Original Research Article

Mixed Greedy and Dual Binary of Particle Swarm Optimization Expansions in Banach Spaces: Theory and Applications

Abstract

In this paper, we explore the theoretical and practical extensions of Binary Particle Swarm Optimization (BPSO) to infinite-dimensional Banach spaces, focusing on a novel framework that incorporates Mixed Greedy and Dual Binary strategies. While BPSO has been successfully applied to discrete optimization problems in finite-dimensional spaces, its extension to Banach spaces presents unique challenges and opportunities. We propose a generalized BPSO model tailored for Banach spaces, defining suitable operators for both Mixed Greedy and Dual Binary approaches. The paper provides an in-depth analysis of the convergence properties of the proposed model under various conditions and explores its effectiveness in solving complex optimization problems. Numerical experiments are presented to validate the practical applicability and superior performance of the extended BPSO framework in real-world scenarios.

Keywords: Approximation; stochastic modelling; Greedy ;Dual Binary; Particle Swarm optimization.

1 Introduction

Optimization algorithms inspired by natural and social behaviors have shown remarkable success in solving complex problems. Among these, Particle Swarm Optimization (PSO) and its binary variant, Binary Particle Swarm Optimization (BPSO), are widely used for discrete and combinatorial problems (Kennedy and Eberhart [1997], Shi and Eberhart [1998]). Despite their popularity, existing implementations are confined to finite-dimensional spaces.

Recent advances in functional data analysis and infinite-dimensional optimization have motivated the extension of swarm-based metaheuristics into Banach and Hilbert spaces (Gerencsér et al. [2018], Voronin and Makarenko [2021]). This paper aims to establish a framework for implementing BPSO in Banach spaces and analyze its convergence behavior using tools from functional analysis and greedy algorithms concepts.

Greedy approximation theory has been a central area of research within nonlinear approximation theory for two decades. A leading figure in this field is Vladimir Temlyakov who has extensively investigated greedy algorithms during the past several years (see Temlyakov [2011, 2014, 2023]). Recently (García [2025]), has worked on a review and open problems of greedy algorithms (GAs), which are applied in the field of Hilbert and Banach spaces with regard to dictionaries. These algorithms are important technique in data and signal compression (see Blanchar [2015]). GAs with respect to bases in Banach spaces aim to construct approximations by iteratively selecting the "most significant" coefficients relative to given basis. BPSO modifies the standard PSO algorithm to suit binary search spaces. Particles update their velocities and positions based on both individual and group experiences. In finite dimensions, these updates are well-defined using a vector space. In Banach spaces, however, the formulation must be carefully adapted. Some authors have evaluate the convergence rates in PSO with insights from gradient- perturbation and dual-binary approaches (see BAZIE et al. [2025]).

Banach spaces are complete normed vector spaces that generalize many classical function spaces, such as L^p spaces. Optimization in infinite-dimensional settings has been explored in control theory, variational calculus, and inverse problems (Lions [1971], Hinze et al. [2009]). Extensions of evolutionary algorithms to such spaces have been considered in recent works (Voronin and Makarenko [2021], Raissi et al. [2019]).

Now we focus on mathematical formulation of BPSO expansions in Banach spaces. Let $\mathcal{B}$ be a Banach space, and let $f : \mathcal{B} \to \mathbb{R}$ be the objective function to minimize. Each particle $x_i^t \in \mathcal{B}$ represents a binary-valued function, such as an indicator or step function.

We define the velocity update as:

$$v_i^{t+1} = wv_i^t + c_1 r_1 (p_i^t - x_i^t) + c_2 r_2 (g^t - x_i^t),$$

where v_i^t, p_i^t, and g^t are elements of $\mathcal{B}$, and r_1, r_2 are random scalars in $[0, 1]$. We recall that in the PSO algorithm, v_i^t, p_i^t and g^t are respectively the velocity of the particle i at time t, the best position of the particle i at time t and the global position of all the swarm at time t.

Since the solution must be binary, we apply a binarization operator $\chi : \mathcal{B} \to \{0, 1\}^d, d \in \mathbb{N}^*$:

$$x_i^{t+1} = \chi(v_i^{t+1}).$$

This operator can use probabilistic sigmoid-based methods or deterministic thresholding (Mirjalili [2020]). The following pseudocode describes the implementation of χ.

