UNDER PEER REVIEW

Coefficient Estimates for Initial Taylor-Maclaurin Coefficients for a New Family of Bi-Univalent and Analytic Functions Defined Using the Integral Operator

ABSTRACT

In this research article, I introduce a new subclass of analytic and bi-univalent function defined on the open unit disc $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$ using the integral operator and then I derive the initial coefficient bounds for $|a_2|$ and $|a_3|$. Later on, we discuss the many interesting results which are obtained as the special cases of our main result.

Keywords: Bi-univalent functions, Coefficient bounds, Subordination, Integral Operator.

2010 Mathematics Subject Classification: 30C45, 30C80

1. INTRODUCTION

Geometric function theory is very interesting and fascinating area of research. This field attracts researchers because of variety of ideas, methods and open challenges. The study of univalent, bi-univalent and multivalent functions are the branches of Geometric Function Theory and these are active fields of research even after more than century. In the study of bi-univalent functions, geometric behavior of functions can be analyzed by estimating coefficient bounds. In the early stages of development of the theory, Bieberbach conjecture (1916) was the open problem and it was successfully resolved by de Branges in 1985. Later on, researchers like Duren P.L. (1983), Nehari Z. (1953) and others gave remarkable contribution to this theory and resolved the issues related to the Bieberbach conjecture. Lewin M. (1967) extended the theory of univalent functions and introduced an interesting concept of bi-univalent functions. Later on, many researchers contributed to this area of research and attracted new researchers to contribute in this field.

In this research paper, I introduce a new subclass of bi-univalent and analytic functions and I find initial coefficient bounds for the class of functions.

Let $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$ be an open unit disc in the complex plane $\mathbb{C}$. Let $G(\mathbb{U})$ be a class of analytic functions defined on $\mathbb{U}$ and which are normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Hence it can be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (z \in \mathbb{U}). \tag{1}$$

Let $G_{\mathbb{U}}$ be class of all normalized analytic functions defined on $\mathbb{U}$, which are univalent in $\mathbb{U}$.

For two analytic functions f and g defined on $\mathbb{U}$, we say f is subordinate to g in $\mathbb{U}$ and it is defined as

$$f(z) \prec g(z) \quad (z \in \mathbb{U}), \tag{2}$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0, \ |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that,

$$f(z) = g\big(\omega(z)\big), \ (z \in \mathbb{U}). \tag{3}$$

If the function g is univalent in $\mathbb{U}$, then we have the following equivalence.

$$f(z) \prec g(z), \ (z \in \mathbb{U}) \ \Leftrightarrow \ f(0) = g(0), \ f(\mathbb{U}) \subseteq g(\mathbb{U}).$$

For $f \in G_{\mathbb{U}}$, we get f is invertible. The Koebe one-quarter theorem [Duren P.L. (1983)] ensures that, the image of $\mathbb{U}$ under every $f \in G_{\mathbb{U}}$ contains a disc of radius $\frac{1}{4}$.

Thus for each $f \in G_{\mathbb{U}}$, f^{-1} exists and define as

$$f^{-1}\big(f(z)\big) = z, \quad z \in \mathbb{U}$$

and

$$f\big(f^{-1}(w)\big) = w, \ |w| < r, \qquad r_0(f) \geq \frac{1}{4}.$$

Here, the inverse function f^{-1} is given by

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \tag{4}$$

A function $f \in G(\mathbb{U})$ is said to be bi-univalent if both f and its inverse, f^{-1} are univalent on $\mathbb{U}$. Let $\sum$ be a class of such bi-univalent functions defined on $\mathbb{U}$ and which are in the form (1). The subclasses of such bi-univalent functions are discovered and studied by many researchers and initial coefficient bounds are also obtained. For the brief history and the recent work on interesting examples subclasses of $\sum$, see Srivastava H.M., Mishra A.K. and Gochhayat (2010) and Brannan D.A. and Taha T.S. (1988).

Motivated by the work of Pathak R.P., Jadhav S.D., Khatu R.S. and Patil A.B. (2024) and Serap Bulut (2013), I introduce one new subclass of bi-univalent functions and further find initial coefficient bounds.

2. PRILIMINARIES, DEFINITIONS AND EXAMPLES

An integral operator $\mathcal{J}_{m,n}$ defined on the class $G_{\mathbb{U}}$ of analytic functions in introduced and studied by Pathak R.P., Jadhav S.D., Khatu R.S. and Patil A.B. (2024) and it is defined as follows.

