

Ranks and Subdegrees of the Action of the Product of Four Alternating Groups on the Cartesian Product of Four Sets of Ordered Pairs

ABSTRACT

The rank and subdegrees of the action of $A_n \times A_n \times A_n \times A_n$ acting on $W^{[2]} \times X^{[2]} \times Y^{[2]} \times Z^{[2]}$ are determined in this paper. Using combinatorics and mathematical induction, the rank of $A_n \times A_n \times A_n \times A_n$, ($\forall n \geq 4$) acting on $W^{[2]} \times X^{[2]} \times Y^{[2]} \times Z^{[2]}$, is 2^4 and the subdegrees are: $1, 4 \left(\binom{n!}{(n-2)!} - 1 \right), 6 \left(\binom{n!}{(n-2)!} - 1 \right)^2, 4 \left(\binom{n!}{(n-2)!} - 1 \right)^3$ and $\left(\binom{n!}{(n-2)!} - 1 \right)^4$.

Keywords: Group action, Cartesian product, Orbit, Rank and Subdegrees.

1. PRELIMINARIES

1.0 Introduction

The Cartesian product of ordered sets of pairs $W_1^{[2]}, X_2^{[2]}, Y_3^{[2]}$ and $Z_4^{[2]}$ is defined as the set of all ordered pairs (w, x, y, z) such that $w \in W^{[2]}, x \in X^{[2]}, y \in Y^{[2]}$ and $z \in Z^{[2]}$. The ordered pairs sets are generated using the Groups, Algorithms and Programming software (GAP).

Higman (1964) worked on finite permutation groups of rank 3 and proved that the rank of any 4-fold transitive group is 3 when considered as a group of permutations of unordered pairs of distinct letters. Higman (1970) established that the action of S_n on 2-element subsets from the set $X = \{1, 2, \dots, n\}$ has rank 3 with subdegrees as: $1, 2(n-1)$ and $\binom{n-1}{2}$.

Nyaga (2018) showed that the action of direct product of A_n , is transitive on the Cartesian product of 3 sets. The rank and subdegrees associated with this action for $n \geq 4$ is 8; and $1, (n-1), (n-1)^2, (n-1)^3$ respectively.

Mutua *et al.*, (2018) found the action of direct product of $S_n \times A_n$ on $P \times Q$ to be transitive and also imprimitive provided $n \geq 3$. The associated rank for this action is 6 when $n = 3$, but is 4 for all $n \geq 3$. The subdegrees are: $1, (n-1), (n-1), (n-1)^2$. Mutua *et al.*, (2018) established that for $n \geq 4$, the sub-orbits are self-paired thus the respective graphs for the suborbits are un-directed.

Maraka et al., (2025) showed that the action of the direct product of three alternating groups acting on the Cartesian product of three sets of ordered φ -tuples has a rank of 8 and the subdegrees are: $1, 3 \left(\binom{n!}{(n-\varphi)!} - 1 \right), 3 \left(\binom{n!}{(n-\varphi)!} - 1 \right)^2$ and $\left(\binom{n!}{(n-\varphi)!} - 1 \right)^3$.

1.1 Definitions and Theorems

Definition 1.1.1. Group action [9]

Given a group G and a non-empty set X , the action of G to the left of X matches a unique element $gx \in X$ if $\forall g \in G$ such that for all $x \in X$ and $g_1, g_2 \in G$:

- (i) $(g_1 g_2)x = g_1(g_2)x$.
- (ii) $e.x = x$, given that e is the identity in G .

When G acts from the right side of X , its action can similarly be denoted as such.

Definition 1.1.2. Transitive group [5]

A group G is termed to act transitively on a set X provided for all $x, y \in X, \exists g \in G: g(x) = y$, that is, the action gives only a one orbit.

Definition 1.1.3. Stabilizer of an Element [12]

Let $x \in X$ and a group G act on X . The stabilizer of x in G is given by $Stab_G(x) = \{g \in G: gx = x\}$.

Definition 1.1.4. Fixed point [9]

Given a non-empty set X and group G acting on X with $g \in G$. The set of elements x fixed by $g \in G$ is referred to as fixed point set of h given by $Fix(g) = \{x \in P: h(x) = x\}$.

Definition 1.2.5. Orbit [5]

For $x \in X$, the action of a group G on X , partitions X into separate equivalence classes known as orbits. Hence, $Orb_G(x) = \{gx: g \in G\}$.

Definition 1.2.6. Rank and Subdegrees [10]

Given that the action of G on a set X is transitive and $x \in X$. The orbits of G_x on X are referred to as the suborbits. The rank of G on X is the number of those suborbits and their sizes are called subdegrees of G on X .