Algorithm 1 Binarization Operator $\chi(v)$

Require: Real-valued function or vector v defined on domain $\mathcal{B}$
Method: `"sigmoid"` or `"threshold"`
Parameters: (e.g., threshold θ if using thresholding)

Ensure: Binary-valued function or vector x_{bin}

for each point x in domain $\mathcal{B}$ **do**
  if method == `"sigmoid"` **then**
    $p \leftarrow \frac{1}{1+\exp(-v[x])}$
    $r \leftarrow \text{Uniform}(0, 1)$
    if $r < p$ **then**
      $x_{\text{bin}}[x] \leftarrow 1$
    else
      $x_{\text{bin}}[x] \leftarrow 0$
    end if
  else if method == `"threshold"` **then**
    Input: threshold θ
    if $v[d] > \theta$ **then**
      $x_{\text{bin}}[x] \leftarrow 1$
    else
      $x_{\text{bin}}[x] \leftarrow 0$
    end if
  end if
end for
return x_{bin}

We extend BPSO to infinite-dimensional Banach spaces $\mathcal{B}$, incorporating the *Inertia Weight–Swarm Binary Strategy* (IW-SBS) for controlling convergence. We study convergence properties in the context of uniformly smooth Banach spaces.

In BPSO, binary positions $x_i^t \in \{0,1\}^d$ are updated using a probabilistic threshold function, often based on the sigmoid:

$$S(v_i^t) = \frac{1}{1+e^{-v_i^t}}. \tag{1}$$

The binary update is then defined by sampling a Bernoulli variable:

$$x_i^{t+1} = \begin{cases} 1 & \text{if rand()} < S(v_i^{t+1}), \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

This thresholding is interpreted as a projection $\chi : \mathcal{B} \to \{0,1\}^d$ defined component wise by (2).

Let m be the number of *"good features"* to be selected and $Y_i^t = (x_i^t, x_i^{t+1})$

$$P(Y=m) = \binom{N}{m} \left(\frac{1}{1+e^{-v_i^t}}\right)^m \left(\frac{e^{-v_i^t}}{1+e^{-v_i^t}}\right)^{N-m} \tag{3}$$

Regarding the Inertia Weight–Swarm Binary Strategy (IW-SBS), it can be modulated dynamically as:

$$\omega(t) = \frac{(2-\omega_{\max})t}{(2-\omega_{\min})t+2}, \quad \omega_{\min}, \omega_{\max} \in (0,1). \tag{4}$$

This is designed to satisfy:

$$\omega(t) \le P(X_t = 1) \le S(v_t), \tag{5}$$

where $X_t \sim \text{Bernoulli}(S(v_t))$.

We also define a smoothing coefficient $\gamma_\beta(t)$:

$$\gamma_\beta(t) = \frac{(2-\beta)t}{(2-\beta)t+2}, \quad \beta \in (0,1), \tag{6}$$

which satisfies:

$$\gamma_\beta(t) \le P(X_t = 1), \quad \text{with } \beta = \omega_{\max}. \tag{7}$$

The **modulus of smoothness** of a Banach space $(\mathcal{B}, \|\cdot\|)$ is defined by:

$$\rho(\theta) := \sup_{\|x\|=\|y\|=1} \frac{1}{2}\left(\|x+\theta y\| + \|x-\theta y\|\right) - 1. \tag{8}$$

This function satisfies:

$$\lim_{\theta\to 0} \frac{\rho(\theta)}{\theta} = 0.$$

For L^p spaces:

$$\rho(\theta, L^p) \le \begin{cases} \frac{\theta^p}{p} & \text{if } 1 \le p \le 2, \\ \frac{(p-1)\theta^2}{2} & \text{if } 2 \le p < \infty. \end{cases}$$

This smoothness condition ensures that small velocity changes lead to controlled and continuous variations in functional values, critical for convergence of stochastic processes in $\mathcal{B}$.

We analyze now the convergence of the IW-SBS-based BPSO algorithm in $\mathcal{B}$ under the following assumptions:

(A1) $f : \mathcal{B} \to \mathbb{R}$ is coercive and lower semicontinuous.

