A Study On Dual Generalized Edouard Numbers

Abstract. In this research, we introduce the generalized dual Edouard numbers, a novel class of number sequences that extends existing recurrence relations into a new mathematical framework. Several special cases of these numbers are examined in detail, including the dual Edouard numbers and the dual Edouard-Lucas numbers, each revealing intriguing combinatorial and algebraic properties.

Explicit expressions for these sequences are derived, such as Binet-type formulas, generating functions, and summation identities, which offer analytical insight into their behavior and structural patterns. In addition, we explore matrix representations associated with these sequences, providing an elegant algebraic tool for further theoretical development and potential applications.

2010 Mathematics Subject Classification. 11B37,11B39, 11B83.

Keywords. Edouard numbers, Edouard-Lucas numbers, dual Edouard numbers, dual Edouard-Lucas numbers,

1. Introduction

In mathematical and geometric contexts, a hypercomplex system refers to a framework that generalizes the principles of complex numbers. These systems possess rich algebraic structures and are frequently studied for their diverse applications in physics and engineering. Below, we provide a concise overview of the key application areas of hypercomplex number systems in these fields.

In contrast to complex numbers, hypercomplex systems provide a more sophisticated framework for representing transformations and symmetries in higher-dimensional spaces. As noted by Kantor in [20], these systems can be viewed as extensions of the real number line, offering algebraic tools tailored to multidimensional analysis. The principal types of hypercomplex number systems encompass complex numbers, hyperbolic numbers, and dual numbers. Complex numbers, defined by a real and an imaginary component, serve as the foundational structure for more advanced hypercomplex systems. Hyperbolic numbers build upon the complex number framework and are employed in diverse mathematical models, particularly those involving Lorentz transformations and spacetime geometries. Dual numbers, distinguished by the presence of a dual unit whose square is zero, are instrumental in various algebraic constructions, including automatic differentiation and kinematic analysis.

The following sections offer more detailed insights into the mathematical properties and application areas of these hypercomplex systems.

- Complex numbers are constructed by extending the real number system through the introduction of an imaginary unit, denoted as "i", which satisfies the identity $i^2 = -1$. A complex number is typically expressed in the form $z = a + bi$, where a and b are real numbers, and i represents the imaginary unit.
- Hyperbolic numbers also referred to as double numbers or split complex numbers extend the real number system by introducing a new unit element j, which satisfies the identity $j^2 = 1$ [25]. These numbers are distinct from real and complex numbers due to their unique algebraic properties. A hyperbolic number is defined as:

  $$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\}.$$

  where a and b are real numbers and j is the hyperbolic unit. This structure enables the modeling of systems with split-signature metrics and has notable applications in areas such as special relativity and signal processing.
- Dual numbers [13] expand the real number system through the incorporation of a new element ε, which satisfies the identity $\varepsilon^2 = 0$. This infinitesimal unit distinguishes dual numbers from other hypercomplex systems and makes them especially valuable in modeling instantaneous rates of change. A dual number is defined as:

  $$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

  where a and b are real numbers, and ε is the nilpotent unit. Dual numbers are commonly used in applications such as automatic differentiation, kinematics, and perturbation analysis, due to their ability to elegantly encode infinitesimal variations.
- Among the non-commutative examples of hypercomplex number systems are quaternions [16]. Quaternions generalize complex numbers by incorporating three distinct imaginary units, typically denoted as i,j, and k. A quaternion has the form as $a_0 + ia_1 + ja_2 + ka_3$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$.These multiplication rules result in a non-commutative structure, meaning the order of multiplication affects the result.The set of quaternion numbers is formally defined as:

  $$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\},$$

- Additional hypercomplex systems include octonions and sedenions, which are discussed in [19] and [26]. The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions), and $\mathbb{S}$ (sedenions) are all constructed as real algebras derived from the real numbers $\mathbb{R}$ using a recursive procedure known as the Cayley–Dickson Process. This technique successively doubles the dimension of each algebra and continues beyond sedenions to produce what are collectively referred to as the 2^n-ions.The following table highlights selected publications from the literature that investigate the properties and applications of these extended number systems.

  For more information on hypercomplex algebra, see [22,17,24,21]

A dual hyperbolic number is a type of hypercomplex number, specifically a member of the hyperbolic number system. A dual hyperbolic number is defined as follows

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$.

$\mathbb{H}_{\mathbb{D}}$, the set of all dual hyperbolic numbers, are generally denoted by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The $\{1, j, \varepsilon, \varepsilon j\}$ is linearly independent, and the algebra $\mathbb{H}_{\mathbb{D}}$ is generated by their span, i.e. $\mathbb{H}_{\mathbb{D}} = sp\{1, j, \varepsilon, \varepsilon j\}$

Therefore, $\{1, j, \varepsilon, \varepsilon j\}$ forms a basis for the dual hyperbolic algebra $\mathbb{H}_{\mathbb{D}}$. For more detail, see [3].

The next properties are holds for the base elements $\{1, j, \varepsilon, \varepsilon j\}$ of dual hyperbolic numbers (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j,\ \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0,\ j^2 = j.j = 1 \\ \varepsilon.j &= j.\varepsilon,\ \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0,\ j(\varepsilon j) = (\varepsilon j)j = \varepsilon \end{aligned}$$

where ε denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j denotes the hyperbolic unit ($j^2 = 1$), and εj denotes the dual hyperbolic unit ($(j\varepsilon)^2 = 0$).

We claim that p and q be two dual hyperbolic numbers that $q = a_0 + ja_1 + \varepsilon a_2 + j\varepsilon a_3$ and $p = b_0 + jb_1 + \varepsilon b_2 + j\varepsilon b_3$ and then we can write the product of p and q as

$$qp = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0) + \varepsilon(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + j\varepsilon(a_0b_3 + a_1b_2 + a_2b_1 + b_0a_3)$$

and we can write the sum dual hyperbolic numbers p and q as componentwise.

The dual hyperbolic numbers form a commutative ring, real vector space and an algebra. $\mathbb{H}_{\mathbb{D}}$ is not field since every dual hyperbolic numbers doesn't have an inverse. For more detail about dual hyperbolic numbers, see [3].

It's known that many author studied the generalized (r, s, t) sequence. One of these sequences is generalized Edouard numbers. Soykan, [30] defined generalized Edouard numbers. Before we present our original study , we recall some proporities related to generalized Edouard numbers such as reccurance relations, Binet's formula, generating function .

A generalized Edouard sequence , with the initial values W_0, W_1, W_2 not all being zero, $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 7W_{n-1} - 7W_{n-2} + W_{n-3};\ W_0, W_1, W_2 \quad (n \geq 3) \tag{1.1}$$

Moreover, we define generalized Edouard sequence given to negative subscripts as follows,

$$W_{-n} = 7W_{-(n-1)} - 7W_{-(n-2)} + W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Thus, recurrence (1.1) is true for all integer n.