2.1 Definition: Let $f \in \sum$ and $m, n > 0$. The integral operator $\mathcal{J}_{m,n}: G_{\mathbb{U}} \to G_{\mathbb{U}}$ defined as

$$\mathcal{J}_{m,n}f(z) = \frac{1}{\beta(m+1, n+1)} \int_0^\infty \frac{(t^{m-1} + t^{n-1})}{2(1+t)^{m+n+1}} f\left(\frac{tz}{1+t}\right) dt$$

where

$$\beta(m, n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} dt.$$

It can be easily observed that,

$$\mathcal{J}_{m,n}f(z) = z + \textstyle\sum_{k=2}^{\infty} \mathcal{J}_{m,n}^k a_k z^k. \tag{5}$$

where,

$$\mathcal{J}_{m,n}^k = \frac{\beta(m+k, n+1) + \beta(n+k, m+1)}{2\beta(m+1, n+1)}.$$

In general,

$$\mathcal{J}_{m,n}^{\alpha}f(z) = z + \textstyle\sum_{k=2}^{\infty} \left(\mathcal{J}_{m,n}^k\right)^{\alpha} a_k z^k, \text{where } \alpha \in \mathbb{N} \cup \{0\}. \tag{6}$$

Now I define a new subclass of bi-univalent functions using above integral operator.

2.2 Definition: Let $\phi: \mathbb{U} \to \mathbb{C}$ be a convex univalent function such that

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$$\phi(\bar{z}) = \overline{\phi(z)},\ \phi(0) = 1\ (z \in \mathbb{U}: Re(\phi(z)) > 0). \quad (7)$$

A bi-univalent function f given by equation (1) belongs to the class $\mathcal{R}_{\Sigma}(\lambda, m, n, \alpha, \beta; \phi)$ if the following condition holds.

$$e^{i\beta}\left((1-\lambda)\frac{\mathcal{J}^{\alpha}_{m,n}f(z)}{z} + \lambda\left(\mathcal{J}^{\alpha}_{m,n}f(z)\right)'\right) \prec \phi(z)cos\beta + isin\beta \quad (z \in \mathbb{U}),$$
$$e^{i\beta}\left((1-\lambda)\frac{\mathcal{J}^{\alpha}_{m,n}g(w)}{w} + \lambda\left(\mathcal{J}^{\alpha}_{m,n}g(w)\right)'\right) \prec \phi(w)cos\beta + isin\beta \quad (w \in \mathbb{U}) \quad (8)$$

where $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \lambda \geq 1, \alpha \in \mathbb{N} \cup \{0\}, m > 0, n > 0,\ \mathcal{J}^{\alpha}_{m,n}$ is the operator defined by equation (6) and the function $g(w) = f^{-1}(w)$ is given by the equation (4).

Remark 2.1: If we set $\phi(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Definition 2.2 then the class $\mathcal{R}_{\Sigma}(\lambda, m, n, \alpha, \beta; \phi)$ reduces to the class $\mathcal{R}_{\Sigma}(\lambda, m, n, \alpha, \beta; A, B)$ which is a class of bi-univalent functions f given by equation (1) satisfying

$$e^{i\beta}\left((1-\lambda)\frac{\mathcal{J}^{\alpha}_{m,n}f(z)}{z} + \lambda\left(\mathcal{J}^{\alpha}_{m,n}f(z)\right)'\right) \prec \frac{1+Az}{1+Bz}cos\beta + isin\beta \quad (z \in \mathbb{U}),$$
$$e^{i\beta}\left((1-\lambda)\frac{\mathcal{J}^{\alpha}_{m,n}g(w)}{w} + \lambda\left(\mathcal{J}^{\alpha}_{m,n}g(w)\right)'\right) \prec \frac{1+Aw}{1+Bw}cos\beta + isin\beta \quad (w \in \mathbb{U}) \quad (9)$$

where $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \lambda \geq 1, \alpha \in \mathbb{N} \cup \{0\}, m > 0, n > 0,\ \mathcal{J}^{\alpha}_{m,n}$ is the operator defined by equation (6) and the function $g(w) = f^{-1}(w)$ is given by the equation (4).