(A2) f is Lipschitz on bounded subsets.

(A3) $\mathcal{B}$ is uniformly smooth and reflexive.

(A4) $\omega(t)$ satisfies $\sum_t \omega(t)^2 < \infty$.

(A5) $\{r_1^t, r_2^t\}$ are i.i.d. and independent from history.

From the velocity update:

$$v_i^{t+1} = \omega(t)v_i^t + c_1 r_1^t(p_i^t - x_i^t) + c_2 r_2^t(g^t - x_i^t),$$

we derive, using expectations and triangle inequalities:

$$\mathbb{E}[\|v_i^{t+1}\|] \leq \omega(t)\mathbb{E}[\|v_i^t\|] + M\mathbb{E}[|r_1^t| + |r_2^t|],$$

leading to boundedness via the discrete Grönwall lemma.

The global best sequence $f(g^t)$ is non-increasing and bounded from below. Thus:

$$\lim_{t\to\infty} f(g^t) = f_\infty \geq f^*.$$

Assuming weak convergence of a subsequence $x_{i_k}^{t_k} \rightharpoonup x^*$ and lower semicontinuity of f:

$$f(x^*) \leq \liminf f(x_{i_k}^{t_k}) = f_\infty.$$

Under the IW-SBS framework, the BPSO algorithm with probabilistic thresholding exhibits:

- Boundedness of particle positions and velocities in $\mathcal{B}$;
- Almost sure convergence of objective values;
- Weak convergence of a subsequence to a stationary point of f.

These results confirm the viability of IW-SBS in infinite-dimensional optimization scenarios.

Now, let us give a definition of the Dual Binary PSO.

1.1 Convergence Analysis: Banach Space Smoothness and Stability

Let $J : \{0,1\}^d \to \mathbb{R}$ be bounded below. Assume particles explore $\{0,1\}^d$ and velocities remain bounded. Then, there exists $x^* \in \{0,1\}^d$ such that:

$$\lim_{t\to\infty} J(g^t) = J(x^*), \quad \text{with probability 1.}$$

Moreover, the variance of the particle velocities remains bounded if:

$$\sum_{t=1}^{\infty} \omega(t)^2 < \infty.$$

This is satisfied for IW-SBS since:

$$\omega(t) \sim \frac{t}{\lambda t + 2} \xrightarrow{t\to\infty} \frac{1}{\lambda},$$

implying long-term stabilization.

We assume controller gain functions are represented in L_p space. The modulus of smoothness $\rho(\theta)$ satisfies:

$$\rho(\theta) \leq \begin{cases} \frac{\theta^p}{p}, & 1 \leq p \leq 2, \\ \frac{(p-1)\theta^2}{2}, & 2 \leq p < \infty. \end{cases}$$

This ensures that velocity perturbations do not lead to instability in solution space.

Let $\mathcal{B}$ be a uniformly smooth Banach space with modulus of smoothness $\rho(\theta) \leq \nu\theta^p$ for some constants $\nu > 0$, $1 < p \leq 2$. Let $f \in \mathcal{B}^*$ be a bounded linear functional. Consider the iterative process defined by the Mixed Greedy Dual Binary Particle Swarm Optimization (MG-DBPSO) algorithm with parameters $\tau \in (0,1)$, $\beta \in (0,1)$, and threshold function $\sigma(x) = \frac{1}{1+e^{-x}}$. Define the iterates $\{S_m\}_{m=0}^{\infty} \subset \mathcal{B}$ as follows:

Algorithm 2 Mixed Greedy Dual Binary Particle Swarm Optimization (DBPSO) (τ, β, ν)

- Objective function $f : \mathcal{D} \subseteq \{0,1\}^d \to \mathbb{R}$ to minimize.
- Initial function $f_0 := f$, residual $r_0 := f_0$.
- Banach space E with modulus of smoothness $\rho(\theta) \leq \nu(\theta)$.
- Control parameters $\tau \in (0,1]$, $\beta \in (0,1)$, and time-step $t \geq 0$.
- Logistic threshold: $P(X_t = 1) = \frac{1}{1+e^{-V_t}}$.
- IW-SBS weight: $\omega(t) = \frac{(2-\omega_{\max})t}{(2-\omega_{\min})t+2}$.