In the Table 1 we give the first some generalized Edouardnumbers with positive subscript and negative subscript

Table 1. A few generalized Edouard numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$7W_0 - 7W_1 + W_2$
2	W_2	$42W_0 - 48W_1 + 7W_2$
3	$W_0 - 7W_1 + 7W_2$	$246W_0 - 287W_1 + 42W_2$
4	$7W_0 - 48W_1 + 42W_2$	$1435W_0 - 1680W_1 + 246W_2$
5	$42W_0 - 287W_1 + 246W_2$	$8365W_0 - 9799W_1 + 1435W_2$
6	$246W_0 - 1680W_1 + 1435W_2$	$48756W_0 - 57120W_1 + 8365W_2$

If we obtain,respectively, $W_0 = 0, W_1 = 1, W_2 = 7$ then$\{W_n\} = \{E_n\}$ is called the Edouard sequence, $W_0 = 3, W_1 = 7, W_2 = 35$ then $\{W_n\} = \{K_n\}$ is called the Edouard-Lucas sequence. Alternatively, Edouard sequence $\{E_n\}_{n\geq 0}$, Edouard-Lucas sequence $\{K_n\}_{n\geq 0}$ are given by the third-order recurrence relations as

$$E_n = 7E_{n-1} - 7E_{n-2} + E_{n-3}, \qquad E_0 = 0, E_1 = 1, E_2 = 7, \tag{1.2}$$

$$K_n = 7K_{n-1} - 7K_{n-2} + K_{n-3}, \qquad K_0 = 3, K_1 = 7, K_2 = 35, \tag{1.3}$$

The sequences given above can be extended to negative subscripts by defining, respectively,

$$\begin{aligned} E_{-n} &= 7E_{-(n-1)} - 7E_{-(n-2)} + E_{-(n-3)}, \\ K_{-n} &= 7K_{-(n-1)} - 7K_{-(n-2)} + K_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, ...$. As a consequence, recurrences (1.2)-(1.3) hold for all integer n.

We can list some important properties of generalized Edouard numbers that are needed.

Binet formula of generalized Edouard sequence can be calculated using its characteristic equation written as

$$x^3 - 7x^2 + 7x - 1 = \left(x^2 - 6x + 1\right)(x - 1) = 0.$$

The roots of the characteristic equation are

$$\begin{aligned} \alpha &= 3+2\sqrt{2}, \\ \beta &= 3-2\sqrt{2}, \\ \gamma &= 1, \end{aligned}$$

By using these roots and the recurrence relation, Binet formula are written below

$$\begin{aligned} W_n &= \frac{z_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{z_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{z_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \\ &= \frac{z_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{z_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} - \frac{z_3}{4} \end{aligned}$$

where

$$\begin{aligned} z_1 &= W_2-(\beta+1)W_1+\beta W_0, \\ z_2 &= W_2-(\alpha+1)W_1+\alpha W_0, \\ z_3 &= W_2-6W_1+W_0. \end{aligned}$$

and

$$\begin{aligned} A_1 &= \frac{W_2-(\beta+1)W_1+\beta W_0}{(\alpha-\beta)(\alpha-\gamma)}, \\ A_2 &= \frac{W_2-(\alpha+1)W_1+\alpha W_0}{(\beta-\alpha)(\beta-\gamma)}, \\ A_3 &= \frac{W_2-6W_1+W_0}{(\gamma-\alpha)(\gamma-\beta)}. \end{aligned} \tag{1.4}$$

Then we present Binet formula of Edouard sequences and Edouard-Lucas sequences, respectively, given below

$$\begin{aligned} E_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-1)} - \frac{1}{4}, \\ K_n &= \alpha^n+\beta^n+1. \end{aligned}$$

After then we can write the generating function of generalized Edouard numbers,

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0+(W_1-7W_0)x+(W_2-7W_1+7W_0)x^2}{1-7x+7x^2-x^3}. \tag{1.5}$$

Next, we give the exponential generating function of $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence W_n.

Lemma 1. *[5, Lemma 1.4]. Suppose that* $f_{GW_n}(x) = \sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ *is the exponential generating function of the generalized Edouard sequence* $\{W_n\}$*. Then*

$$\sum_{n=0}^{\infty} W_n \frac{x^n}{n!} = \frac{(W_2-(\beta+1)W_1+\beta W_0)}{(\alpha-\beta)(\alpha-1)} e^{\alpha x} + \frac{(W_2-(\alpha+1)W_1+\alpha W_0)}{(\beta-\alpha)(\beta-1)} e^{\beta x} - \frac{(W_2-6W_1+W_0)}{4} e^{x}.$$

The previous Lemma gives the following results as particular examples.

Corollary 2. *Exponential generating function of Edouard and Edouard-Lucas numbers are*

a):

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-1)} - \frac{1}{4}) \frac{x^n}{n!} = \frac{\alpha e^{\alpha x}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta e^{\beta x}}{(\beta-\alpha)(\beta-1)} - \frac{1}{4} e^x.$$

b):

$$\sum_{n=0}^{\infty} K_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (\alpha^n + \beta^n + 1) \frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^x.$$

For more details, see [30].

Now, we are presenting information about specific number systems, including the hypercomplex system, which encompasses complex numbers, hyperbolic numbers, and dual numbers. We note that hyperbolic numbers will play a crucial role in our work. Moreover hyperbolic functions and numbers find applications in various branches of engineering, such as electrical engineering (e.g., transmission lines), control systems (e.g., system dynamics), signal processing (e.g., filter design), and diverse fields of engineering physics, including special relativity, wave propagation, fluid dynamics, optics, and heat conduction. It's important to note that while hyperbolic numbers have interesting mathematical properties, their adoption in practical applications depends on the specific problem at hand and whether they offer advantages over other number systems in a given context.

Initially, we discuss hypercomplex number systems, which are extensions of real numbers, for more detail see [20]. In addition that some commutative special cases of hypercomplex number systems include complex numbers, hyperbolic numbers, and dual numbers. These systems are widely used in various branches of mathematics and physics. We will now present these number systems sequentially, as outlined below.

- Complex numbers simplest form of hypercomplex numbers. Complex numbers defined as $z = a+ib$, where a and b real numbers and i imaginary unit that satisfy $i^2 = -1$. In addition that a and b named, respectively, $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ Consequently, the definition of complex numbers given by,

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\}.$$

- Hyperbolic (double, split-complex) numbers, for more detail see [25], Split-complex numbers, commonly recognized as hyperbolic numbers, defined as $h = a + jb$ where a and b real numbers and j hyperbolic unit that satisfy $j^2 = 1$. In addition that a and b named, respectively, $Re(h)$ and $Hyp(h)$. Thus, the definition of hyperbolic numbers given by,

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\},$$

- Dual numbers, see [13], defined as $d = a + \varepsilon b$ where a and b real numbers and ε dual unit that satisfy $\varepsilon^2 = 0$. Furthermore, a and b called, respectively, $Re(d)$ and $Du(d)$. Thus, defination of dual numbers given by,

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

- A dual hyperbolic number, specifically within the hyperbolic number system, constitutes a distinct type of hypercomplex number. A dual hyperbolic number is defined by,

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$ and the set of all dual hyperbolic numbers are defined by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon ja_3 : a_0, a_1, a_2, a_3 \in \mathbb{R},\ j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The $\{1, j, \varepsilon, \varepsilon j\}$ is linear independent and $\mathbb{H}_{\mathbb{D}} = sp\{1, j, \varepsilon, \varepsilon j\}$ so that $\{1, j, \varepsilon, \varepsilon j\}$ is a basis of $\mathbb{H}_{\mathbb{D}}$. For more detail see, [3]

The next properties are true for the base elements $\{1, j, \varepsilon, \varepsilon j\}$ (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j,\ \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0,\ j^2 = j.j = 1 \\ \varepsilon.j &= j.\varepsilon,\ \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0,\ j(\varepsilon j) = (\varepsilon j)j = \varepsilon \end{aligned}$$

where ε satisfy the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j satisfy the hyperbolic unit ($j^2 = 1$), and εj satisfy the dual hyperbolic unit ($(j\varepsilon)^2 = 0$).