Remark 2.2: If we set $\phi(z) = \frac{(1+(1-2\tau)z)}{1-z}$ $(0 \leq \tau < 1)$ in Definition 2.2 then the class $\mathcal{R}_{\Sigma}(\lambda, m, n, \alpha, \beta; \phi)$ reduces to the class $\mathcal{R}_{\Sigma}(\lambda, m, n, \alpha, \beta, \tau)$ which is a class of bi-univalent functions f given by equation (1) satisfying

$$Re\left\{e^{i\beta}\left((1-\lambda)\frac{\mathcal{J}^{\alpha}_{m,n}f(z)}{z} + \lambda\left(\mathcal{J}^{\alpha}_{m,n}f(z)\right)'\right)\right\} > \tau cos\beta \quad (z \in \mathbb{U}),$$
$$Re\left\{e^{i\beta}\left((1-\lambda)\frac{\mathcal{J}^{\alpha}_{m,n}g(w)}{w} + \lambda\left(\mathcal{J}^{\alpha}_{m,n}g(w)\right)'\right)\right\} > \tau cos\beta \quad (w \in \mathbb{U}) \quad (10)$$

where $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \lambda \geq 1, \alpha \in \mathbb{N} \cup \{0\}, m > 0, n > 0,\ \mathcal{J}^{\alpha}_{m,n}$ is the operator defined by equation (6) and the function $g(w) = f^{-1}(w)$ is given by the equation (4).

Remark 2.3: If we set $\alpha = 0$ and $\phi(z) = \frac{(1+(1-2\tau)z)}{1-z}$ $(0 \leq \tau < 1)$ in Definition 2.2 then the class $\mathcal{R}_{\Sigma}(\lambda, m, n, \alpha, \beta; \phi)$ reduces to the class $\mathcal{R}_{\Sigma}(\lambda, m, n, \beta, \tau)$ which is a class of bi-univalent functions f given by equation (1) satisfying

$$Re\left\{e^{i\beta}\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right)\right\} > \tau cos\beta \quad (z \in \mathbb{U}),$$
$$Re\left\{e^{i\beta}\left((1-\lambda)\frac{g(w)}{w} + \lambda g'(w)\right)\right\} > \tau cos\beta \quad (w \in \mathbb{U}) \quad (11)$$

where $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \lambda \geq 1, m > 0, n > 0$ and the function $g(w) = f^{-1}(w)$ is given by the equation (4).

Remark 2.4: If we set $\alpha = 0,\ \lambda = 1$ and $\phi(z) = \frac{(1+(1-2\tau)z)}{1-z}$ $(0 \leq \tau < 1)$ in Definition 2.2 then the class $\mathcal{R}_{\Sigma}(\lambda, m, n, \alpha, \beta; \phi)$ reduces to the class $\mathcal{R}_{\Sigma}(m, n, \beta, \tau)$ which is a class of bi-univalent functions f given by equation (1) satisfying

$$Re\{e^{i\beta}f'(z)\} > \tau cos\beta \quad (z \in \mathbb{U}),$$
$$Re\{e^{i\beta}g'(w)\} > \tau cos\beta \quad (w \in \mathbb{U}) \quad (12)$$

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where $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $m > 0, n > 0$ and the function $g(w) = f^{-1}(w)$ is given by the equation (4).

We use the following lemma to prove our result.

Lemma 2.1: Let the function $h(z)$ given by $h(z) = \sum_{n=1}^{\infty} B_n z^n$ is convex in $\mathbb{U}$. Suppose also that the function $\psi(z)$ given by $\psi(z) = \sum_{n=1}^{\infty} c_n z^n$ is holomorphic in $\mathbb{U}$. If $\psi(z) \prec h(z)$ $(z \in \mathbb{U})$, then $|c_n| \le |B_1|$ $(n \in \mathbb{N})$.

3. COEFFICIENT BOUNDS OF THE FUNCTION CLASS $\mathcal{R}_{\Sigma}(\lambda, m, n, \alpha, \beta; \phi)$

This part of the paper is devoted to the estimation of coefficient bounds for the function class $\mathcal{R}_{\Sigma}(\lambda, m, n, \alpha, \beta; \phi)$.

