterion of M.RIESZ. A set $\mathscr{N}$ in $L^2(\mathbb{R})$ is relatively compact iff
(a) $\int |\psi(\xi)|^2 d\xi \leq C, \quad \forall \psi \in \mathscr{N}$,
(b)

$$\lim_{\upsilon\to 0} \int |\psi(\xi+\upsilon)-\psi(\xi)|^2 d\xi = 0$$

uniformly for $\psi \in \mathscr{N}$,
(c)

$$\lim_{\sigma\to\infty} \int\limits_{|\xi|\geq\sigma} |\psi(\xi)|^2 d\xi = 0$$

uniformly for $\psi \in \mathscr{N}$.
See the verification of (a), (b) and (c) in [7].
We will now prove the

Theorem 1. *If $a(x,\xi)$ is a symbol, the pseudo-differential type operator $A(x,\Delta')-\mathscr{A}(x,\Delta')$ is compact in $L^2(\mathbb{R})$.*

Proof. Let Θ be a bounded set in $L^2(\mathbb{R})$. We define $U(x,\Delta') = A(x,\Delta') - \mathscr{A}(x,\Delta')$. To show that $U(\Theta)$ is relatively compact in $L^2(\mathbb{R})$. Or $\widehat{U}(\Theta) = \{\widehat{U\varphi}, \varphi \in \Theta\}$ is relatively compact in $L^2(\mathbb{R})$.
We have

$$^{\alpha}\|\left(A(x,\Delta') - \mathscr{A}(x,\Delta')\right)\varphi\|_{H^1(\mathbb{R})} \leq C(\alpha)^{\alpha}\|\varphi\|_{H^1(\mathbb{R})};$$

therefore, for $\varphi \in \Theta$, the set $\{U\varphi\}_{\varphi\in\Theta}$ is bounded in $H^1(\mathbb{R})$. Hence the set $\widehat{U(\Theta)}$ is bounded in $L^2(\mathbb{R})$.
We have to show that for every $\sigma > 0$, it is

$$\lim_{|\upsilon|\to 0} \int_{|\xi|\leq\sigma} \left|\widehat{U}\varphi(\xi+\upsilon) - \widehat{U}\varphi(\xi)\right|^2 d\xi = 0$$

uniformly for $\varphi \in \Theta$.

Lemma 2. *The symbol $a(x,\xi)$ is such that $a(\infty,\xi) \equiv 0$, in this case to prove that*

$$\lim_{|\upsilon|\to 0} \int_{|\xi|\leq\sigma} \left|\widehat{A}(x,\Delta')\varphi(\xi+\upsilon) - \widehat{A}(x,\Delta')\varphi(\xi)\right|^2 d\xi = 0$$

uniformly for $\varphi \in \Theta \cap \mathscr{G}$.

Proof. We have

$$\left[\mathscr{F}_\alpha\left(A(x,\Delta')\right)\varphi\right](\xi) = C'_\alpha \int e^{-i(\eta-\xi)\eta\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\xi)\widehat{\varphi}_\alpha(\eta)\mathrm{d}\eta$$

and

$$\left[\mathscr{F}_\alpha\left(A(x,\Delta')\right)\varphi\right](\xi+\upsilon) = C'_\alpha \int e^{-i(\eta-\xi-\upsilon)\eta\cot\alpha}\hat{a}'_\alpha(\xi+\upsilon-\eta,\xi)\widehat{\varphi}_\alpha(\eta)\mathrm{d}\eta.$$

Now

$$\begin{aligned}
& \left|\left[\mathscr{F}_\alpha\left(A(x,\Delta')\right)\varphi\right](\xi+\upsilon) - \left[\mathscr{F}_\alpha\left(A(x,\Delta')\right)\varphi\right](\xi)\right| \\
\leq\ & \left|C'_\alpha \int e^{-i(\eta-\xi-\upsilon)\eta\cot\alpha}\hat{a}'_\alpha(\xi+\upsilon-\eta,\xi)\widehat{\varphi}_\alpha(\eta)\mathrm{d}\eta\right| \\
+\ & \left|C'_\alpha \int e^{-i(\eta-\xi)\eta\cot\alpha}\hat{a}'_\alpha(\xi-\eta,\xi)\widehat{\varphi}_\alpha(\eta)\mathrm{d}\eta\right| \\
\leq\ & \left|C'_\alpha\right| \int \left|\hat{a}'_\alpha(\xi+\upsilon-\eta,\xi)\right|\left|\widehat{\varphi}_\alpha(\eta)\right|\mathrm{d}\eta \\
+\ & \left|C'_\alpha\right| \int \left|\hat{a}'_\alpha(\xi-\eta,\xi)\right|\left|\widehat{\varphi}_\alpha(\eta)\right|\mathrm{d}\eta \\
\leq\ & \frac{1}{\sqrt{2\pi\sin\alpha}}\left(\int \left|\hat{a}'_\alpha(\xi+\upsilon-\eta,\xi)\right|^2\mathrm{d}\eta\right)^{\frac{1}{2}}\left(\int \left|\widehat{\varphi}_\alpha(\eta)\right|^2 d\eta\right)^{\frac{1}{2}} \\
+\ & \frac{1}{\sqrt{2\pi\sin\alpha}}\left(\int \left|\hat{a}'_\alpha(\xi-\eta,\xi)\right|^2\mathrm{d}\eta\right)^{\frac{1}{2}}\left(\int \left|\widehat{\varphi}_\alpha(\eta)\right|^2 d\eta\right)^{\frac{1}{2}}.
\end{aligned}$$