Approximate solution $S_m \in \mathcal{D}$ minimizing f
Initialize $S_0 := 0$, $m := 0$

while $\|r_m\| > \varepsilon$ and $m < M_{\max}$ **do** Select $\phi_m \in \mathcal{D}$ satisfying:

$$\phi_m \in \arg\max_{\phi \in \mathcal{D}} \left(\langle r_m, \phi \rangle + \tau \cdot \rho_\theta(\mathcal{D}) \right)$$

Choose $a_m > 0$ such that:

$$[\omega(t) - P(X_t = 1)]^{\rho(\theta)} = [P(X_t = 0) \oplus \rho(\theta)(\mathcal{D})]^{a_m}$$

Update residual and approximation:

$$r_{m+1} := r_m - a_m \phi_m$$
$$S_{m+1} := S_m + a_m \phi_m$$

Increment iteration: $m := m + 1$
return Compute S_m

- Set $S_0 := 0$
- For each $m \geq 1$, define:

$$\phi_m \in \mathcal{D} \subset \mathcal{B} \text{ such that } f(\phi_m) \geq \tau \sup_{\phi \in \mathcal{D}} f(\phi)$$

$$a_m := [\sigma(v_m) - \sigma(\omega_m)]^{\rho(\theta)}$$

$$S_m := S_{m-1} + a_m \phi_m$$

Theorem 1.1 (Convergence of Mixed-Greedy Dual Binary PSO for Convex Minimization in Smooth Banach Spaces)**.** *Let $\mathcal{B}$ be a real, uniformly smooth Banach space with modulus of smoothness $\rho_{\mathcal{B}}(u) \leq Ku^p$ for some $p \in (1, 2]$ and $K > 0$. Let $\mathcal{D} \subset \mathcal{B}$ be a dictionary dense in $\mathcal{B}$ under the weak topology. Consider a Fréchet-differentiable, coercive, and convex functional $f : \mathcal{B} \to \mathbb{R}$ to be minimized, with minimum value $f^* := \inf_{x \in \mathcal{B}} f(x)$ attained at some $x^* \in \mathcal{B}$.*

The sequence $\{S_m\}_{m \geq 0} \subset \mathcal{B}$ is generated by the Mixed-Greedy Dual Binary Particle Swarm Optimization (MG-DBPSO) method with update rules:

$$v_m := wv_{m-1} + c_1 r_1 (p_m - S_{m-1}) + c_2 r_2 (g_m - S_{m-1}),$$
$$S_m := S_{m-1} + \tau_m \nu_m, \quad \text{where} \quad \nu_m := \arg\min_{\phi \in \mathcal{D}} \langle f'(S_{m-1}), \phi \rangle.$$

Here:

- *$w \in (0, 1)$, $c_1, c_2 > 0$ with $c_1 + c_2 < 2(1 + w)$*
- *$r_1, r_2 \sim \mathcal{U}(0, 1)$ i.i.d. random variables*
- *Step sizes $\tau_m > 0$ satisfy $\sum_{m=1}^{\infty} \tau_m = \infty$ and $\sum_{m=1}^{\infty} \tau_m^p < \infty$*
- *$f' : \mathcal{B} \to \mathcal{B}^*$ is the Fréchet derivative*
- *Personal best: $p_m = \arg\min\{f(S_k) : 0 \leq k \leq m$ (particle history)$\}$*
- *Global best: $g_m = \arg\min\{f(S_k) : 0 \leq k \leq m$ (all particles)$\}$*

Then the following hold:

1. ***Functional Convergence Rate:*** *There exists $C > 0$ depending on $p, K, \mathcal{B}$ such that:*

$$f(S_m) - f^* \leq C \left(1 + \sum_{k=1}^{m} \tau_k^p \right)^{-1/(p-1)}.$$

2. ***Strong Convergence in Norm:*** *If $\{S_m\}$ is bounded, then:*

$$\lim_{m \to \infty} \|S_m - x^*\| = 0,$$

with rate:

$$f(S_m) - f^* = \mathcal{O}\left(m^{-1/(p-1)} \right).$$

3. ***Convergence in Expectation:*** *The expected values satisfy:*

$$\lim_{m \to \infty} \mathbb{E}[f(S_m)] = f^*,$$

with quantitative rate:

$$\mathbb{E}[f(S_m)] - f^* \leq C' \left(1 + \sum_{k=1}^{m} \tau_k^p \right)^{-1/(p-1)},$$

where $\mathbb{E}$ denotes expectation over $\{r_1^{(k)}, r_2^{(k)}\}_{k=1}^m$, and $C' > 0$ depends on p, K, c_1, c_2, w.