In addition that the other number sytems are quarternions, octonions and sedenions given below, respectively,

- Quaternion numbers, non-commutative examples of hypercomplex number systems, are a four-dimensional extension of complex numbers. They are expressed as $a_0 + ia_1 + ja_2 + ka_3$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$, and i, j, and k are the quaternion units that satisfy specific multiplication rules. For more detail see [16]. Quaternion numbers are defined by

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\},$$

- Octonions is a set, every element of the set linear combinations of unit octonions $\{e_i : i = 0, 1.2, ..., 7\}$, doneted as $\mathbb{O}$. Octonions are defined by,

$$\mathbb{O} = \{\sum_{i=0}^{7} a_i e_i : a_i \in \mathbb{R}, e_0 e_i = e_i e_0 = e_i,\ e_i e_j = -\delta_{ij} e_0 + \varepsilon_{ijk} e_k\ \}$$

where $e_e = 1$, δ_{ij} is Kroneker delta (equal to 1 if and only if $i = j$), ε_{ijk} is anti-symetric tensor. For more detail see [19, 33]

- Sedenions is a set, every element of the set linear combinations of unit sedenions $\{e_i : i = 0, 1.2, ..., 15\}$, denoted by $\mathbb{S}$. It can be seen from here that ever sedenion can be written as

$$\sum_{i=0}^{15} a_i e_i$$

where a_i is real number. For more detail see, [26, 33].

Next we give some proporties on two hyperbolic numbers, $h_1 = a + jb$ and $h_2 = c + jd$, as

$$
\begin{aligned}
h_1 + h_2 &= (a+b) + j(c+d), \\
h_1 . h_2 &= (ac+bd) + j(ad+bc), \\
\overline{h_1} &= a - jb \\
\frac{h_1}{h_2} &= \frac{(ac-bd)+j(cb-ad)}{c^2-d^2}, \\
h_1 &= h_2 \text{ if only if } a = c \text{ and } b = d, \\
\langle h_1, h_2 \rangle &= (ac+bd) + j(bc+ad), \\
\|h_1\| &= \sqrt{|a^2-b^2|}, \text{ called norm of } h_1, \\
\text{if } \left|a^2-b^2\right| &> 0, \; h_1 \text{ is named spacelike vector}, \\
\text{if } \left|a^2-b^2\right| &< 0, \; h_1 \text{ is named timelike vector}, \\
\text{if } \left|a^2-b^2\right| &= 0, \; h_1 \text{ is named null(light-like) vector}.
\end{aligned}
$$

Note that$\{\mathbb{R}^2, H, \langle, \rangle\}$ is called Lorentz plane and denoted as $\mathbb{R}_1^2$. There is an isomorphism relationship between the Lorentz plane and hyperbolic numbers. For more detail, see [33].

Hence the algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are real algebras attained from the real numbers $\mathbb{R}$ by a doubling procedure known as the Cayley-Dickson Process. This doubling process can be extended beyond the sedenions to form what are known as the 2^n-ions (see for example [6, 16, 18, 23, 15].

Some authors have conducted studies about the dual, hyperbolic, dual hyperbolic and other special numbers. Now we give some information published papers in litarature.

- Cockle [10] explored hyperbolic numbers with complex coefficients, contributing to the early development of hypercomplex algebra.
- Eren and Soykan [12] studied the generalized Generalized Woodall Numbers.
- Cheng and Thompson [8] introduced dual numbers with complex coefficients, expanding the algebraic versatility of dual number systems for applications in polynomial equations and transformation theory.
- Akar at al [3] introduced the concept of dual hyperbolic numbers, combining characteristics of dual and hyperbolic systems into a unified algebraic structure.

Next, we present some information on hyperbolic numbers presented in literature.

- Aydın [1] presented hyperbolic Fibonacci numbers given by

$$\widetilde{F}_n = F_n + hF_{n+1},$$

where Fibonacci numbers are given by $F_{n+2} = F_{n+1} + F_n$, with the initial conditation $F_0 = 0$, $F_1 = 1$.

- Soykan and Taşdemir [28] studied hyperbolic generalized Jacobsthal numbers given by

$$\widetilde{V}_n = V_n + hV_{n+1}$$

where generalized Jacobsthal numbers are $V_{n+2} = V_{n+1} + 2V_n$ with the initial conditation $V_0 = a$, $V_1 = b$.

- Taş [32] studied hyperbolic Jacobsthal-Lucas sequence written by

$$HJ_n = J_n + hJ_{n+1}$$

where Jacobsthal-Lucas numbers given by $J_{n+2} = J_{n+1} + 2J_n$ with the inintial conditation $J_0 = 2$, $J_1 = 1$.

- Dikmen and Altınsoy, [11] studied On Third Order Hyperbolic Jacobsthal Numbers given by

$$\begin{aligned} \widehat{J}_n^{(3)} &= J_n^{(3)} + hJ_{n+1}^{(3)}, \\ \widehat{j}_n^{(3)} &= j_n^{(3)} + hj_{n+1}^{(3)} \end{aligned}$$

where Jacobsthal numbers, respectively, given by $J_n^{(3)} = J_{n-1}^{(3)} + J_{n-2}^{(3)} + 2J_{n-3}^{(3)}$, $J_0^{(3)} = 0$, $J_1^{(3)} = 1$, $J_2^{(3)} = 1$, $j_n^{(3)} = j_{n-1}^{(3)} + j_{n-2}^{(3)} + 2j_{n-3}^{(3)}$, $j_0^{(3)} = 2$, $j_1^{(3)} = 1$, $j_2^{(3)} = 5$.

Following this, we provide details on dual hyperbolic sequences as they are presented in literature.

- Soykan et al [27] presented dual hyperbolic generalized Pell numbers given by

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}$$

where generalized Pell numbers, with the initial values V_0, V_1 not all being zero, are given by $V_n = 2V_{n-1} + V_{n-2}$, $V_0 = a$, $V_1 = b$ $(n \geq 2)$.

- Cihan et al [2] studied dual hyperbolic Fibonacci and Lucas numbers given by, respectively,

$$\begin{aligned} DHF_n &= F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3}, \\ DHL_n &= L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3} \end{aligned}$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$, $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$.

- Soykan et al [28] studied dual hyperbolic generalized Jacopsthal numbers given by

$$\widehat{J}_n = J_n + jJ_{n+1} + \varepsilon J_{n+2} + j\varepsilon J_{n+3}$$

where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = a$, $J_1 = b$.