We get

$$\begin{aligned}
& \left| \left[\mathscr{F}_\alpha \left(A(x,\Delta')\right)\varphi\right](\xi+\upsilon) - \left[\mathscr{F}_\alpha \left(A(x,\Delta')\right)\varphi\right](\xi)\right| \\
\leq\ & \frac{1}{\sqrt{2\pi\sin\alpha}}({}^{\alpha}\|\varphi\|_0)^2 D_l \left(\int (1+(\xi+\upsilon-\eta)^2\csc^2\alpha)^{-l}\mathrm{d}\eta\right)^{\frac{1}{2}} \\
+\ & \frac{1}{\sqrt{2\pi\sin\alpha}}({}^{\alpha}\|\varphi\|_0)^2 D'_l \left(\int (1+(\xi-\eta)^2\csc^2\alpha)^{-l}\mathrm{d}\eta\right)^{\frac{1}{2}}, \quad \text{where } l = 1,2,3,\text{....to } \infty.
\end{aligned}$$

For every fixed $\sigma > 0$; we have

$$\begin{aligned}
& \int_{|\xi|\leq\sigma} \left|[\widehat{A}(x,\Delta')\varphi](\xi+\upsilon) - [\widehat{A}(x,\Delta')\varphi](\xi)\right|^2 \mathrm{d}\xi \\
\leq\ & \frac{1}{\pi\sin\alpha}({}^{\alpha}\|\varphi\|_0)^2 D_l^2 \int_{|\xi|\leq\sigma}\int_{|\xi|\leq\sigma} (1+(\xi+\upsilon-\eta)^2\csc^2\alpha)^{-l}\mathrm{d}\eta\,\mathrm{d}\xi \\
+\ & \frac{1}{\pi\sin\alpha}({}^{\alpha}\|\varphi\|_0)^2 D_l'^2 \int_{|\xi|\leq\sigma}\int_{|\xi|\leq\sigma} (1+(\xi-\eta)^2\csc^2\alpha)^{-l}\mathrm{d}\eta\,\mathrm{d}\xi \\
\leq\ & \frac{1}{\pi\sin\alpha}({}^{\alpha}\|\varphi\|_0)^2 D_l^2 C_1 + \frac{1}{\pi\sin\alpha}({}^{\alpha}\|\varphi\|_0)^2 D_l'^2 C_2,
\end{aligned}$$

where l is sufficiently large.
We take $\varepsilon > 0$, such that $\frac{1}{\pi\sin\alpha}({}^{\alpha}\|\varphi\|_0)^2 D_l^2 C_1 < \frac{\varepsilon}{2}$ and $\frac{1}{\pi\sin\alpha}({}^{\alpha}\|\varphi\|_0)^2 D_l'^2 C_2 < \frac{\varepsilon}{2}$.
Therefore,

$$\int_{|\xi|\leq\sigma} \left|[\widehat{A}(x,\Delta')\varphi](\xi+\upsilon) - [\widehat{A}(x,\Delta')\varphi](\xi)\right|^2 \mathrm{d}\xi < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence

$$\lim_{|\upsilon|\to 0} \int\limits_{|\xi|\leq\sigma} \left|\widehat{A}(x,\Delta')\varphi(\xi+\upsilon) - \widehat{A}(x,\Delta')\varphi(\xi)\right|^2 d\xi = 0.$$

This completes the proof of Lemma 2. □

Lemma 3. *The symbol* $a(x,\xi)$ *is such that* $a(\infty,\xi)\equiv 0$*, in this case to prove that*

$$\lim_{|\upsilon|\to 0} \int\limits_{|\xi|\leq\sigma} \left|\widehat{\mathscr{A}\varphi}(\xi+\upsilon) - \widehat{\mathscr{A}\varphi}(\xi)\right|^2 d\xi = 0$$

uniformly for $\varphi \in \Theta \cap \mathscr{G}$, $\quad\forall\quad$ *fixed* $\sigma > 0$.

Proof. We have

$$\left[\left(\widehat{\mathscr{A}}(x,\Delta')\right)\varphi\right](\xi) = \int \hat{a}_\alpha(\xi-\eta,\eta)\hat{\varphi}_\alpha(\eta)d\eta$$

and

$$\left[\left(\widehat{\mathscr{A}(x,\Delta')}\right)\varphi\right](\xi+\upsilon) = \int \hat{a}_\alpha(\xi+\upsilon-\eta,\eta)\hat{\varphi}_\alpha(\eta)d\eta$$

$$\begin{aligned}\left[\left(\widehat{\mathscr{A}(x,\Delta')}\right)\varphi\right](\xi+\upsilon) - \left[\left(\widehat{\mathscr{A}(x,\Delta')}\right)\varphi\right](\xi) &= \int \hat{a}_\alpha(\xi+\upsilon-\eta,\eta)\hat{\varphi}_\alpha(\eta)d\eta \\ &- \int \hat{a}_\alpha(\xi-\eta,\eta)\hat{\varphi}_\alpha(\eta)d\eta.\end{aligned}$$