Definition 1.1.7. Direct product action [2]

Given (G_1, X) and (G_2, Y) as permutation groups. The direct product $G_1 \times G_2$ acts on the separate union $X \cup Y$ by the law $x(g_1, g_2) = \begin{cases} xg_1, & x \in X \\ yg_2, & y \in Y \end{cases}$ and on Cartesian product $X \times Y$ by the law $(x, y)(g_1, g_2) = (xg_1, yg_2)$.

Theorem 1.1.8. [6]

The action of $A_{n_1} \times \dots \times A_{n_m}, (\forall m < \infty)$ on $X_1^{[\varphi]} \times \dots \times X_m^{[\varphi]}$ is transitive, $\forall n - \varphi \geq 2$.

Theorem 1.1.9. Orbit – Stabilizer Theorem [12]

Given G acts on a set X , $|Orb_G(x)| = |G:Stab_G(x)|$.

Theorem 1.1.10 [1]

The $G_1 \times G_2$ -orbit containing $(x, y) \in X \times Y$ is given by $Orb_{G_1}(x) \times Orb_{G_2}(y)$ and the stabilizer of (x, y) is given by $Stab_{G_1}(x) \times Stab_{G_2}(y)$.

2. MAIN RESULTS

Theorem 2.1 (Rank)

The rank of $G = A_n \times A_n \times A_n \times A_n, (n \geq 4)$ acting on $W^{[2]} \times X^{[2]} \times Y^{[2]} \times Z^{[2]}$ is 2^4 .

Proof:

Suppose G acts on $W^{[2]} \times X^{[2]} \times Y^{[2]} \times Z^{[2]}$. Let $[w_1, w_2] \in W^{[2]}, [x_1, x_2] \in X^{[2]}, [y_1, y_2] \in Y^{[2]}$ and $[z_1, z_2] \in Z^{[2]}$.

Let $Q = \{[w_1, w_2], [x_1, x_2], [y_1, y_2], [z_1, z_2]\}$.

Then, G_Q has orbits with exactly four, three, two, one or no ordered pairs elements from Q .

There is only one way of choosing an element with exactly 4 ordered pairs from Q , that is, $4C_4 = 1$.

There are only four possible ways of selecting an element with exactly three ordered pairs from Q , that is, $4C_3 = 4$. So, there are 4 orbits of G_Q containing exactly 3 –ordered pairs elements from Q .

There is only 6 possible ways of selecting an element with exactly two ordered pairs elements from Q , that is, $4C_2 = 6$. Thus, there are 6 orbits with exactly 2-ordered pairs from Q .

There is only four ways selecting an element with exactly one ordered pair of elements from Q , that is, $4C_1 = 4$. So, there are four orbits with exactly 1-ordered pair from Q .

Lastly, there is only one way of selecting an element with no ordered pairs from Q . That is, there are $4C_0 = 1$ orbit with no ordered pair from Q .

The rank of G on $W^{[2]} \times X^{[2]} \times Y^{[2]} \times Z^{[2]}$ is;

$$R(G) = 4C_4 + 4C_3 + 4C_2 + 4C_1 + 4C_0 = 1 + 4 + 6 + 4 + 1.$$

$$R(G) = 16 = 2^4.$$

Theorem 2.2 (Subdegrees).

The subdegrees of $G = A_n \times A_n \times A_n \times A_n$ acting on $W^{[2]} \times X^{[2]} \times Y^{[2]} \times Z^{[2]}$ are; 1, $4 \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right)$, $6 \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right)^2$, $4 \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right)^3$ and $\left(\left(\frac{n!}{(n-2)!} \right) - 1 \right)^4$.

Proof:

Let $Q = \{[w_1, w_2], [x_1, x_2], [y_1, y_2], [z_1, z_2]\}$.

Suppose $G = A_n \times A_n \times A_n \times A_n$, acts on $W^{[2]} \times X^{[2]} \times Y^{[2]} \times Z^{[2]}$. Then, $W^{[2]} = \{[w_1, w_2], \dots, [w_n, w_{n-1}]\}$, $X^{[2]} = \{[x_1, x_2], \dots, [x_n, x_{n-1}]\}$, $Y^{[2]} = \{[y_1, y_2], \dots, [y_n, y_{n-1}]\}$ and $Z^{[2]} = \{[z_1, z_2], \dots, [z_n, z_{n-1}]\}$.

Let $Q = \{[w_1, w_2], [x_1, x_2], [y_1, y_2], [z_1, z_2]\}$.

By Armstrong Theorem, we have; $Orb_{G_Q} = Orb_{G_W} \times Orb_{G_X} \times Orb_{G_Y} \times Orb_{G_Z}$.