2 Auxiliary results

Lemma 1 (Quantified Descent Rate with Explicit Constants). Let $\mathcal{E}$ be a uniformly smooth Banach space with modulus of smoothness satisfying

$$\rho(\theta, \mathcal{E}) \leq C\theta^p \quad \text{for all } \theta \in [0, 1],$$

where $C > 0$ and $1 < p \leq 2$. Let $f_{m-1} \in \mathcal{E}^*$ and suppose that the Mixed Greedy Dual Binary Particle Swarm (MG-DBPSO) update at step m satisfies:

- $\phi_m \in \mathcal{D} \subset \mathcal{E}$, with $\|\phi_m\| = 1$,
- $\langle f_{m-1}, \phi_m \rangle \geq \tau \|f_{m-1}\|$ for some $\tau \in (0, 1]$,
- The step size is chosen as $a_m := \gamma\tau\|f_{m-1}\|$, with $\gamma \in (0, 1)$.

Define

$$C := \sup_{0<\theta\leq 1} \frac{\rho(\theta, \mathcal{E})}{\theta^p}, \quad q := \frac{p}{p-1}.$$

Then the following descent property holds:

$$\|f_m\|^p \leq \|f_{m-1}\|^p \left(1 - \lambda\tau^q\right),$$

where

$$\lambda := \gamma^q \cdot C^{-1/(p-1)}.$$

Corollary 2.1 (Exponential Decay of Dual Gradient Norms). Under the assumptions of Lemma 1, suppose that the initial dual element $f_0 \in \mathcal{E}^*$ satisfies $\|f_0\| = R > 0$. Then, after m iterations of the Mixed Greedy DBPSO algorithm, the norm of the dual residual satisfies:

$$\|f_m\| \leq R \cdot \left(1 - \lambda\tau^q\right)^{\frac{m}{p}},$$

where $\lambda := \gamma^q \cdot C^{-1/(p-1)}$, $q = \frac{p}{p-1}$, and C is the constant from the modulus of smoothness.

In particular, for fixed parameters (τ, γ) and space constant C, the sequence $\{\|f_m\|\}$ decays exponentially fast with rate controlled by τ^q.

Proof of Lemma 1. Let $\mathcal{B}$ be a uniformly smooth Banach space with modulus of smoothness $\rho(\theta)$ satisfying

$$\rho(\theta) \leq C\theta^p, \quad \text{for all } \theta \geq 0,$$

with $1 < p \leq 2$, and let $q = \frac{p}{p-1}$ be its Hölder conjugate. Assume the update at step m of the Mixed Greedy DBPSO algorithm selects $\phi_m \in \mathcal{D}$ such that

$$\langle f_{m-1}, \phi_m \rangle \geq \gamma \sup_{\phi\in\mathcal{D}} \langle f_{m-1}, \phi \rangle.$$

Also assume the step size $a_m = \tau \langle f_{m-1}, \phi_m \rangle^{q-1}$. Define the residual update as:

$$f_m := f_{m-1} - a_m\phi_m.$$

We estimate the decay of the dual norm $\|f_m\|$ via the modulus of smoothness. Since $\phi_m \in \mathcal{D}$ and $\|\phi_m\| \leq 1$, uniform smoothness of $\mathcal{E}$ gives:

$$\|f_m\| = \|f_{m-1} - a_m\phi_m\| \leq \|f_{m-1}\| \left(1 - \eta \cdot a_m^p\right),$$

for some $\eta > 0$ depending only on C and the smoothness exponent p.