Now,

$$\begin{aligned} & \left|\left[\left(\widehat{\mathscr{A}(x,\Delta')}\right)\varphi\right](\xi+\upsilon) - \left[\left(\widehat{\mathscr{A}(x,\Delta')}\right)\varphi\right](\xi)\right|^2 \\ = & \left|\int \left[\hat{a}_\alpha(\xi+\upsilon-\eta,\eta) - \hat{a}_\alpha(\xi-\eta,\eta)\right]\hat{\varphi}_\alpha(\eta)d\eta\right|^2 \\ \leq & \left(\int |\hat{\varphi}_\alpha(\eta)|^2 d\eta\right)\left(\int \left|\hat{a}_\alpha(\xi+\upsilon-\eta,\eta) - \hat{a}_\alpha(\xi-\eta,\eta)\right|^2 d\eta\right) \\ = & ({}^\alpha\|\varphi\|_0)^2 \int \left|\hat{a}_\alpha(\xi+\upsilon-\eta,\eta) - \hat{a}_\alpha(\xi-\eta,\eta)\right|^2 d\eta. \end{aligned}$$

Apply Taylor's formula; we obtain, if $\hat{a}_\alpha = \hat{a}_\alpha(\lambda,\eta)$, the relation

$$\hat{a}_\alpha(\xi+\upsilon-\eta,\eta) - \hat{a}_\alpha(\xi-\eta,\eta) = (\upsilon, grad_\lambda \hat{a}_\alpha(\xi-\eta+\theta\upsilon,\eta)), \qquad 0<\theta<1.$$

It implies that

$$\left|\hat{a}_\alpha(\xi+\upsilon-\eta,\eta) - \hat{a}_\alpha(\xi-\eta,\eta)\right| \leq |\upsilon||grad_\lambda \hat{a}_\alpha(\xi-\eta+\theta\upsilon,\eta)|.$$

From (6),

$$\left|(1+|\lambda|^2)^l D_\lambda \hat{a}_\alpha(\lambda,\eta)\right| \leq C_l, \qquad \eta \neq 0, \lambda \in \mathbb{R}.$$

$$\therefore \left|grad_\lambda \hat{a}_\alpha(\xi-\eta+\theta\upsilon,\eta)\right| \leq C_l(1+|\xi-\eta+\theta\upsilon|^2)^{-l}, \qquad \forall\, l=1,2,3,\ldots\text{to } \infty.$$

$$\left|\left[\left(\widehat{\mathscr{A}(x,\Delta')}\right)\varphi\right](\xi+\upsilon) - \left[\left(\widehat{\mathscr{A}(x,\Delta')}\right)\varphi\right](\xi)\right|^2 \leq ({}^\alpha\|\varphi\|_0)^2 \int |\upsilon|^2 C_l^2 (1+|\xi-\eta+\theta\upsilon|^2)^{-2l} d\eta.$$

Terefore, taking integration on both sides with limit $|\xi| \leq \sigma$, we have

$$\begin{aligned} & \int\limits_{|\xi|\leq\sigma} \left|\left[\left(\widehat{\mathscr{A}(x,\Delta')}\right)\varphi\right](\xi+\upsilon) - \left[\left(\widehat{\mathscr{A}(x,\Delta')}\right)\varphi\right](\xi)\right|^2 d\xi \\ \leq & ({}^\alpha\|\varphi\|_0)^2 \int\limits_{|\xi|\leq\sigma}\int\limits_{|\xi|\leq\sigma} |\upsilon|^2 C_l^2 (1+|\xi-\eta+\theta\upsilon|^2)^{-2l} d\eta d\xi \qquad (12) \end{aligned}$$

Since $\varphi \in \Theta \cap \mathscr{G}$ we have $({}^{\alpha}\|\varphi\|_0)^2 \leq M$. We take $\varepsilon > 0$, and choose at first $\rho_0(\varepsilon)$ and $\upsilon_0(\varepsilon)$ such that

$$M \int\limits_{|\xi|\leq\sigma_0} \int\limits_{|\xi|\leq\sigma_0} |\upsilon_0(\varepsilon)|^2 C_l^2 (1+|\xi-\eta+\theta\upsilon_0(\varepsilon)|^2)^{-2l} d\eta d\xi \leq \varepsilon, \tag{13}$$

where l is sufficiently large.
From the (12) and (13), we get

$$\int\limits_{|\xi|\leq\sigma} |\left[\left(\widehat{\mathscr{A}(x,\Delta')}\right)\varphi\right](\xi+\upsilon) - \left[\left(\widehat{\mathscr{A}(x,\Delta')}\right)\varphi\right](\xi)|^2 d\xi \leq \varepsilon.$$

Therefore

$$\lim_{|\upsilon|\to 0} \int\limits_{|\xi|\leq\sigma} |\left[\left(\widehat{\mathscr{A}(x,\Delta')}\right)\varphi\right](\xi+\upsilon) - \left[\left(\widehat{\mathscr{A}(x,\Delta')}\right)\varphi\right](\xi)|^2 d\xi = 0.$$