Let; $W^{[2]'} = \{[w_1, w_2]\} \setminus W^{[2]}$, $X^{[2]'} = \{[x_1, x_2]\} \setminus X^{[2]}$, $Y^{[2]'} = \{[y_1, y_2]\} \setminus Y^{[2]}$ and $Z^{[2]'} = \{[z_1, z_2]\} \setminus Z^{[2]}$.

$$|W^{[2]}| = |X^{[2]}| = |Y^{[2]}| = |P_4^{[2]}| = \frac{n!}{(n-2)!}.$$

Therefore, $|W^{[2]'}| = |X^{[2]'}| = |Y^{[2]'}| = |Z^{[2]'}| = \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right)$.

The 2^4 orbits of Orb_{G_P} are:

(a) The suborbit containing exactly four ordered pairs of elements from Q .

There is only one orbit of G_Q containing exactly 4 –ordered pairs elements from Q .

$$\Delta_0 = Orb_{G_Q} \{[w_1, w_2], [x_1, x_2], [y_1, y_2], [z_1, z_2]\}$$

$$|\Delta_0| = |[w_1, w_2]| \times |[x_1, x_2]| \times |[y_1, y_2]| \times |[z_1, z_2]|$$

Thus, $|\Delta_0| = 1$.

(b) Suborbits containing exactly three ordered pairs of elements from Q .

There are 4 orbits of G_Q containing exactly 3 –ordered pairs elements from Q .

$$\Delta_1 = Orb_{G_Q} \{w', [x_1, x_2], [y_1, y_2], [z_1, z_2]\}: w' \in W^{[2]'}$$

$$\begin{aligned} |\Delta_1| &= |W^{[2]'}| \times |[x_1, x_2]| \times |[y_1, y_2]| \times |[z_1, z_2]| \\ &= |W^{[2]'}| \times 1 \times 1 \times 1 \\ &= \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right) \times 1 \times 1 \times 1 \end{aligned}$$

Thus, $|\Delta_1| = \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right)$.

$$\Delta_2 = Orb_{G_Q} \{[w_1, w_2], x', [y_1, y_2], [z_1, z_2]\}: x' \in X^{[2]'}$$

$$\begin{aligned} |\Delta_2| &= |[w_1, w_2]| \times |X^{[2]'}| \times |[y_1, y_2]| \times |[z_1, z_2]| \\ &= 1 \times |X^{[2]'}| \times 1 \times 1 \\ &= 1 \times \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right) \times 1 \times 1 \end{aligned}$$

Thus, $|\Delta_3| = \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right)$.

$$\Delta_3 = Orb_{G_Q} \{[w_1, w_2], [x_1, x_2], y', [z_1, z_2]\}: y' \in Y^{[2]'}$$

$$\begin{aligned} |\Delta_3| &= |[w_1, w_2]| \times |[x_1, x_2]| \times |Y^{[2]'}| \times |[z_1, z_2]| \\ &= 1 \times 1 \times |Y^{[2]'}| \times 1 \\ &= 1 \times 1 \times \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right) \times 1 \end{aligned}$$

Thus, $|\Delta_3| = \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right)$.

$$\Delta_4 = Orb_{G_Q} \{[w_1, w_2], [x_1, x_2], [y_1, y_2], z'\}: z' \in Z^{[2]'}$$

$$\begin{aligned} |\Delta_4| &= |[w_1, w_2]| \times |[x_1, x_2]| \times |[y_1, y_2]| \times |Z^{[2]'}| \\ &= 1 \times 1 \times 1 \times |Z^{[2]'}| \\ &= 1 \times 1 \times 1 \times \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right) \end{aligned}$$

Thus, $|\Delta_4| = \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right)$.

Therefore, $|\Delta_1| = |\Delta_2| = |\Delta_3| = |\Delta_4|$.

(c) **Suborbits containing exactly two ordered pairs of elements from Q .**

There are 6 orbits with exactly 2-ordered pairs from Q .

$$\Delta_5 = Orb_{G_Q} \{w', x', [y_1, y_2], [z_1, z_2]\}: w' \in W^{[2]', y' \in Y^{[2]'}$$

$$|\Delta_5| = |W^{[2]'}| \times |Y^{[2]'}| \times |[x_1, x_2]| \times |[y_1, y_2]|$$

$$= \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right) \times \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right) \times 1 \times 1$$

Thus, $|\Delta_5| = \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right)^2$.

But, $|\Delta_5| = |\Delta_6| = |\Delta_7| = |\Delta_8| = |\Delta_9| = |\Delta_{10}| = \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right)^2$.