- Yılmaz and Soykan [34] introduced dual hyperbolic generalized Guglielmo numbers are

$$\widehat{T}_0 = T_0 + jT_1 + \varepsilon T_2 + j\varepsilon T_3$$

where $T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, T_0 = 0, T_1 = 1, T_2 = 3.$

- Ayrılma and Soykan [4] studied dual hyperbolic generalized Edouard number and Edouard-Lucas number given by

$$\begin{aligned} \widehat{E}_0 &= E_0 + jE_1 + \varepsilon E_2 + j\varepsilon E_3, \\ \widehat{K}_0 &= K_0 + jK_1 + \varepsilon K_2 + j\varepsilon K_3, \end{aligned}$$

where $E_n = 7E_{n-1} - 7E_{n-2} + E_{n-3}$, $E_0 = 0, E_1 = 1, E_2 = 7$ and $K_n = 7K_{n-1} - 7K_{n-2} + K_{n-3}$, $K_0 = 3, K_1 = 7, K_2 = 35.$

- Bród et al [7] studied dual hyperbolic generalized balancing numbers as

$$DHB_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}$$

where $B_n = 6B_{n-1} - B_{n-2}$, $B_0 = 0$, $B_1 = 1$.

Next section, we define the dual generalized Edouard numbers and some special properties, generating function and Binet's formula , of these numbers.

2. Dual Generalized Edouard Numbers and their Generating Functions and Binet's Formulas

In this section, we define dual generalized Edouard numbers then we present generating functions and Binet formulas for these numbers.

On the set of $\mathbb{H}_{\mathbb{D}}$, we will now explore dual generalized Edouard numbers on $\mathbb{D}$.The nth generalized dual Edouard numbers, with DW_0, DW_1, DW_2 being the initial conditions, are defined as follows

$$DW_n = W_n + \varepsilon W_{n+1}. \tag{2.1}$$

in addition (2.1) can be written to negative subscripts by defining,

$$DW_{-n} = W_{-n} + \varepsilon W_{-n+1} \tag{2.2}$$

so identity (2.1) holds for all integers n.

Now we define some special cases of dual generalized Edouard numbers. The nth dual edouard numbers, the nth dual Edouard-Lucas numbers, respectively, are given as

the nth generalized dual Edouard numbers $DE_n = E_n + \varepsilon E_{n+1}$, with DK_0, DK_1, DK_2 being the initial conditions, are defined as follows

$$DE_n = E_n + \varepsilon E_{n+1}$$

where

$$\begin{aligned} DE_0 &= E_0+\varepsilon E_1, \\ DE_1 &= E_1+\varepsilon E_2, \\ DE_2 &= E_2+\varepsilon E_3, \end{aligned}$$

the nth generalized dual Edouard-Lucas numbers $DK_n = K_n+\varepsilon K_{n+1}$, with DK_0, DK_1, DK_2 being the initial conditions, are defined as follows

$$DK_n = K_n+\varepsilon\ K_{n+1}$$

where

$$\begin{aligned} DK_0 &= K_0+\varepsilon\ K_1, \\ DK_1 &= K_1+\varepsilon\ K_2, \\ DK_2 &= K_2+\varepsilon\ K_3, \end{aligned}$$

For dual Edouard numbers, taking $W_n = E_n$, $E_0 = 0$, $E_1 = 1$, $E_2 = 7$, we get

$$\begin{aligned} DE_0 &= \varepsilon, \\ DE_1 &= 1+7\varepsilon, \\ DE_2 &= 7+42\varepsilon, \end{aligned}$$

for dual Edouard-Lucas numbers, taking $W_n = K_n$, $K_0 = 3$, $K_1 = 7$, $K_2 = 35$, we get

$$\begin{aligned} DK_0 &= 3+7\varepsilon, \\ DK_1 &= 7+35\varepsilon, \\ DK_2 &= 35+199\varepsilon, \end{aligned}$$

Thus, by using (2.1), we can formulate the following identity for non-negative integers n,

$$DW_n = 7DW_{n-1} - 7DW_{n-2} + DW_{n-3}. \tag{2.3}$$

Hence the sequence $\{DW_n\}_{n\geq 0}$ can be given as

$$DW_{-n} = 7DW_{-(n-1)} - 7DW_{-(n-2)} + DW_{-(n-3)},$$

for $n\in\{1,2,3....\}$ by using (2.2). Accordingly, recurrence (2.3) is true for all integer n.

In the Table 2, We provide the initial dual generalized Edouard numbers with both positive and negative subscripts.

Table 2. Some dual generalized Edouard numbers

n	DW_n	DW_{-n}
0	DW_0	DW_0
1	DW_1	$7DW_0 - 7DW_1 + DW_2$
2	DW_2	$42DW_0 - 48DW_1 + 7DW_2$
3	$DW_0 - 7DW_1 + 7DW_2$	$246DW_0 - 287DW_1 + 42DW_2$
4	$7DW_0 - 48DW_1 + 42DW_2$	$1435DW_0 - 1680DW_1 + 246DW_2$
5	$42DW_0 - 287DW_1 + 246DW_2$	$8365DW_0 - 9799DW_1 + 1435DW_2$
6	$246DW_0 - 1680DW_1 + 1435DW_2$	$48756DW_0 - 57120DW_1 + 8365DW_2$

Note that

$$DW_0 = W_0 + \varepsilon W_1,\ DW_1 = W_1 + \varepsilon W_2,\ DW_2 = W_2 + \varepsilon W_3.$$

Some dual Edouard numbers, dual Edouard-Lucas numbers with positive or negative subscripts are presented tables which is given below .

Table 3. dual Edouard numbers

n	DE_n	DE_{-n}
0	ε	
1	$1 + 7\varepsilon$	0
2	$7 + 42\varepsilon$	1
3	$42 + 246\varepsilon$	$7 + \varepsilon$
4	$246 + 1435\varepsilon$	$42 + 7\varepsilon$
5	$1435 + 8365\varepsilon$	$246 + 42\varepsilon$

Table 4. dual Edouard-Lucas numbers

n	DK_n	DK_{-n}
0	$3 + 7\varepsilon$	
1	$7 + 35\varepsilon$	$7 + 3\varepsilon$
2	$35 + 199\varepsilon$	$35 + 7\varepsilon$
3	$199 + 1155\varepsilon$	$199 + 35\varepsilon$
4	$1155 + 6727\varepsilon$	$1155 + 199\varepsilon$
5	$6727 + 39203\varepsilon$	$6727 + 1155\varepsilon$

Now, we will establish Binet's formula for the dual generalized Edouard numbers, and for the remainder of the study, we will utilize the following notations:

$$\widetilde{\alpha} = 1 + \varepsilon\alpha, \tag{2.4}$$

$$\widetilde{\beta} = 1 + \varepsilon\beta, \tag{2.5}$$

$$\widetilde{\gamma} = 1 + \varepsilon. \tag{2.6}$$

Note that the following identities are true.

$$\begin{aligned}
\widetilde{\alpha}^2 &= 1 + 2\alpha\varepsilon, \\
\widetilde{\beta}^2 &= 1 + 2\beta\varepsilon, \\
\widetilde{\gamma}^2 &= 1 + 2\varepsilon, \\
\widetilde{\alpha}\widetilde{\beta} &= 1 + \varepsilon(\alpha + \beta), \\
\widetilde{\alpha}\widetilde{\gamma} &= 1 + \varepsilon(\alpha + \gamma), \\
\widetilde{\gamma}\widetilde{\beta} &= 1 + \varepsilon(\gamma + \beta).
\end{aligned}$$

THEOREM 3. *(Binet's Formula) For any integer n, the nth dual generalized Edouard number can be expressed as follows*

$$DW_n = \widetilde{\alpha} A_1 \alpha^n + \widetilde{\beta} A_2 \beta^n + \widetilde{\gamma} A_3 \tag{2.7}$$

where $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\gamma}$ are given as (2.4),(2.5),(2.6).