The proof of the Lemma 3 is completed. □

Lemma 4. *We have in the case* $a(\infty,\xi) \equiv 0$ *that* $\forall \sigma > 0$

$$\lim_{|\upsilon|\to 0} \int\limits_{|\xi|\leq\sigma} |\left[\left(\widehat{A}(x,\Delta')\right)\varphi\right](\xi+\upsilon) - \left[\left(\widehat{A}(x,\Delta')\right)\varphi\right](\xi)|^2 d\xi = 0$$

and

$$\lim_{|\upsilon|\to 0} \int\limits_{|\xi|\leq\sigma} |\left[\left(\widehat{\mathscr{A}(x,\Delta')}\right)\varphi\right](\xi+\upsilon) - \left[\left(\widehat{\mathscr{A}(x,\Delta')}\right)\varphi\right](\xi)|^2 d\xi = 0$$

uniformly for $\varphi \in \Theta$*-bounded set in* $L^2(\mathbb{R})$.

Proof. Since $\mathscr{G}$ is dense in $L^2(\mathbb{R})$. Given $\varepsilon > 0$, howeversmall and Θ a bounded set in $L^2(\mathbb{R})$, there is $\forall \varphi \in \Theta$, an element $\varphi_\varepsilon \in \mathscr{G}$, such that ${}^{\alpha}\|\varphi - \varphi_\varepsilon\|_0 < \sqrt{\frac{\varepsilon}{4c}}$. Hence, for $\varphi \in \Theta$ we have ${}^{\alpha}\|\varphi\|_0 \leq L$, and ${}^{\alpha}\|\varphi_\varepsilon\|_0 = {}^{\alpha}\|\varphi_\varepsilon - \varphi + \varphi\|_0 \leq {}^{\alpha}\|(-1)(\varphi - \varphi_\varepsilon)\|_0 + {}^{\alpha}\|\varphi\|_0 = {}^{\alpha}\|\varphi - \varphi_\varepsilon\|_0 + {}^{\alpha}\|\varphi\|_0 \leq \varepsilon + L \leq L+1$, and therefore the set $\{\varphi_\varepsilon : \varphi \in \Theta\}$ is a set Θ_1, bounded in $L^2(\mathbb{R})$ and included in $\mathscr{G}$. Here we have, for $|\upsilon| \leq |\upsilon_0(\varepsilon)|$ that in case $a(\infty,\xi) \equiv 0$

$$\int\limits_{|\xi|\leq\sigma} |\left[\left(\widehat{A}(x,\Delta')\right)\varphi\right](\xi+\upsilon) - \left[\left(\widehat{A}(x,\Delta')\right)\varphi\right](\xi)|^2 d\xi \leq \frac{\varepsilon}{2}, \quad \forall \varphi_\varepsilon \in \Theta_1$$

and

$$\int\limits_{|\xi|\leq\sigma} |\left[\left(\widehat{\mathscr{A}}(x,\Delta')\right)\varphi\right](\xi+\upsilon)-\left[\left(\widehat{\mathscr{A}}(x,\Delta')\right)\varphi\right](\xi)|^2 d\xi \leq \frac{\varepsilon}{2}, \quad \forall\varphi_\varepsilon\in\Theta_1.$$

Hence, we deduce the inequalities

$$\begin{aligned}
& \int\limits_{|\xi|\leq\sigma} |\left[\left(\widehat{A}(x,\Delta')\right)\varphi\right](\xi+\upsilon)-\left[\left(\widehat{A}(x,\Delta')\right)\varphi\right](\xi)|^2 d\xi \\
= & \int\limits_{|\xi|\leq\sigma} |\left[\left(\widehat{A}(x,\Delta')\right)\varphi\right](\xi+\upsilon)-\left[\left(\widehat{A}(x,\Delta')\right)\varphi_\varepsilon\right](\xi+\upsilon)+\left[\left(\widehat{A}(x,\Delta')\right)\varphi_\varepsilon\right](\xi+\upsilon) \\
- & \left[\left(\widehat{A}(x,\Delta')\right)\varphi_\varepsilon\right](\xi)+\left[\left(\widehat{A}(x,\Delta')\right)\varphi_\varepsilon\right](\xi)-\left[\left(\widehat{A}(x,\Delta')\right)\varphi\right](\xi)|^2 d\xi \\
\leq & \int\limits_{|\xi|\leq\sigma} |\left[\left(\widehat{A}(x,\Delta')\right)\varphi\right](\xi+\upsilon)-\left[\left(\widehat{A}(x,\Delta')\right)\varphi_\varepsilon\right](\xi+\upsilon)|^2 d\xi \\
+ & \int\limits_{|\xi|\leq\sigma} |\left[\left(\widehat{A}(x,\Delta')\right)\varphi_\varepsilon\right](\xi+\upsilon)-\left[\left(\widehat{A}(x,\Delta')\right)\varphi_\varepsilon\right](\xi)|^2 d\xi \\
+ & \int\limits_{|\xi|\leq\sigma} |\left[\left(\widehat{A}(x,\Delta')\right)\varphi_\varepsilon\right](\xi)-\left[\left(\widehat{A}(x,\Delta')\right)\varphi\right](\xi)|^2 d\xi \\
= & ({}^\alpha\|A(x,\Delta')(\varphi-\varphi_\varepsilon)\|)_0^2+\int\limits_{|\xi|\leq\sigma} |\left[\left(\widehat{A}(x,\Delta')\right)\varphi_\varepsilon\right](\xi+\upsilon)-\left[\left(\widehat{A}(x,\Delta')\right)\varphi_\varepsilon\right](\xi)|^2 d\xi \\
+ & ({}^\alpha\|A(x,\Delta')(\varphi-\varphi_\varepsilon)\|_0)^2 \\
= & 2({}^\alpha\|A(x,\Delta')(\varphi-\varphi_\varepsilon)\|_0)^2+\int\limits_{|\xi|\leq\sigma} |\left[\left(\widehat{A}(x,\Delta')\right)\varphi_\varepsilon\right](\xi+\upsilon)-\left[\left(\widehat{A}(x,\Delta')\right)\varphi_\varepsilon\right](\xi)|^2 d\xi \\
\leq & 2c({}^\alpha\|\varphi-\varphi_\varepsilon\|_0)^2+\frac{\varepsilon}{2} \\
\leq & 2c\left(\sqrt{\frac{\varepsilon}{4c}}\right)^2+\frac{\varepsilon}{2}=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