(d) Suborbits containing exactly one ordered pair of elements from Q .

There are 4 orbits with exactly 2-ordered pairs from Q .

$$\Delta_{11} = Orb_{G_Q} \{[w_1, w_2], x', y', z'\} : x' \in X^{[2]'}, y' \in Y^{[2]'}, z' \in Z^{[2]}'$$

$$|\Delta_{11}| = |[w_1, w_2]| \times |X^{[2]'}| \times |Y^{[2]'}| \times |Z^{[2]}'|$$

$$= 1 \times \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right) \times \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right) \times \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right)$$

Thus, $|\Delta_{11}| = \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right)^3$.

But, $|\Delta_{11}| = |\Delta_{12}| = |\Delta_{13}| = |\Delta_{14}| = \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right)^3$.

(e) Suborbits with no ordered pair of elements from Q .

There is only one orbit of G_Q having no ordered pairs elements from Q .

$$\Delta_{15} = Orb_{G_Q} \{w', x', y', z'\} : w' \in W^{[2]'}, x' \in X^{[2]'}, y' \in Y^{[2]'}, z' \in Z^{[2]}'$$

$$|\Delta_{15}| = |W^{[2]'}| \times |X^{[2]'}| \times |Y^{[2]'}| \times |Z^{[2]}'|$$

$$= \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right) \times \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right) \times \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right) \times \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right)$$

Thus, $|\Delta_{15}| = \left(\left(\frac{n!}{(n-2)!} \right) - 1 \right)^4$.

Table 1 : Subdegrees of $A_n \times A_n \times A_n \times A_n$ acting on $W^{[2]} \times X^{[2]} \times Y^{[2]} \times Z^{[2]}$

Length of suborbit	1	$\left(\left(\frac{n!}{(n-2)!}\right) - 1\right)$	$\left(\left(\frac{n!}{(n-2)!}\right) - 1\right)^2$	$\left(\left(\frac{n!}{(n-2)!}\right) - 1\right)^3$	$\left(\left(\frac{n!}{(n-2)!}\right) - 1\right)^4$
Number of suborbits	1	4	6	4	1

Theorem 2.3 (Summation of subdegrees)

The sum of the number of elements in all the 16 Q –orbits equals the cardinality of

$$W^{[2]} \times X^{[2]} \times Y^{[2]} \times Z^{[2]}, \text{ that is; } 1 + 4\left(\left(\frac{n!}{(n-2)!}\right) - 1\right) + 6\left(\left(\frac{n!}{(n-2)!}\right) - 1\right)^2 + 4\left(\left(\frac{n!}{(n-2)!}\right) - 1\right)^3 + \left(\left(\frac{n!}{(n-2)!}\right) - 1\right)^4 = |W^{[2]} \times X^{[2]} \times Y^{[2]} \times Z^{[2]}|.$$

Proof:

The cardinality, $|W^{[2]} \times X^{[2]} \times Y^{[2]} \times Z^{[2]}| = \left(\frac{n!}{(n-2)!}\right)^4$. It suffices to show that sum of the subdegrees is equivalent to the cardinality of $W^{[2]} \times X^{[2]} \times Y^{[2]} \times Z^{[2]}$.

Let $t = \left(\frac{n!}{(n-2)!}\right)$. So, we have;

$$1 + 4(t - 1) + 6\{(t - 1)^2\} + 4\{(t - 1)^3\} + (t - 1)^4 = 1 + 4t - 4 + 6t^2 - 12t + 6 + 4t^3 - 12t^2 + 12t - 4 + t^4 - 4t^3 + 6t^2 - 4t + 1 = t^4$$

But, $t = \left(\frac{n!}{(n-2)!}\right)$, so $t^4 = \left(\frac{n!}{(n-2)!}\right)^4 = |W^{[2]} \times X^{[2]} \times Y^{[2]} \times Z^{[2]}|$.

3. CONCLUSION

From this research, it can be concluded that the rank of $A_n \times A_n \times A_n \times A_n, (\forall n \geq 4)$ acting

on $W^{[2]} \times X^{[2]} \times Y^{[2]} \times Z^{[2]}$, is 2^4 and the subdegrees are: $1, 4\left(\left(\frac{n!}{(n-2)!}\right) - 1\right), 6\left(\left(\frac{n!}{(n-2)!}\right) - 1\right)^2, 4\left(\left(\frac{n!}{(n-2)!}\right) - 1\right)^3$ and $\left(\left(\frac{n!}{(n-2)!}\right) - 1\right)^4$.

CONFLICT OF INTEREST

The author hereby declares that NO conflict of interest is present in this work.

Disclaimer (Artificial intelligence)

The author hereby declares that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

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