Proof. Using Binet's formula of the generalized Edouard numbers given below

$$W_n = A_1 \alpha^n + A_2 \beta^n + A_3$$

where A_1, A_2, A_3 are given (1.4) we get

$$\begin{aligned} DW_n &= W_n + \varepsilon W_{n+1}, \\ &= A_1 \alpha^n + A_2 \beta^n + A_3 + (A_1 \alpha^{n+1} + A_2 \beta^{n+1} + A_3)\varepsilon \\ &= \widetilde{\alpha} A_1 \alpha^n + \widetilde{\beta} A_2 \beta^n + \widetilde{\gamma} A_3. \end{aligned}$$

This proves (2.7). □

As special cases, for any integer n, the Binet's Formula of nth dual Edouard numbers, the Binet's Formula of nth dual Edouard-Lucas numbers, respectively, are

$$\begin{aligned} E_n &= \frac{\widetilde{\alpha} \alpha^{n+1}}{(\alpha - \beta)(\alpha - 1)} + \frac{\widetilde{\beta} \beta^{n+1}}{(\beta - \alpha)(\beta - 1)} - \frac{\widetilde{\gamma}}{4}, \\ K_n &= \widetilde{\alpha} \alpha^n + \widetilde{\beta} \beta^n + \widetilde{\gamma}, \end{aligned}$$

Next, we will introduce the generating function of the dual generalized Edouard numbers.

THEOREM 4. *The generating function for the dual generalized Edouard numbers is*

$$f_{DW_n}(x) = \frac{DW_0 + (DW_1 - 7DW_0)x + (DW_2 - 7DW_1 + 7DW_0)x^2}{(1 - 7x + 7x^2 - x^3)}. \tag{2.8}$$

Proof. Let the generating function of the dual generalized Edouard numbers is given below

$$f_{DW_n}(x) = \sum_{n=0}^{\infty} DW_n x^n$$

Following that, by utilizing the definition of the dual generalized Edouard numbers, and substracting $7xg(x)$ and $-7x^2g(x)$and $x^3g(x)$ from $g(x)$, we get

$$\begin{aligned}
(1-7x+7x^2-x^3)f_{GDW_n}(x) &= \sum_{n=0}^{\infty} DW_n x^n - 7x\sum_{n=0}^{\infty} DW_n x^n + 7x^2\sum_{n=0}^{\infty} DW_n x^n - x^3\sum_{n=0}^{\infty} DW_n x^n, \\
&= \sum_{n=0}^{\infty} DW_n x^n - 7\sum_{n=0}^{\infty} DW_n x^{n+1} + 7\sum_{n=0}^{\infty} DW_n x^{n+2} - \sum_{n=0}^{\infty} DW_n x^{n+3}, \\
&= \sum_{n=0}^{\infty} DW_n x^n - 7\sum_{n=1}^{\infty} DW_{n-1} x^n + 7\sum_{n=2}^{\infty} DW_{n-2} x^n - \sum_{n=3}^{\infty} DW_{n-3} x^n, \\
&= (DW_0 + DW_1x + DW_2x^2) - 7(DWx + DW_1x^2) + 7GW_0x^2 \\
&\quad + \sum_{n=3}^{\infty}(DW_n - 7DW_{n-1} + 7DW_{n-2} - DW_{n-3})x^n, \\
&= DW_0 + DW_1x + DW_2x^2 - 7DW_0x - 7DW_1x^2 + 7DW_0x^2, \\
&= DW_0 + (DW_1 - 7DW_0)x + (DW_2 - 7DW_1 + 7DW_0)x^2.
\end{aligned}$$

Note that we use the recurrence relation $DW_n = 7DW_{n-1} - 7DW_{n-2} + DW_{n-3}$. We rearrange equation which is given above then we obtain (2.8). □

As specific cases, the generating functions of the dual Edouard, Edouard-Lucas are given by

$$\begin{aligned}
f_{DE_n}(x) &= \frac{\varepsilon + x}{(1-7x+7x^2-x^3)}, \\
f_{DK_n}(x) &= \frac{7\varepsilon + 3 + (-14\varepsilon - 14)\,x + (3\varepsilon + 7)\,x^2}{(1-7x+7x^2-x^3)},
\end{aligned}$$

respectively. □

Next, we give the exponential generating function of $\sum\limits_{n=0}^{\infty} DW_n \frac{x^n}{n!}$ of the sequence DW_n.

LEMMA 5. *Suppose that* $f_{DW_n}(x) = \sum\limits_{n=0}^{\infty} DW_n \frac{x^n}{n!}$ *is the exponential generating function of the dual generalized Edouard sequence* $\{DW_n\}$.

Then $\sum\limits_{n=0}^{\infty} DW_n \frac{x^n}{n!}$ is given by

$$\begin{aligned}
\sum_{n=0}^{\infty} DW_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty} DW_n \frac{x^n}{n!} + \varepsilon\sum_{n=0}^{\infty} DW_{n+1} \frac{x^n}{n!} \\
&= \frac{(W_2 - (\beta+1)W_1 + \beta W_0)}{(\alpha-\beta)(\alpha-1)} e^{\alpha x} + \frac{(W_2 - (\alpha+1)W_1 + \alpha W_0)}{(\beta-\alpha)(\beta-1)} e^{\beta x} - \frac{(W_2 - 6W_1 + W_0)}{4} e^{x} \\
&\quad + \varepsilon\Big(\frac{(W_2 - (\beta+1)W_1 + \beta W_0)\alpha}{(\alpha-\beta)(\alpha-1)} e^{\alpha x} + \frac{(W_2 - (\alpha+1)W_1 + \alpha W_0)\beta}{(\beta-\alpha)(\beta-1)} e^{\beta x} - \frac{(W_2 - 6W_1 + W_0)}{4} e^{x}\Big).
\end{aligned}$$

Proof: Note that we have

$$\sum_{n=0}^{\infty} DW_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (DW_n + \varepsilon DW_{n+1}) \frac{x^n}{n!}.$$

Then using the Binet's formula of dual generalized Edouard numbers or exponential generating function of the generalized Edouard sequence we get the required identy.

The previous Lemma gives the following results as particular examples.

Corollary 6. *Exponential generating function of dual Edouard and dual Edouard-Lucas numbers are*

a):

$$\sum_{n=0}^{\infty} DE_n \frac{x^n}{n!} = \frac{\alpha e^{\alpha x}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta e^{\beta x}}{(\beta-\alpha)(\beta-1)} - \frac{1}{4}e^x + \varepsilon(\frac{\alpha^2 e^{\alpha x}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta^2 e^{\beta x}}{(\beta-\alpha)(\beta-1)} - \frac{1}{4}e^x).$$

b):

$$\sum_{n=0}^{\infty} DK_n \frac{x^n}{n!} = e^{\alpha x} + e^{\beta x} + e^x + \varepsilon(\alpha e^{\alpha x} + \beta e^{\beta x} + e^x).$$

3. Deriving Binet's Formula From the Generating Function

Next ,by using generating function $f_{DW_n}(x)$, we investigate Binet formula of $\{DW_n\}$.