Hence,

$$\lim_{|\upsilon|\to 0}\int\limits_{|\xi|\leq\sigma} |\left[\left(\widehat{A}(x,\Delta')\right)\varphi\right](\xi+\upsilon)-\left[\left(\widehat{A}(x,\Delta')\right)\varphi\right](\xi)|^2 d\xi=0.$$

Similary, we can prove that

$$\lim_{|\upsilon|\to 0}\int\limits_{|\xi|\leq\sigma} |\left[\left(\widehat{\mathscr{A}}(x,\Delta')\right)\varphi\right](\xi+\upsilon)-\left[\left(\widehat{\mathscr{A}}(x,\Delta')\right)\varphi\right](\xi)|^2 d\xi=0.$$

This completes the proof of Lemma 4. □

Thus, applying Lemma 2, Lemma 3, Lemma 4, Theorem 1 is proved. □

Theorem 2. *If $a(x,\xi)$, $b(x,\xi)$ and $c(x,\xi) = a(x,\xi)b(x,\xi)$ are symbols, then $A(x,\Delta')B(x,\Delta') - C(x,\Delta') : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a compact operator.*

Proof. Let $T(x,\Delta') = A(x,\Delta')B(x,\Delta') - C(x,\Delta')$ be an operator. Hence, if $\varphi \in \Theta$ where Θ is a bounded set in $L^2(\mathbb{R})$, then $\widehat{T}(x,\Delta')(\Theta)$ is bounded in $L^2(\mathbb{R})$. Therefore, we have to prove that, $\forall\sigma > 0$

$$\lim_{|\upsilon|\to 0} \int\limits_{|\xi|\le\sigma} \left| \left[\widehat{T}(x,\Delta')\varphi\right](\xi+\upsilon) - \left[\widehat{T}(x,\Delta')\varphi\right](\xi)\right|^2 = 0 \tag{14}$$

uniformly for $\varphi \in \Theta$.
When $a(\infty,\xi) \equiv b(\infty,\xi) \equiv c(\infty,\xi) \equiv 0$, then using Theorem 1 we get, $\forall\sigma > 0$

$$\lim_{|\upsilon|\to 0} \int\limits_{|\xi|\le\sigma} \left| \left[\widehat{C}(x,\Delta')\varphi\right](\xi+\upsilon) - \left[\widehat{C}(x,\Delta')\varphi\right](\xi)\right|^2 = 0 \tag{15}$$

uniformly for $\varphi \in \Theta$.
It is only to find

$$\lim_{|\upsilon|\to 0} \int\limits_{|\xi|\le\sigma} \left| \left[\widehat{AB}(x,\Delta')\varphi\right](\xi+\upsilon) - \left[\widehat{AB}(x,\Delta')\varphi\right](\xi)\right|^2 = 0. \tag{16}$$

From Lemma 4, then $\forall\varepsilon > 0$, $\exists\ \delta_L(\varepsilon)$, such that

$$\int\limits_{|\xi|\le\sigma} \left|\widehat{A}\psi(\xi+\upsilon) - \widehat{A}\psi(\xi)\right|^2 d\xi \le \varepsilon,\ \ if\ \ |\upsilon| < \delta_L(\varepsilon)\ \ and\ \ {}^{\alpha}\|\psi\|_0 \le L.$$

If φ is arbitrary in $L^2(\mathbb{R})$ and ${}^{\alpha}\|\varphi\|_0 \le \frac{M^2}{c}$, where $M, c \in \mathbb{R}_+$. Then $\frac{\varphi}{{}^{\alpha}\|\varphi\|_0}$ is of norm 1, therefore

$$\int\limits_{|\xi|\le\sigma} \left|\widehat{A}\frac{\varphi}{{}^{\alpha}\|\varphi\|_0}(\xi+\upsilon) - \widehat{A}\frac{\varphi}{{}^{\alpha}\|\varphi\|_0}(\xi)\right|^2 d\xi < \frac{\varepsilon}{M^2}\ \ \ if\ |\upsilon| < \delta_1(\varepsilon),$$