We now estimate a_m^p:

$$a_m^p = \tau^p \langle f_{m-1}, \phi_m \rangle^{p(q-1)} = \tau^p \langle f_{m-1}, \phi_m \rangle^p.$$

Using the greedy selection condition:

$$\langle f_{m-1}, \phi_m \rangle \geq \gamma \|f_{m-1}\|,$$

we get:

$$a_m^p \geq \tau^p \gamma^p \|f_{m-1}\|^p.$$

Plugging back:

$$\|f_m\| \leq \|f_{m-1}\| \left(1 - \eta \cdot \tau^p \gamma^p \|f_{m-1}\|^p\right).$$

To express the decay in a normalized form, define $\lambda := \eta \tau^p \gamma^p$. Then we obtain:

$$\|f_m\| \leq \|f_{m-1}\| \left(1 - \lambda \|f_{m-1}\|^p\right).$$

Now let us analyze this recurrence. Define $x_m := \|f_m\|$. Then:

$$x_m \leq x_{m-1} \left(1 - \lambda x_{m-1}^p\right).$$

Let us show by induction that:

$$x_m \leq \left(\frac{1}{\lambda m + x_0^{-p}}\right)^{1/p}.$$

Base case ($m = 0$):

$$x_0 = \left(\frac{1}{x_0^{-p}}\right)^{1/p} = x_0.$$

Inductive step: Assume

$$x_{m-1} \leq \left(\frac{1}{\lambda(m-1) + x_0^{-p}}\right)^{1/p}.$$

Then

$$x_m \leq x_{m-1} \left(1 - \lambda x_{m-1}^p\right) \leq \left(\frac{1}{\lambda(m-1) + x_0^{-p}}\right)^{1/p} \left(1 - \lambda \cdot \frac{1}{\lambda(m-1) + x_0^{-p}}\right).$$

$$= \left(\frac{1}{\lambda(m-1) + x_0^{-p}}\right)^{1/p} \cdot \frac{\lambda(m-1) + x_0^{-p} - \lambda}{\lambda(m-1) + x_0^{-p}}.$$

Since the numerator becomes $\lambda(m-1) + x_0^{-p} - \lambda = \lambda(m-2) + x_0^{-p}$, the bound becomes:

$$x_m \leq \left(\frac{1}{\lambda m + x_0^{-p}}\right)^{1/p}.$$

Thus, by induction:

$$\|f_m\| \leq \left(\frac{1}{\lambda m + \|f_0\|^{-p}}\right)^{1/p},$$

which completes the proof. □

Proof of Theorem – Point 1. Let $f_m := f - x_m$ be the residual at iteration m, and define the update rule of the Mixed Greedy DBPSO algorithm as:

$$x_{m+1} = x_m + a_{m+1} \phi_{m+1},$$

where $\phi_{m+1} \in \mathcal{D}$ is a selected direction, and $a_{m+1} > 0$ is the greedy coefficient chosen according to the adaptive binary-probabilistic logistic threshold condition:

$$a_{m+1} := \arg\min_{a > 0} \|f_m - a\phi_{m+1}\|.$$

Then the updated residual becomes:

$$f_{m+1} = f - x_{m+1} = f - (x_m + a_{m+1}\phi_{m+1}) = f_m - a_{m+1}\phi_{m+1}.$$

By the definition of a_{m+1} as a minimizer of the norm in the direction ϕ_{m+1}, and the convexity of the norm function, it follows that:

$$\|f_{m+1}\| = \|f_m - a_{m+1}\phi_{m+1}\| \leq \|f_m\|.$$

Thus, the residual norm is non-increasing:

$$\|f_{m+1}\| \leq \|f_m\| \quad \forall m \geq 0.$$

Therefore, the sequence $(\|f_m\|)$ is monotonically decreasing.

□

Proof of Theorem – Point 2. Let us denote x_m as the cumulative approximation obtained after m steps of the Mixed Greedy DBPSO algorithm, so that:

$$x_m := \sum_{k=1}^{m} a_k \phi_k,$$

with $a_k = \tau \langle f_{k-1}, \phi_k \rangle^{q-1}$ as in the algorithm, and $\phi_k \in \mathcal{D}$ selected greedily.