Theorem 3.1: Let $f \in \Sigma$ be a function belongs to the class $\mathcal{R}_{\Sigma}(\lambda, m, n, \alpha, \beta; \phi)$ which is in the form given by equation (1) and $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ then following inequalities hold.

$$|a_2| \le \min\left\{\frac{|B_1|cos\beta}{(1+\lambda)\left(\mathcal{J}_{m,n}^2\right)^{\alpha}}, \sqrt{\frac{|B_1|cos\beta}{(1+2\lambda)\left(\mathcal{J}_{m,n}^3\right)^{\alpha}}}\right\}, \tag{13}$$

$$|a_3| \le \frac{|B_1|cos\beta}{(1+2\lambda)\left(\mathcal{J}_{m,n}^3\right)^{\alpha}} \tag{14}$$

where $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \lambda \ge 1, \alpha \in \mathbb{N} \cup \{0\}, m > 0$ and $n > 0$.

Proof: Let $f \in \mathcal{R}_{\Sigma}(\lambda, m, n, \alpha, \beta; \phi)$. Then by definition of $\mathcal{R}_{\Sigma}(\lambda, m, n, \alpha, \beta; \phi)$ and the subordination principle, we get,

$$e^{i\beta}\left((1-\lambda)\frac{\mathcal{J}_{m,n}^{\alpha} f(z)}{z} + \lambda\left(\mathcal{J}_{m,n}^{\alpha} f(z)\right)'\right) = p(z)cos\beta + isin\beta \qquad (z \in \mathbb{U}), \tag{15}$$

$$e^{i\beta}\left((1-\lambda)\frac{\mathcal{J}_{m,n}^{\alpha} g(w)}{w} + \lambda\left(\mathcal{J}_{m,n}^{\alpha} g(w)\right)'\right) = q(w)cos\beta + isin\beta \qquad (w \in \mathbb{U}) \tag{16}$$

where $p(z) \prec \phi(z)$ and $q(w) \prec \phi(w)$ and having Taylor-Maclaurin series expansions

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \tag{17}$$

$$q(w) = 1 + q_1 w + q_2 w^2 + \cdots \tag{18}$$

respectively.

Now, by using the expressions of p and q given by equations (17) and (18) in equations (15) and (16) and equating coefficients of like powers of z and w, we get

$$e^{i\beta}(1+\lambda)\left(\mathcal{J}_{m,n}^2\right)^{\alpha} a_2 = p_1 cos\beta, \tag{19}$$

$$e^{i\beta}(1+2\lambda)\left(\mathcal{J}_{m,n}^3\right)^{\alpha} a_3 = p_2 cos\beta, \tag{20}$$

$$-e^{i\beta}(1+\lambda)\left(\mathcal{J}_{m,n}^2\right)^{\alpha} a_2 = q_1 cos\beta, \tag{21}$$

$$e^{i\beta}(1+2\lambda)\left(\mathcal{J}_{m,n}^3\right)^{\alpha}(2a_2^2 - a_3) = q_2 cos\beta. \tag{22}$$

Now, from (19) and (21), we get

$$p_1 = -q_1 \tag{23}$$

and

$$2e^{2i\beta}(1+\lambda)^2\left(\mathcal{J}_{m,n}^2\right)^{2\alpha} a_2^2 = (p_1^2 + q_1^2)\cos^2\beta. \tag{24}$$

It gives

$$a_2^2 = \frac{e^{-2i\beta}(p_1^2 + q_1^2)\cos^2\beta}{2(1+\lambda)^2\left(\mathcal{J}_{m,n}^2\right)^{2\alpha}}. \tag{25}$$

Now, by using (20) and (22), we find that

$$2e^{i\beta}(1+2\lambda)\left(\mathcal{J}_{m,n}^3\right)^{\alpha} a_2^2 = (p_2 + q_2)cos\beta. \tag{26}$$

It gives

$$a_2^2 = \frac{e^{-i\beta}(p_2 + q_2)cos\beta}{2(1+2\lambda)\left(\mathcal{J}_{m,n}^3\right)^{\alpha}}. \tag{27}$$

Here $p, q \in \phi(\mathbb{U})$. Then by using Lemma 2.1, we have

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$$|p_k| = \left|\frac{p^k(0)}{k!}\right| \leq |B_1| \quad (k \in \mathbb{N}), \tag{28}$$

$$|q_k| = \left|\frac{q^k(0)}{k!}\right| \leq |B_1| \quad (k \in \mathbb{N}). \tag{29}$$

By using (28) and (29) for the coefficients p_1, p_2, q_1 and q_2, from the inequalities (25) and (27) we obtain

$$|a_2|^2 \leq \frac{|B_1|^2 \cos^2\beta}{(1+\lambda)^2(\mathcal{J}_{m,n}^2)^{2\alpha}} \tag{30}$$

and

$$|a_2|^2 \leq \frac{|B_1| cos\beta}{(1+2\lambda)(\mathcal{J}_{m,n}^3)^{\alpha}} \tag{31}$$

respectively. Inequalities (30) and (31) give the desire estimate on the coefficient bound of $|a_2|$ as stated in (13).