$$\int\limits_{|\xi|\le\sigma} \left|\widehat{A}\varphi(\xi+\upsilon) - \widehat{A}\varphi(\xi)\right|^2 d\xi < ({}^{\alpha}\|\varphi\|_0)^2\frac{\varepsilon}{M^2}\ \ \ if\ |\upsilon| < \delta_1(\varepsilon), \quad \forall\varphi \in L^2(\mathbb{R}), \tag{17}$$

using (17) for $AB\varphi$, we get

$$\int\limits_{|\xi|\leq\sigma} \left|\widehat{AB}\varphi(\xi+\upsilon)-\widehat{AB}\varphi(\xi)\right|^2 d\xi < ({}^{\alpha}\|B\varphi\|_0)^2\frac{\varepsilon}{M^2} \quad if\ |\upsilon|<\delta_1(\varepsilon), \quad \forall\varphi\in L^2(\mathbb{R}). \tag{18}$$

Since the pseudo-differential type operator $B(x,\Delta')$ is bounded. Therefore, ${}^{\alpha}\|B\varphi\|_0 \leq c\ {}^{\alpha}\|\varphi\|_0$, where $c>0$ is a certain constant.

$$\therefore \int\limits_{|\xi|\leq\sigma} \left|\widehat{AB}\varphi(\xi+\upsilon)-\widehat{AB}\varphi(\xi)\right|^2 d\xi \leq c^2({}^{\alpha}\|\varphi\|_0)^2\frac{\varepsilon}{M^2} \leq c^2\frac{M^2}{c^2}\frac{\varepsilon}{M^2} = \varepsilon$$

$|\upsilon| < \delta_1(\varepsilon), \quad \forall\varphi\in L^2(\mathbb{R})$.

It implies that

$$\lim_{|\upsilon|\to 0}\int\limits_{|\xi|\leq\sigma} \left|\widehat{AB}\varphi(\xi+\upsilon)-\widehat{AB}\varphi(\xi)\right|^2 d\xi = 0. \tag{19}$$

Using (16) and (19), we get

$$\lim_{|\upsilon|\to 0}\int\limits_{|\xi|\leq\sigma} \left|\left(\widehat{AB}-\widehat{C}\right)\varphi(\xi+\upsilon)-\left(\widehat{AB}-\widehat{C}\right)\varphi(\xi)\right|^2 d\xi = 0.$$

Hence, $A(x,\Delta')B(x,\Delta')-C(x,\Delta'): L^2(\mathbb{R})\to L^2(\mathbb{R})$ is a compact operator. □

This completes the proof of the Theorem 2.

Theorem 3. *Let $a(x,\xi)$ be a symbol, $A(x,\Delta')$ and $\mathscr{A}(x,\Delta')$ the pseudo-differential type operators. Then $A(x,\Delta')-\mathscr{A}(x,\Delta'): L^2(\mathbb{R})\to L^2(\mathbb{R})$ is a compact linear operator.*

Proof. Using Theorem 1 and Theorem 2, Theorem 3 is proved. □

6 Norms of pseudo-differential type operator modulo compact operators

Theorem 4. *Let $A(x,\Delta')$ be pseudo-differential type operator associated with the symbol $a(x,\xi)$. Then, for every $\varepsilon>0$ there is a semi-norm $\|\ \|'$ on $L^2(\mathbb{R})$, dependent of ε, such that every L^2-bounded sequence contains a subsequence convergent in the semi-norm $\|\ \|'$, such that the inequality*

$$\|A(x,\Delta')\varphi\|_0 \leq (M+\varepsilon)^{\alpha}\|\varphi\|_0 + {}^{\varepsilon}\|\varphi\|, \qquad \forall\varphi\in L^2(\mathbb{R})$$

is proved.

Proof. We consider $b_\varepsilon(x,\xi) = (M^2 - \overline{a}(x,\xi)a(x,\xi) + \varepsilon)^{\frac{1}{2}}$. It is obvious that

$$b_\varepsilon(x,\xi) = \overline{b_\varepsilon}(x,\xi), \quad \varepsilon > 0, \; x,\xi \neq 0 \in \mathbb{R}.$$

Let us consider the $B_\varepsilon(x,\Delta')$, $\mathscr{B}_\varepsilon(x,\Delta')$ associated with $b_\varepsilon(x,\xi)$ and $\overline{\mathscr{A}}(x,\Delta')$ associated with $\overline{a}(x,\xi)$.

Lemma 5. *The linear operator*

$$U_\varepsilon = (M^2+\varepsilon)I - \overline{\mathscr{A}} \cdot A - \mathscr{B}_\varepsilon \cdot B_\varepsilon : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \tag{20}$$

is compact.