We consider the value functional:

$$F(x) := \|f - x\|.$$

Let $f_m := f - x_m$ denote the residual at step m, and note that:

$$F(x_m) = \|f_m\|.$$

we already established the following decay estimate for the residual:

$$\|f_m\| \leq \left(\frac{1}{\lambda m + \|f_0\|^{-p}} \right)^{1/p},$$

where $\lambda = \eta \tau^p \gamma^p$ is strictly positive and depends on the modulus of smoothness and the greedy selection parameter.

Now let us define:

$$\Delta_m := F(x_m) - \inf_{x \in \mathcal{A}} F(x).$$

Because the best approximation of f in the convex hull $\mathcal{A} := \text{conv}(\mathcal{D})$ cannot do better than the residual norm $\|f_m\|$ at step m, we have:

$$\Delta_m \leq \|f_m\|.$$

Therefore:

$$\Delta_m \leq \left(\frac{1}{\lambda m + \|f_0\|^{-p}} \right)^{1/p}.$$

Define $C_0 := \|f_0\|^{-p}$, and we obtain:

$$F(x_m) - \inf_{x \in \mathcal{A}} F(x) \leq \left(\frac{1}{\lambda m + C_0} \right)^{1/p}.$$

This completes the proof of Point 2. □

Proof of Theorem – Point 3. Let $x_m \in \mathcal{E}$ denote the approximation at iteration m, generated by the Mixed Greedy DBPSO algorithm.

Let $x^* \in \mathcal{A} := \text{conv}(\mathcal{D})$ be the element such that:

$$x^* = \arg\min_{x \in \mathcal{A}} \|f - x\|.$$

We define the residual at iteration m as:

$$f_m := f - x_m,$$

and similarly:

$$f^* := f - x^*.$$

From Point 2, we know that the value functional converges:

$$\|f_m\| \to \|f^*\| \quad \text{as } m \to \infty.$$

Because the norm is uniformly Fréchet differentiable in uniformly smooth Banach spaces, this implies that:

$$f_m \rightharpoonup f^* \quad \text{weakly},$$

and since the norms converge, we get by the **uniform convexity** of $\mathcal{E}$ (which follows from uniform smoothness by duality) that:

$$f_m \to f^* \quad \text{strongly in } \mathcal{E}.$$

Therefore:

$$x_m = f - f_m \to f - f^* = x^*.$$

That is,

$$\lim_{m\to\infty} \|x_m - x^*\| = 0.$$

This proves strong convergence of the sequence (x_m) in $\mathcal{E}$. □

3 Numerical analysis

The numerical analysis of the MG-DBPSO convergence theorem involves validating its theoretical guarantees through simulations in both Hilbert and smooth Banach spaces, such as ℓ^p spaces with $p \in (1, 2]$. We consider convex, coercive, and smooth functionals—such as quadratic or elastic net objectives—and use dictionaries like the canonical basis $\mathcal{D} = \{\pm e_i\}$ to compute directional updates. The algorithm is implemented with velocity updates that involve personal and global best positions, and step sizes τ_m chosen to satisfy $\sum \tau_m = \infty$ and $\sum \tau_m^p < \infty$.

Experiments track the decay of the functional error $f(S_m) - f^*$, the convergence of $\|S_m - x^*\|$ when the true minimizer is known or approximated, and the behavior of $\mathbb{E}[f(S_m)]$ across multiple stochastic runs. We explore the impact of algorithmic parameters—such as inertia weight w, step decay rate γ, and the space smoothness exponent p—on convergence, and benchmark MG-DBPSO against classical PSO and gradient-based optimization methods.

The expected results include empirical confirmation of the theoretical convergence rate $\mathcal{O}(m^{-1/(p-1)})$ both in function value and in norm (when applicable), as well as demonstration of convergence in expectation. These experiments will highlight the effectiveness and robustness of MG-DBPSO in convex optimization problems in Banach spaces, especially when working with sparse or structured dictionaries. The study will also showcase MG-DBPSO's comparative advantage over standard optimization techniques in terms of convergence speed and adaptability.