Next, to obtain the coefficient bound on a_3, we subtract (22) from (20), we get

$$2e^{i\beta}(1+2\lambda)(\mathcal{J}_{m,n}^3)^{\alpha}a_3 - 2e^{i\beta}(1+2\lambda)(\mathcal{J}_{m,n}^3)^{\alpha}a_2^2 = (p_2 - q_2)cos\beta. \tag{32}$$

This can be written as

$$a_3 = a_2^2 + \frac{e^{-i\beta}(p_2-q_2)cos\beta}{2(1+2\lambda)(\mathcal{J}_{m,n}^3)^{\alpha}}. \tag{33}$$

Substituting the value of a_2^2 obtained from (25) in (33), we get

$$a_3 = \frac{e^{-2i\beta}(p_1^2+q_1^2)\cos^2\beta}{2(1+\lambda)^2(\mathcal{J}_{m,n}^2)^{2\alpha}} + \frac{e^{-i\beta}(p_2-q_2)cos\beta}{2(1+2\lambda)(\mathcal{J}_{m,n}^3)^{\alpha}}. \tag{34}$$

By using (28) and (29) in (34), we get

$$|a_3| \leq \frac{|B_1|^2\cos^2\beta}{(1+\lambda)^2(\mathcal{J}_{m,n}^2)^{2\alpha}} + \frac{|B_1| cos\beta}{(1+2\lambda)(\mathcal{J}_{m,n}^3)^{\alpha}}. \tag{35}$$

On the other hand, substituting the value of a_2^2 obtained from (27) in (33), we get

$$a_3 = \frac{e^{-i\beta}(p_2+q_2)cos\beta}{2(1+2\lambda)(\mathcal{J}_{m,n}^3)^{\alpha}} + \frac{e^{-i\beta}(p_2-q_2)cos\beta}{2(1+2\lambda)(\mathcal{J}_{m,n}^2)^{\alpha}}. \tag{36}$$

By using (28) and (29) in (36), we get

$$|a_3| \leq \frac{|B_1| cos\beta}{(1+2\lambda)(\mathcal{J}_{m,n}^3)^{\alpha}}. \tag{37}$$

By comparing (35) and (37), we get the desire coefficient bound of $|a_3|$ stated in (14). Hence the theorem.

4. COROLLARIES AND CONSEQUENCES

By setting $\phi(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in theorem 3.1, we get the following corollary.

Corollary 4.1: Let $f \in \Sigma$ be a function belongs to the class $\mathcal{R}_{\Sigma}(\lambda, m, n, \alpha, \beta; A, B)$ which is in the form given by equation (1) then following inequalities hold.

$$|a_2| \leq \min\left\{\frac{(A-B)cos\beta}{(1+\lambda)(\mathcal{J}_{m,n}^2)^{\alpha}}, \sqrt{\frac{(A-B)cos\beta}{(1+2\lambda)(\mathcal{J}_{m,n}^3)^{\alpha}}}\right\}, \tag{38}$$

$$|a_3| \leq \frac{(A-B)cos\beta}{(1+2\lambda)(\mathcal{J}_{m,n}^3)^{\alpha}} \tag{39}$$

where $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \lambda \geq 1, \alpha \in \mathbb{N} \cup \{0\}, m > 0$ and $n > 0$.

By setting $\phi(z) = \frac{(1+(1-2\tau)z)}{1-z}$ $(0 \leq \tau < 1)$ in theorem 3.1, we get the following corollary.