Proof. We define a relation

$$\mathscr{B}_\varepsilon \cdot B_\varepsilon = (\mathscr{B}_\varepsilon - B_\varepsilon) \cdot B_\varepsilon + B_\varepsilon^2 = U_1 + B_\varepsilon^2, \tag{21}$$

where $U_1 = (\mathscr{B}_\varepsilon - B_\varepsilon) \cdot B_\varepsilon$ is a compact operator according to Theorem 3. From (20) and (21), we get

$$U_\varepsilon = (M^2+\varepsilon)I - \overline{\mathscr{A}} \cdot A - U_1 - B_\varepsilon^2.$$

We consider

$$\overline{\mathscr{A}} \cdot A = (\overline{\mathscr{A}} - \overline{A}) \cdot A + \overline{A} \cdot A = U_2 + \overline{A} \cdot A, \qquad \text{(say)};$$

where, $U_2 = (\overline{\mathscr{A}} - \overline{A}) \cdot A : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a compact operator, according to the Theorem 3 and hence we get

$$U_\varepsilon = (M^2+\varepsilon)I - \overline{\mathscr{A}} \cdot A - B_\varepsilon^2 - (U_1 + U_2).$$

We have: $B_\varepsilon \cdot B_\varepsilon - (M^2 + \varepsilon - (\overline{a} \cdot a)(x,\Delta')) = U_3 : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a compact operator. Hence, we derive

$$B_\varepsilon(x,\Delta') \cdot B_\varepsilon(x,\Delta') = B_\varepsilon^2(x,\Delta') = M^2 + \varepsilon - (\overline{a} \cdot a)(x,\Delta') + U_3,$$

and therefore

$$\begin{aligned} U_\varepsilon &= M^2 + \varepsilon - \overline{A} \cdot A - (M^2+\varepsilon) + (\overline{a} \cdot a)(x,\Delta') - (U_1 + U_2 + U_3) \\ &= (\overline{a} \cdot a)(x,\Delta') - \overline{A} \cdot A - (U_1 + U_2 + U_3) = U_0 \qquad (\textit{say}), \end{aligned}$$

where, $U_0 : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a compact linear operator by Theorem 3.

Hence, Lemma 5 is proved. □

Lemma 6. *Given arbitrary $\varepsilon > 0$, we have the relation*

$$Re(U_\varepsilon \varphi, \varphi)_0 + \varepsilon \|\varphi\|_0^2 \geq -\frac{1}{4\varepsilon}\|U_\varepsilon \varphi\|_0^2, \qquad \forall \varphi \in L^2(\mathbb{R}).$$

Proof. In fact, we have

$$\begin{aligned}
|Re(U_\varepsilon \varphi, \varphi)_0| &\leq \|U_\varepsilon \varphi\|_0 \|\varphi\|_0 \\
&= 2 \cdot \frac{1}{2\sqrt{\varepsilon}}\|U_\varepsilon \varphi\|_0 \cdot \sqrt{\varepsilon}\|\varphi\|_0 \\
&\leq \left(\frac{1}{2\sqrt{\varepsilon}}\|U_\varepsilon \varphi\|_0\right)^2 + \left(\sqrt{\varepsilon}\|\varphi\|_0\right)^2 \\
&= \frac{1}{4\varepsilon}\|U_\varepsilon \varphi\|_0^2 + \varepsilon \|\varphi\|_0^2
\end{aligned}$$

and consequently

$$Re(U_\varepsilon \varphi, \varphi)_0 \geq -\frac{1}{4\varepsilon}\|U_\varepsilon \varphi\|_0^2 - \varepsilon \|\varphi\|_0^2,$$

$$Re(U_\varepsilon \varphi, \varphi)_0 + \varepsilon \|\varphi\|_0^2 \geq -\frac{1}{4\varepsilon}\|U_\varepsilon \varphi\|_0^2.$$

This completes the proof. □

Lemma 7. *We have the relation, $\forall \varepsilon > 0$*

$$\|A(x, \Delta')\varphi\|_0^2 \leq (M^2 + 2\varepsilon)\|\varphi\|_0^2 + \frac{1}{4\varepsilon}\|U_\varepsilon \varphi\|_0^2, \qquad \forall \varphi \in L^2(\mathbb{R}).$$

Proof. We have

$$(U_\varepsilon \varphi, \varphi)_0 = (M^2 + 2\varepsilon)\|\varphi\|_0^2 - \|A(x, \Delta)\varphi\|_0^2 - \|B_\varepsilon(x, \Delta')\varphi\|_0^2.$$

Using Lemaa 6, the estimate

$$(M^2 + 2\varepsilon)\|\varphi\|_0^2 - \|A(x, \Delta')\varphi\|_0^2 - \|B_\varepsilon(x, \Delta')\varphi\|_0^2 \geq -\frac{1}{4\varepsilon}\|U_\varepsilon \varphi\|_0^2$$

and therefore

$$\|A(x, \Delta')\varphi\|_0^2 + \|B_\varepsilon(x, \Delta')\varphi\|_0^2 \leq (M^2 + 2\varepsilon)\|\varphi\|_0^2 + \frac{1}{4\varepsilon}\|U_\varepsilon \varphi\|_0^2$$

and hence a fortiori

$$\|A(x, \Delta')\varphi\|_0^2 \leq (M^2 + 2\varepsilon)\|\varphi\|_0^2 + \frac{1}{4\varepsilon}\|U_\varepsilon \varphi\|_0^2. \tag{22}$$

This proves the Lemma 7. □

Applying $\sqrt{a+b} \leq \sqrt{a}+\sqrt{b}$, a *and* b > 0 in (23), we get

$$\|A(x,\Delta^{'})\varphi\|_0 \leq (M+\sqrt{2\varepsilon})\|\varphi\|_0 + \frac{1}{2\sqrt{\varepsilon}}\|U_\varepsilon\varphi\|_0. \tag{23}$$