Theorem 7. *(Binet formula of dual generalized Edouard numbers)*

$$DW_n = \widetilde{\alpha} A_1 \alpha^n + \widetilde{\beta} A_2 \beta^n + \widetilde{\gamma} A_3. \tag{3.1}$$

Proof. We write

$$\sum_{n=0}^{\infty} DW_n x^n = \frac{DW_0 + (DW_1 - 7DW_0)x + (DW_2 - 7DW_1 + 7DW_0)x^2}{(1-7x+7x^2-x^3)} = \frac{d_1}{(1-\alpha x)} + \frac{d_2}{(1-\beta x)} + \frac{d_3}{(1-x)}, \tag{3.2}$$

so that

$$\begin{aligned} \sum_{n=0}^{\infty} DW_n x^n &= \frac{d_1}{(1-\alpha x)} + \frac{d_2}{(1-\beta x)} + \frac{d_3}{(1-x)}, \\ &= \frac{d_1(1-x)(1-\beta x) + d_2(1-\alpha x)(1-x) + d_3(1-\alpha x)(1-\beta x)}{(x^2-6x+1)(1-x)}, \end{aligned}$$

then, we get

$$DW_0 + (DW_1 - 7DW_0)x + (DW_2 - 7DW_1 + 7DW_0)x^2 = d_1 + d_2 + d_3 + (-d_2 - \alpha d_2 - \beta d_1 - \alpha d_3 - \beta d_3)x + (\alpha d_2 + \beta d_1 + \alpha\beta d_3)x^2.$$

By equation the coefficients of corresponding powers of x in the above equation, we get

$$\begin{aligned} DW_0 &= d_1 + d_2 + d_3, \\ DW_1 - 7DW_0 &= -d_2 - \alpha d_2 - \beta d_1 - \alpha d_3 - \beta d_3, \\ DW_2 - 7DW_1 + 7DW_0 &= \alpha d_2 + \beta d_1 + \alpha\beta d_3. \end{aligned} \tag{3.3}$$

If we solve (3.3) we obtain

$$
\begin{aligned}
d_1 &= \frac{DW_0\alpha^2 + (DW_1 - 7DW_0)\alpha + (DW_2 - 7DW_1 + 7DW_0)}{(\alpha-\beta)(\alpha-\gamma)}, \\
d_2 &= \frac{DW_0\beta^2 + (DW_1 - 7DW_0)\beta + (DW_2 - 7DW_1 + 7DW_0)}{(\beta-\alpha)(\beta-\gamma)}, \\
d_3 &= \frac{DW_0 + (DW_1 - 7DW_0) + (DW_2 - 7DW_1 + 7DW_0)}{(\gamma-\alpha)(\gamma-\beta)},
\end{aligned}
$$

Thus (3.2) stated as follows

$$
\begin{aligned}
\sum_{n=0}^{\infty} DW_n x^n &= d_1 \sum_{n=0}^{\infty} \alpha^n x^n + d_2 \sum_{n=0}^{\infty} \beta^n x^n + d_3 \sum_{n=0}^{\infty} x^n, \\
&= \sum_{n=0}^{\infty} (d_1\alpha^n + d_2\beta^n + d_3) x^n, \\
&= \sum_{n=0}^{\infty} \Big(\frac{DW_2 - (\beta+1)DW_1 + \beta DW_0}{(\alpha-\beta)(\alpha-\gamma)}\alpha^n + \frac{DW_2 - (\alpha+1)DW_1 + \alpha DW_0}{(\beta-\alpha)(\beta-\gamma)}\beta^n + \frac{DW_2 - 6DW_1 + DW_0}{(\gamma-\alpha)(\gamma-\beta)}\Big) x^n.
\end{aligned}
$$

Hence, we get

$$
DW_n = \widetilde{\alpha} A_1 \alpha^n + \widetilde{\beta} A_2 \beta^n + \widetilde{\gamma} A_3. \ \square
$$

4. Some Identities Related to Dual Generalized Edouard numbers

We will now introduce some specific identities, i.e Simpson's formula, for the dual generalized Edouard sequence $\{DW_n\}$. The next theorem gives the Simpson's formula for the dual generalized Edouard numbers.

THEOREM 8. *(Simpson's formula for dual generalized Edouard numbers) For all integers n we have,*

$$
\begin{vmatrix} DW_{n+2} & DW_{n+1} & DW_n \\ DW_{n+1} & DW_n & DW_{n-1} \\ DW_n & DW_{n-1} & DW_{n-2} \end{vmatrix} = \begin{vmatrix} DW_2 & DW_1 & DW_0 \\ DW_1 & DW_0 & DW_{-1} \\ DW_0 & DW_{-1} & DW_{-2} \end{vmatrix}. \qquad (4.1)
$$

Proof. First we assume that $n \geqslant 0$. For the proof, we employ mathematical induction on n. For $n = 0$ identity (4.1) is true. Now we take (4.1) is true for $n = k$. Therfore, the following identity can be written

$$
\begin{vmatrix} DW_{k+2} & DW_{k+1} & DW_k \\ DW_{k+1} & DW_k & DW_{k-1} \\ DW_k & DW_{k-1} & DW_{k-2} \end{vmatrix} = \begin{vmatrix} DW_2 & DW_1 & DW_0 \\ DW_1 & DW_0 & DW_{-1} \\ DW_0 & DW_{-1} & DW_{-2} \end{vmatrix}.
$$

If we take $n = k + 1$, we can get

$$\begin{aligned}
\begin{vmatrix} DW_{k+3} & DW_{k+2} & DW_{k+1} \\ DW_{k+2} & DW_{k+1} & DW_{k} \\ DW_{k+1} & DW_{k} & DW_{k-1} \end{vmatrix} &= \begin{vmatrix} 7DW_{k+2} - 7DW_{k+1} + DW_{k} & DW_{k+2} & DW_{k+1} \\ 7DW_{k+1} - 7DW_{k} + DW_{k-1} & DW_{k+1} & DW_{k} \\ 7DW_{k} - 7DW_{k-1} + DW_{k-2} & DW_{k} & DW_{k-1} \end{vmatrix} \\
&= 7\begin{vmatrix} DW_{k+2} & DW_{k+2} & DW_{k+1} \\ DW_{k+1} & DW_{k+1} & DW_{k} \\ DW_{k} & DW_{k} & DW_{k-1} \end{vmatrix} - 7\begin{vmatrix} DW_{k+1} & DW_{k+2} & DW_{k+1} \\ DW_{k} & DW_{k+1} & DW_{k} \\ DW_{k-1} & DW_{k} & DW_{k-1} \end{vmatrix} \\
&\quad + \begin{vmatrix} DW_{k} & DW_{k+2} & DW_{k+1} \\ DW_{k-1} & DW_{k+1} & DW_{k} \\ DW_{k-2} & DW_{k} & DW_{k-1} \end{vmatrix} \\
&= \begin{vmatrix} DW_{k+2} & DW_{k+1} & DW_{k} \\ DW_{k+1} & DW_{k} & DW_{k-1} \\ DW_{k} & DW_{k-1} & DW_{k-2} \end{vmatrix}
\end{aligned}$$

Attention that if we take $n < 0$ the proof can be conducted in a similarly. Thus, the proof is concluded.□

From Theorem 4.1, we get following corollary.