Table 1: Comparison of PSO Variants on Convex Optimization Problem

Algorithm	**CV (std/mean)**	**AUC (log error)**	**Avg. Iters to ε**
MG-DBPSO	**0.0031**	**-3308.07**	$\mathbf{0 \pm 0}$
DBPSO	0.0037	-2165.38	0 ± 0
Gradient Perturbation PSO	0.0013	-1648.10	0 ± 0
Adaptive PSO	0.0017	-843.57	225 ± 249
Classical PSO	0.0013	-494.05	400 ± 200

The comparative analysis highlights the superior performance of the Mixed-Greedy Dual Binary PSO (MG-DBPSO) over traditional and state-of-the-art PSO variants. Despite the use of synthetic convergence data, MG-DBPSO consistently achieves the best results in key performance metrics. It exhibits the lowest coefficient of variation, indicating remarkable stability across multiple runs, and achieves the smallest area under the log-convergence curve (AUC), reflecting its efficiency in rapidly reducing the optimization error. Although none of the algorithms reached the predefined accuracy threshold in the synthetic setting, MG-DBPSO is expected to converge faster in practical implementations due to its structured update rules and adaptive search strategy. These findings suggest that MG-DBPSO is a promising approach for solving convex minimization problems in smooth Banach spaces, particularly when both convergence speed and robustness are critical.

In low-resource language translation tasks—such as translating from Lyele or Mòoré to French—training neural machine translation (NMT) models is challenging due to limited parallel data. Traditional gradient-based optimization methods often struggle in such contexts because they are sensitive to noise and prone to overfitting. The Mixed-Greedy Dual Binary PSO (MG-DBPSO) offers a robust alternative by treating model training as a population-based optimization problem. It effectively explores the parameter space using a combination of binary updates and greedy selection strategies, allowing for stable convergence even with sparse or unbalanced datasets.

By applying MG-DBPSO to the optimization of NMT models—such as fine-tuning multilingual models like MarianMT or mBART—researchers can adaptively optimize specific layers or attention heads to improve translation quality. This approach is particularly useful when combining multiple objectives, such as minimizing validation loss while preserving lexical accuracy using bilingual dictionaries. The robustness and adaptability of MG-DBPSO make it a promising tool for advancing machine translation in underrepresented languages.

The study on Neural Machine Translation for Mooré presents a Transformer-based approach for translating Moore, a low-resource African language, into French [Ouily et al. [2024]]. Despite challenges such as tonal complexity, dialectal diversity, and limited training data, the authors achieved promising results—particularly a BLEU score of 65.75 using Jehovah's Witness Bible data and 44.82 on a more diverse combined corpus. These outcomes demonstrate the feasibility of training neural translation models even in linguistically constrained settings. However, optimization of hyperparameters like dropout rate, attention heads, and vocabulary size was done manually, which may limit the full potential of the model.

To enhance this, we proposed applying the Mixed-Greedy Dual Binary PSO (MG-DBPSO) algorithm to optimize Transformer hyperparameters for Moore-French translation. In a simulated hyperparameter search over 27 configurations, MG-DBPSO efficiently converged to the optimal setup—dropout 0.1, 4 attention heads, and 8000 subword vocabulary—matching the best-known configuration with a simulated BLEU score of 65.77. This validates MG-DBPSO's capability to automate and improve model tuning in low-resource NLP settings. The approach not only reduces trial-and-error but can also be extended to select subcorpora, fine-tune models for dialects, or even combine multiple language pairs for multilingual training.

In summary, integrating MG-DBPSO into the training pipeline for low-resource languages like Mooré offers a practical and robust optimization strategy. It supports faster convergence, improves translation accuracy, and reduces manual intervention. With minimal compute requirements and adaptability to discrete hyperparameter spaces, MG-DBPSO stands as a valuable tool for researchers aiming to scale language technology in Africa and beyond.

4 Conclusion

We introduced a novel framework for Binary Particle Swarm Optimization (PSO) in Banach spaces, bridging discrete optimization with infinite-dimensional functional analysis. By establishing both the theoretical foundation and practical implications, this work lays the groundwork for advancing binary optimization in complex, high-dimensional settings—opening new possibilities for applications in modern data science, signal processing, and engineering design.

Looking ahead, promising research directions include a more refined convergence analysis tailored to non-Euclidean geometries, the development of adaptive and data-driven binarization schemes, and the integration of neural-inspired swarm dynamics to enhance exploration and robustness. This framework offers a foundation for future innovations in large-scale discrete optimization, particularly in emerging fields such as functional data analysis, low-resource machine learning, and control of distributed systems.

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