Putting semi-norm ${}^{\varepsilon}\|\varphi\| = \frac{1}{2\sqrt{\varepsilon}}\|U_\varepsilon\varphi\|_0$ in (6.4), we have

$$\|A(x,\Delta^{'})\varphi\|_0 \leq (M+\sqrt{2\varepsilon})\|\varphi\|_0 + {}^{\varepsilon}\|\varphi\|.$$

This proves the theorem. □

Theorem 5. *Let H be hilbertian; on H is defined a seminorm* $|\ |$ *such that*
(i) $|\varphi| \leq c\|\varphi\|_H$, $\varphi \in H$,
(ii) for every bounded sequence $(\varphi_n)_1^\infty$ *there exists a Cauchy subsequence with respect to* $|\ |$.
Then: $\forall\varepsilon > 0$, *there exists* $H_\varepsilon -$*a closed linear subspace of H, such that* $H\Theta H_\varepsilon = H_\varepsilon^\perp$ *is of finite dimension and* $|\varphi| \leqq \varepsilon\|\varphi\|_H, \forall\varphi \in H_\varepsilon$.

Proof. See [7]. □

Theorem 6. *Let* $H^s(\mathbb{R})$, $\forall s \in \mathbb{R}$ *be a Hilbertian space and* $A(x,\Delta^{'}) \in \mathscr{L}(H^s : H^s)$. *Let us assume that* $\forall\varepsilon > 0$, *there exists a seminorm* ${}^{\varepsilon}\|\ \|$ *on* $H^s(\mathbb{R})$ *such that* $\|\ \|_s$ *is relatively compact with respect to* ${}^{\varepsilon}\|\ \|$ *and such that* ${}^{\varepsilon}\|\varphi\| \leq c\|\varphi\|_s$, $\forall\varphi \in H^s(\mathbb{R})$ *and*

$$\|A(x,\Delta^{'})\varphi\|_{H^s} \leq (M+\varepsilon)\|\varphi\|_s + {}^{\varepsilon}\|\varphi\|, \qquad \forall\varphi \in H^s(\mathbb{R}).$$

Then:

$$inf\{\|A+U\|_{\mathscr{L}(H^s:H^s)} : U \in \mathfrak{S}_c\} \leq M.$$

Proof. It is sufficient to prove that for every $\varepsilon > 0$, we find a compact operator U_ε in H^s, such that

$$\|(A+U_\varepsilon)\varphi\| \leq (M+\varepsilon)\|\varphi\|, \qquad \forall\varphi \in H^s.$$

Let be $\mathscr{H}_\varepsilon \subset H^s$; for $\varphi \in \mathscr{H}_\varepsilon$ we have, ${}^{\varepsilon}\|\varphi\| \leq \varepsilon\|\varphi\|$ and $\mathscr{H}_\varepsilon^\perp$ of dimension n_ε finite. Let us put $\mathscr{P}_\varepsilon$ the orthogonal projection on $\mathscr{H}_\varepsilon$; hence, $(I-\mathscr{P}_\varepsilon)$ projects on a finite dimensional space.

Therefore, $I-\mathscr{P}_\varepsilon : H^s(\mathbb{R}) \to H^s(\mathbb{R})$ is compact.

Hence, we put $-U_\varepsilon = A(I-\mathscr{P}_\varepsilon)$; this is obviously compact. We have

$$\|(A+U_\varepsilon)\varphi\| = \|A\mathscr{P}_\varepsilon\varphi\|, \qquad \forall\varphi \in H^s(\mathbb{R}).$$

By the hypothesis of the theorem, we get

$$\|(A+U_\varepsilon)\varphi\| \leq (M+\varepsilon)\|\mathscr{P}_\varepsilon\varphi\| + {}^{\varepsilon}\|\mathscr{P}\varphi\|, \qquad \forall\varphi \in H^s(\mathbb{R}).$$

Being now, we get: $\mathscr{P}_\varepsilon \varphi \in \mathscr{H}_\varepsilon$, we get

$$^{\varepsilon}\|\mathscr{P}\varphi\| \leq \varepsilon \|\mathscr{P}_\varepsilon \varphi\| \leq \varepsilon \|\varphi\|$$

therefore we get,

$$\|(A+U_\varepsilon)\varphi\| \leq (M+2\varepsilon)\|\varphi\|, \qquad \forall \varphi \in H^s(\mathbb{R}).$$

Hence,

$$\inf\{\|A+U\|_{\mathscr{L}(H^s:H^s)} : U \in \mathfrak{S}_c\} \leq M.$$

This completes the proof of Theorem 6. □

Theorem 7. *Let $a(x,\xi)$ be a symbol such that $M = max\{|a(x,\xi)| : x \in \mathbb{R}$ and $|\xi| = 1\}$; let $A(x,\Delta^{'})$ be pseudo-differential type operator associated with $a(x,\xi)$. Let $\mathfrak{S}_c = \{U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a linear compact operator $\}$.*
Then we have the upper estimates

$$inf\,\{\|A(x,\Delta^{'})+U\| : U \in \mathfrak{S}_c\} \leq M, \quad inf\,\{\|\mathscr{A}(x,\Delta^{'})+U\| : U \in \mathfrak{S}_c\} \leq M.$$

Proof. Using Theorem 6 and Theorem 5, Theorem 7 is proved. □

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