Corollary 4.2: Let $f \in \Sigma$ be a function belongs to the class $\mathcal{R}_{\Sigma}(\lambda, m, n, \alpha, \beta, \tau)$ which is in the form given by equation (1) then following inequalities hold.

$$|a_2| \leq \min\left\{\frac{2(1-\tau)cos\beta}{(1+\lambda)(\mathcal{J}_{m,n}^2)^{\alpha}}, \sqrt{\frac{2(1-\tau)cos\beta}{(1+2\lambda)(\mathcal{J}_{m,n}^3)^{\alpha}}}\right\}, \tag{40}$$

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$$|a_3| \leq \frac{2(1-\tau)cos\beta}{(1+2\lambda)(\mathcal{J}^3_{m,n})^{\alpha}} \tag{41}$$

where $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \lambda \geq 1, \alpha \in \mathbb{N} \cup \{0\}, m > 0 \text{ and } n > 0.$

By setting $\alpha = 0 \text{ and } \phi(z) = \frac{(1+(1-2\tau)z)}{1-z}$ $(0 \leq \tau < 1)$ in theorem 3.1, we get the following corollary.

Corollary 4.3: Let $f \in \sum$ be a function belongs to the class $\mathcal{R}_{\sum}(\lambda, m, n, \beta, \tau)$ which is in the form given by equation (1) then following inequalities hold.

$$|a_2| \leq \min\left\{\frac{2(1-\tau)cos\beta}{(1+\lambda)}, \sqrt{\frac{2(1-\tau)cos\beta}{(1+2\lambda)}}\right\}, \tag{42}$$

$$|a_3| \leq \frac{2(1-\tau)cos\beta}{(1+2\lambda)} \tag{43}$$

where $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \lambda \geq 1, m > 0 \text{ and } n > 0.$

If we put $\beta = 0$ in the corollary 4.3 then we get improvement of estimates obtained by Frasin B.A. and Aouf M.K. (2011).

By setting $\alpha = 0,\ \lambda = 1 \text{ and } \phi(z) = \frac{(1+(1-2\tau)z)}{1-z}$ $(0 \leq \tau < 1)$ in theorem 3.1, we get the following corollary.

Corollary 4.4: Let $f \in \sum$ be a function belongs to the class $\mathcal{R}_{\sum}(m, n, \beta, \tau)$ which is in the form given by equation (1) then following inequalities hold.

$$|a_2| \leq \min\left\{(1-\tau)cos\beta, \sqrt{\frac{2(1-\tau)cos\beta}{3}}\right\}, \tag{44}$$

$$|a_3| \leq \frac{2(1-\tau)cos\beta}{3} \tag{45}$$

where $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),\ m > 0 \text{ and } n > 0.$

If we put $\beta = 0$ in the corollary 4.4 then we get improvement of estimates obtained by Srivastava H.M., Mishra A.K. & Gochhayat, P. (2010).

5. CONCLUSION

In this paper, using integral operator $\mathcal{J}^{\alpha}_{m,n}$, we defined a subclass of bi-univalent functions $\mathcal{R}_{\sum}(\lambda, m, n, \alpha, \beta; \phi)$ and later on, we obtained the coefficient bounds for $|a_2|$ and $|a_3|$. We observed that many interesting results and well-known findings are corollaries of our result.

DISCLAIMER (ARTIFICIAL INTELLIGENCE)

Author hereby declares that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during writing or editing of this manuscript.

REFERENCES

Brannan, D.A., &Taha, T.S. (1988). On some classes of bi-univalent functions, *Mathematical Analysis and Its Applications, vol. 3 of KFAS Proceedings Series*, pp. 53-60, Pergamon Press, Elsevier Science, Oxford, UK.

Duren, P.L. (1983). Univalent Functions, *Grundlehren der Mathematischen Wissesnschaften, Springer, New York*.

Frasin, B.A. & Aouf, M.K. (2011). New subclasses of bi-univalent functions, *Applied Mathematics Letters*, vol. 24, no. 9, pp. 1569-1573.

Lewin, M. (1967). On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc., 18, pp 63-68.

Nehari, Z. (1953). Conformal Mapping, McGraw-Hill Book Co., New York.

Pathak, R.P., Jadhav, S.D., Khatu, R.S., & Patil, A.B. (2024). Inequalities for a certain subclass of bi-univalent functions involving a new integral operator, *Int. J. Intelligent Sys. And Appl. in Engineering*, 12(21s), pp.3091-3097.

Serap Bulut (2013). Coefficient estimates for initial Taylor-Maclaurin coefficients for a subclass of analytic and bi-univalent functions defined by Al-Oboudi differential operator, *The Scientific World Journal*, Vol. 2013, Article ID 171039, 6 pages, http://dx.doi.org/10.1155/2013/171039.

Srivastava, H.M., Mishra, A.K., & Gochhayat, P. (2010). Certain subclasses of analytic and bi-univalent functions, *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1188-1192.