Corollary 9.

(a): $$\begin{vmatrix} DE_{n+2} & DE_{n+1} & DE_{n} \\ DE_{n+1} & DE_{n} & DE_{n-1} \\ DE_{n} & DE_{n-1} & DE_{n-2} \end{vmatrix} = -7\varepsilon - 1$$

(b): $$\begin{vmatrix} DK_{n+2} & DK_{n+1} & DK_{n} \\ DK_{n+1} & DK_{n} & DK_{n-1} \\ DK_{n} & DK_{n-1} & DK_{n-2} \end{vmatrix} = 3584\varepsilon + 512$$

Theorem 10. *We assume that n and m are integers, E_n is Edouard numbers, the following identity is true:*

$$DW_{m+n} = E_{m-1}DW_{n+2} + (E_{m-3} - 7E_{m-2})DW_{n+1} + E_{m-2}DW_{n}. \tag{4.2}$$

Proof. The identity (10) can be proved by mathematical induction on m. First we take $n, m \geqslant 0$. If $m = 0$ we get

$$DW_{n} = E_{-1}DW_{n+2} + (E_{-3} - 7E_{-2})DW_{n+1} + E_{-2}DW_{n}$$

which is true by seeing that $E_{-1}=0, E_{-2}=1, E_{-3}=7$. We assume that the identity given holds for $m=k$. For $m=k+1$, we get

$$\begin{aligned}
DW_{(k+1)+n} &= 7DW_{n+k}-7DW_{n+k-1}+DW_{n+k-2} \\
&= 7(E_{k-1}DW_{n+2}+(E_{k-3}-7E_{k-2})DW_{n+1}+E_{k-2}DW_n) \\
&\quad -7(E_{k-2}DW_{n+2}+(E_{k-4}-7E_{k-3})DW_{n+1}+E_{k-3}DW_n) \\
&\quad +(E_{k-3}DW_{n+2}+(E_{k-5}-7E_{k-4})DW_{n+1}+E_{k-4}DW_n) \\
&= (7E_{k-1}-7E_{k-2}+E_{k-3})DW_{n+2}+((7E_{k-3}-7E_{k-4}+E_{k-5}) \\
&\quad -7(7E_{k-2}-7E_{k-3}+E_{k-4}))DW_{n+1}+(7E_{k-2}-7E_{k-3}+E_{k-4})DW_n \\
&= E_kDW_{n+2}+(E_{k-2}-7E_{k-1})DW_{n+1}+E_{k-1}DW_n \\
&= E_{(k+1)-1}DW_{n+2}+(E_{(k+1)-3}-7E_{(k+1)-2})DW_{n+1}+E_{(k+1)-2}DW_n.
\end{aligned}$$

The other cases on n,m the proof can be done easily. Consequently, by mathematical induction on m, this proves (10). □

5. Linear Sum Formulas of Dual Generalized Edouard Numbers

In this section, we give the summation formulas of the dual generalized Edouard numbers with subscripts.

PROPOSITION 11. *For the generalized Edouard numbers, we have the following formulas:*

(a): $\sum_{k=0}^{n} W_k = \frac{1}{4}(-(n+3)W_n+(n+2)(7W_{n+1}-W_{n+2})-(n+1)W_{n+1}+2W_2-13W_1+7W_0)$.

(b): $\sum_{k=0}^{n} W_{2k} = \frac{1}{32}(-(n+3)W_{2n}+(n+2)(-7W_{2n+2}+48W_{2n+1}-7W_{2n})-(n+1)W_{2n+2}+15W_2-96W_1+49W_0)$.

(c): $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{32}(-(n+3)W_{2n+1}+(n+2)(-W_{2n+2}+42W_{2n+1}-7W_{2n})-(n+1)(7W_{2n+2}-7W_{2n+1}+W_{2n})+9W_2-56W_1+15W_0)$.

Proof. It is given in Soykan [31, Theorem 3.3]. □

Now, we will introduce the formulas that allow us to find the sum of dual generalized Edouard numbers.

THEOREM 12. *For $n\geq 0$, dual generalized Edouard numbers have the following formulas:*

(a): $\sum_{k=0}^{n} DW_k = \frac{1}{4}(-(n+3)DW_n+(n+2)(7DW_{n+1}-DW_{n+2})-(n+1)DW_{n+1}+2DW_2-13DW_1+7DW_0)$.

(b): $\sum_{k=0}^{n} DW_{2k} = \frac{1}{32}(-(n+3)DW_{2n}+(n+2)(-7DW_{2n+2}+48DW_{2n+1}-7DW_{2n})-(n+1)DW_{2n+2}+15DW_2-96DW_1+49DW_0)$.

(c): $\sum_{k=0}^{n} DW_{2k+1} = \frac{1}{32}(-(n+3)DW_{2n+1}+(n+2)(-DW_{2n+2}+42DW_{2n+1}-7DW_{2n})-(n+1)(7DW_{2n+2}-7DW_{2n+1}+DW_{2n})+9DW_2-56DW_1+15DW_0)$.

Proof.

(a): Note that using (2.1), we get

$$\sum_{k=0}^{n} DW_k = \sum_{k=0}^{n} W_k + \varepsilon \sum_{k=0}^{n} W_{k+1}$$

and using Proposition 11 the proof can be done easily.

(b): Note that using (2.1), we get

$$\sum_{k=0}^{n} DW_{2k} = \sum_{k=0}^{n} W_{2k} + \varepsilon \sum_{k=0}^{n} W_{2k+1}$$

and using Proposition 11 the proof can be done easily.

(c): Note that using (2.1), we get

$$\sum_{k=0}^{n} DW_{2k+1} = \sum_{k=0}^{n} W_{2k+1} + \varepsilon \sum_{k=0}^{n} W_{2k+2}$$

and using Proposition 11 the proof can be done easily. □

As a special case of the Theorem 12 (a), we present the following corollary.

Corollary 13.

(a): $\sum_{k=0}^{n} DE_k = \frac{1}{4}(-(n+3)DE_n + (n+2)(7DE_{n+1} - DE_{n+2}) - (n+1)DE_{n+1} + 1)$.

(b): $\sum_{k=0}^{n} DK_k = \frac{1}{4}(-(n+3)DK_n + (n+2)(7DK_{n+1} - DK_{n+2}) - (n+1)DK_{n+1} - 8\varepsilon)$.

As a special case of the Theorem 12 (b), we present the following corollary.

Corollary 14.

(a): $\sum_{k=0}^{n} DE_{2k} = \frac{1}{32}(-(n+3)DE_{2n} + (n+2)(-7DE_{2n+2} + 48DE_{2n+1} - 7DE_{2n}) - (n+1)DE_{2n+2} + 7\varepsilon + 9)$.

(b): $\sum_{k=0}^{n} DK_{2k} = \frac{1}{32}(-(n+3)DK_{2n} + (n+2)(-7DK_{2n+2} + 48DK_{2n+1} - 7DK_{2n}) - (n+1)DK_{2n+2} - 32\varepsilon)$.

As a special case of the Theorem 12 (c), we present the following corollary.

Corollary 15.

(a): $\sum_{k=0}^{n} DE_{2k+1} = \frac{1}{32}(-(n+3)DE_{2n+1} + (n+2)(-DE_{2n+2} + 42DE_{2n+1} - 7DE_{2n}) - (n+1)(7DE_{2n+2} - 7DE_{2n+1} + DE_{2n}) + \varepsilon + 7)$.

(b): $\sum_{k=0}^{n} DK_{2k+1} = \frac{1}{32}(-(n+3)DK_{2n+1} + (n+2)(-DK_{2n+2} + 42DK_{2n+1} - 7DK_{2n}) - (n+1)(7DK_{2n+2} - 7DK_{2n+1} + DK_{2n}) - 64\varepsilon - 32)$.

6. Matrices related with Dual Generalized Edouard Numbers

In this part of our study we give some identities on some matrices linked to dual Edouard numbers.By using the $\{E_n\}$ which is defined by the third-order recurrence relation as follows

$$E_n = 7E_{n-1} - 7E_{n-2} + E_{n-3}$$

with the initial conditions $E_0 = 0$, $E_1 = 1$, $E_2 = 7$ we present the square matrix A of order 3 as

$$A = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 1$. Then, we give the following Lemma.

Lemma 16. *For all integers n the following identity is tru*

$$\begin{pmatrix} DW_{n+2} \\ DW_{n+1} \\ DW_n \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}. \tag{6.1}$$

Proof. First, we get $n \geq 0$. Lemma 16 can be given by mathematical induction on n. If $n = 0$ we get

$$\begin{pmatrix} DW_2 \\ DW_1 \\ DW_0 \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}$$

which is true. We claim that the identity (6.1) given holds for $n = k$. Thus the following identity is true.

$$\begin{pmatrix} DW_{k+2} \\ DW_{k+1} \\ DW_k \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}.$$

For $n=k+1$, we get

$$\begin{aligned}
\begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} DW_2 \\ DW_1 \\ DW_0 \end{pmatrix} &= \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} DW_2 \\ DW_1 \\ DW_0 \end{pmatrix} \\
&= \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} DW_{k+2} \\ DW_{k+1} \\ DW_k \end{pmatrix} \\
&= \begin{pmatrix} 7DW_{k+2} - 7DW_{k+1} + DW_k \\ DW_{k+2} \\ DW_{k+1} \end{pmatrix} \\
&= \begin{pmatrix} DW_{k+3} \\ DW_{k+2} \\ DW_{k+1} \end{pmatrix}.
\end{aligned}$$

For the other case $n<0$ the proof is easily attainable. Consequently, using mathematical induction on n, the proof is completed.

Note that

$$A^n = \begin{pmatrix} E_{n+1} & -7E_n + E_{n-1} & E_n \\ E_n & -7E_{n-1} + E_{n-2} & E_{n-1} \\ E_{n-1} & -7E_{n-2} + E_{n-3} & E_{n-2} \end{pmatrix}.$$

For the proof and more detail see [29].

THEOREM 17. *If we define the matrices N_{DW} and S_{DW} as follow*

$$\begin{aligned}
N_{DW} &= \begin{pmatrix} DW_2 & DW_1 & DW_0 \\ DW_1 & DW_0 & DW_{-1} \\ DW_0 & DW_{-1} & DW_{-2} \end{pmatrix}, \\
S_{DW} &= \begin{pmatrix} DW_{n+2} & DW_{n+1} & DW_n \\ DW_{n+1} & DW_n & DW_{n-1} \\ DW_n & DW_{n-1} & DW_{n-2} \end{pmatrix}.
\end{aligned}$$

then the following identity is true:

$$A^n N_{DW} = S_{DW}.$$

Proof. For the proof, we can use the following identities

$$\begin{aligned} A^n N_{DW} &= \begin{pmatrix} E_{n+1} & -7E_n + E_{n-1} & E_n \\ E_n & -7E_{n-1} + E_{n-2} & E_{n-1} \\ E_{n-1} & -7E_{n-2} + E_{n-3} & E_{n-2} \end{pmatrix} \begin{pmatrix} DW_2 & DW_1 & DW_0 \\ DW_1 & DW_0 & DW_{-1} \\ DW_0 & DW_{-1} & DW_{-2} \end{pmatrix}, \\ &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} a_{11} &= DW_2 E_{n+1} + DW_1 (E_{n-1} - 7E_n) + DW_0 E_n, \\ a_{12} &= DW_1 E_{n+1} + DW_0 (E_{n-1} - 7E_n) + DW_{-1} E_n, \\ a_{13} &= DW_0 E_{n+1} + DW_{-1} (E_{n-1} - 7E_n) + DW_{-2} E_n, \\ a_{21} &= DW_2 E_n + DW_1 (E_{n-2} - 7E_{n-1}) + DW_0 E_{n-1}, \\ a_{22} &= DW_1 E_n + DW_0 (E_{n-2} - 7E_{n-1}) + DW_{-1} E_{n-1}, \\ a_{23} &= DW_0 E_n + DW_{-1} (E_{n-2} - 7E_{n-1}) + DW_{-2} E_{n-1}, \\ a_{31} &= DW_2 E_{n-1} + DW_1 (E_{n-3} - 7E_{n-2}) + DW_0 E_{n-2}, \\ a_{32} &= DW_1 E_{n-1} + DW_0 (E_{n-3} - 7E_{n-2}) + DW_{-1} E_{n-2}, \\ a_{33} &= DW_0 E_{n-1} + DW_{-1} (E_{n-3} - 7E_{n-2}) + DW_{-2} E_{n-2}, \end{aligned}$$

Using the Theorem 10 the proof is done. □

From Theorem 17, the following corollary can be written.

COROLLARY 18.

(a): *Let the matrices N_{DE} and S_{DE} are defined as the following*

$$N_{DE} = \begin{pmatrix} DE_2 & DE_1 & DE_0 \\ DE_1 & DE_0 & DE_{-1} \\ DE_0 & DE_{-1} & DE_{-2} \end{pmatrix},$$

$$S_{DE} = \begin{pmatrix} DE_{n+2} & DE_{n+1} & DE_n \\ DE_{n+1} & DE_n & DE_{n-1} \\ DE_n & DE_{n-1} & DE_{n-2} \end{pmatrix},$$

so that the identity given below is true for A^n, N_{DE}, S_{DE},

$$A^n N_{DE} = S_{DE},$$

(b): *Let the matrices N_{DK} and S_{DK} are defined as the following*

$$N_{DK}=\begin{pmatrix} DK_2 & DK_1 & DK_0 \\ DK_1 & DK_0 & DK_{-1} \\ DK_0 & DK_{-1} & DK_{-2} \end{pmatrix},$$

$$S_{DK}=\begin{pmatrix} DK_{n+2} & DK_{n+1} & DK_n \\ DK_{n+1} & DK_n & DK_{n-1} \\ DK_n & DK_{n-1} & DK_{n-2} \end{pmatrix},$$

so that the following identity is true for A^n, N_{DK}, S_{DK},

$$A^n N_{DK} = S_{DK}.$$

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